SURVEYING

THEORY AND PRACTICE

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PREFACE TO THE SECOND EDITION

The widespread use of the first edition of this book has resulted in many valuable suggestions, both as to arrangement and as to additional subject matter. In order the better to meet the requirements of the users of the book, the text has been entirely rewritten, embodying the detailed suggestions of more than a hundred practicing engineers and teachers of surveying. To meet the needs of students, portions of the text have been considerably condensed and detailed explanations have been set in small type. By this arrangement the quality of thoroughness, which was one of the distinguishing features of the first edition, has been preserved without lessening the usefulness of the text. The chapters have been arranged in the interest of more orderly presentation of the subject, and chapters have been added on Hydrographic Surveying and Flow Measurement and Map Projection.

A variety of new material has been included to bring the text thoroughly up to date. New developments in aerial photographic surveying are discussed, and additional information having to do with city, cadastral and geodetic surveying is given. New methods of reproducing drawings are described, and the subjects of curves and earthwork are more completely covered. Astronomical tables and the isogonic chart have been brought up to date, and tables of weir coefficients have been added.

The authors desire to express appreciation of the valuable services rendered by J. W. Kelly, Research Engineer, University of California, in the preparation and editing of the manuscript and in the reading of the proof. Special acknowledgment is also due to R. C. Sloane, Professor of Civil Engineering, Ohio State University, for the preparation of the manuscript for the chapter on Hydrographic Surveying and Flow Measurement, and to Fred C. Scobey, Senior Irrigation Engineer, U. S. Department of Agriculture, for the review of

the manuscript for that chapter and for valuable suggestions which have been incorporated therein.

It is also desired to acknowledge the services of the U. S. Geological Survey in furnishing tables of weir coefficients and photographs of instruments shown in Chapter XXVI.

The authors desire to express their gratification at the reception of the first edition and their appreciation of the interest of those many individuals whose valuable suggestions have greatly aided in the preparation of this, the second edition.

R. E. D.
F. S. F.
W. H. R.

May, 1934.

PREFACE TO THE FIRST EDITION

This book is intended primarily as a text for use in surveying classes as ordinarily conducted in engineering schools during the freshman or sophomore year, but it has also been the aim of the authors to produce in a single volume a treatise on the subject of surveying sufficiently comprehensive to be found of value to practicing engineers and surveyors.

To the end that it may be most useful to students, the more elementary phases of the theory and practice of surveying have been treated in considerable detail, and no pains have been spared to produce a book which will meet the needs of the classroom. The advice of teachers of surveying has been sought frequently and has been given freely.

In order that it may be found valuable to engineers and surveyors, the more advanced phases of the subject have been discussed, methods used on extensive surveys under a variety of conditions have been described, and the relative advantages of the various methods as affected by field conditions have been considered. Not only have the authors drawn upon their own experience but they have made free use of the works of others. The more specialized portions of the book have been reviewed by authorities in these fields.

The book is divided into three parts:

Section 1 is an introduction to the subject, designed to give the student a general familiarity with the fundamentals involved before proceeding to details, to furnish him with a speaking acquaintance with the rudimentary operations of field and office, and to present the general subject of errors, an understanding of which is so essential to good surveying practice. It is believed that this section will be found of particular value in those classes where field and office practice are given concurrently with the work of the classroom. It

may also prove instructive to inexperienced men who are coming in contact with the practice of surveying for the first time.

Section 2, dealing with the elements of surveying, treats in detail the fundamental operations common to all branches of surveying, involving the instruments for and methods of measuring angles, distances, and differences in elevation in the field, and of calculating and plotting these quantities in the office. The material of this section is essentially prerequisite to all surveying practice. It is, so to speak, the framework or skeleton about which any survey takes form. Here special attempt has been made thoroughly to develop principles and to describe the operations involved in sound surveying practice without resorting to statements of details which may best be learned through experience. Where there is more than one method of performing an operation, the aim has been to describe each method and to compare the several methods as to their relative advantages under varying conditions.

Section 3 is concerned with the practice of surveying as extended to entire surveys, including those for establishing boundaries of rural and urban lands; for locating railways, highways, and other routes; for the location and development of mining properties; and for topographic and photographic mapping, both terrestrial and aerial. Much of the material has been drawn from private sources, government publications, and periodicals.

In the preparation of this book, the authors have given careful study as to an arrangement and scope of subject matter which would result in a teachable text for the classroom, and, at the same time, would include additional material of great importance and value to those called upon to plan and carry out actual surveys, but which is ordinarily not given in elementary courses in surveying by reason of the limitation of time. In this study, they are happy to acknowledge the material assistance of experienced teachers by expressions of opinion of what ought to be included in a treatise of this kind. In order to render the greatest possible continuity to the book as a whole, all material relating to a given division of the subject has been placed in a single chapter, but those portions

which are not normally included in the ordinary course in surveying are indicated either by articles or paragraphs in smaller type than that of the body of the text or are placed near the end of the chapter. Thus, in Chap. VIII, the elementary operations of differential leveling are given, following which precise methods are described, the precision of leveling measurements is considered, and methods of adjusting level circuits are shown. By this arrangement, it is felt that assignments for study in the most elementary course may be made to cover readily the desired ground without confusion to the student, even though the material of the course be considerably reduced from that presented in the text.

Experience has demonstrated the desirability of field and office exercises, involving the elementary operations of surveying, to be taken concurrently with the study of the text. At the close of each chapter dealing with these operations, there is outlined a series of problems which it is believed will be found useful in conducting classes in field and office practice. Many of the minor details of field and office procedure not mentioned in the text are given in these problem statements.

Though a knowledge of errors is so vital to sound surveying practice, many surveyors display an ignorance of errors and their effects upon measurements that is truly lamentable; for this reason, it has seemed desirable that greater emphasis be put upon the subject of errors. To this end, Chap. V is devoted entirely to the consideration of errors, with particular reference to accidental errors and the law of probability, and throughout the body of the text, in connection with the various operations of surveying, the causes, kinds, and distribution of errors are discussed.

An understanding of field astronomy makes necessary certain new conceptions which are usually grasped with difficulty by the beginner, and are often not easily imparted by the instructor. It has been the view of the authors that in the classroom the development of principles is of greater importance than the mere recitation of facts. For these reasons, the subject of field astronomy has been treated at somewhat greater length than is ordinarily the case for a text on surveying, and considerable space has been devoted to

describing in simple language the fundamental concepts of the celestial sphere. Thus, Chap. XVII is devoted to developing the principles of astronomy, and Chap. XVIII is devoted to a description of the common methods of determining latitude, longitude, and azimuth, which are applicable to surveys of ordinary precision.

In the chapters on Topographic Surveying, special attention has been given to methods applicable to intermediate- and large-scale mapping, and effort has been made to indicate the dependence of methods upon conditions as regards character of terrain, scale of map, extent of survey, etc.

Terrestrial photographic mapping has long been recognized as a particularly efficient method of portraying the relief of mountainous regions. With the rapid improvement in the instruments and methods employed in aerial surveying, it seems probable that, except where the areas involved are comparatively small, much of the topographic mapping of the future will be accomplished by aerial photography. Chapter XXVII presents the principles of photographic surveying and more especially describes in some detail the processes of aerial map making. Much of the material for this chapter has been secured from unpublished notes, and here appears for the first time in textbook form.

In the preparation of this book, the authors have received suggestions, advice, and material from many sources. They are grateful for this assistance, without which this text would have been most incomplete. They especially desire to make the following acknowledgments:

Professor S. Einarsson, Department of Astronomy, University of California, has reviewed the manuscript for Chaps. XVII and XVIII and has made many suggestions which have resulted in substantially strengthening the material on astronomy.

The Surveyor General of the General Land Office has kindly furnished data for tables bearing on the U. S. system of subdividing the public lands described in Chap. XX, and the *Manual of Instructions* of the General Land Office has been freely quoted.

The Director of the U. S. Coast and Geodetic Survey has supplied data for Tables IV and V and has furnished photo-

graphs from which have been prepared a considerable number of the cuts for Chaps. XXIV, XXV, and XXVIII.

The Director of the U. S. Geological Survey has furnished photographs for numerous halftones for Chaps. XXIII, XXIV, XXV, and XXVII; and C. H. Birdseye, Chief Topographic Engineer and W. H. Herron, Geographer in Charge of Central Division, both of the U. S. Geological Survey, have rendered very generous help in the preparation of those portions of the manuscript dealing with topographic and photographic mapping.

F. H. Peters, Surveyor General of the Topographical Survey of Canada has gone to some trouble to supply suitable photographs from which Figs. 526*a*, 552*a*, and 552*b* have been prepared and has rendered valuable assistance in furnishing material on photographic and aerial mapping.

Major J. W. Bagley, Corps of Engineers, U. S. Army, formerly of the U. S. Geological Survey, has reviewed the manuscript for Chap. XXVII, dealing with photographic surveying; he has made extensive contributions of subject matter, and has suggested important improvements in arrangement.

To John Wiley & Sons, Inc., credit is due for permission to use Table IX, which is from Johnson's "Theory and Practice of Surveying," and Table X, which is from Searles and Ives' "Field Engineering."

It is desired to acknowledge the use of photographs of aerial surveying cameras and other apparatus which have been supplied by Brock and Weymouth, Philadelphia, and Fairchild Aerial Surveys, Inc., New York.

Scattered throughout the text are halftones of surveying instruments, the photographs having been furnished by various manufacturers. In this connection it is desired to acknowledge the assistance of W. and L. E. Gurley, Troy, N. Y.; Keuffel and Esser Company, New York; the A. Lietz Company, San Francisco, Calif.; and C. L. Berger and Sons, Boston, Mass.

R. E. D.
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SECTION I

INTRODUCTION TO SURVEYING

CHAPTER I

FUNDAMENTAL CONCEPTS

1. Surveying.—Surveying has to do with the determination of the relative position of points on or near the surface of the earth. It is the art of measuring horizontal and vertical distances between terrestrial objects, of measuring angles between terrestrial lines, of determining the direction of lines, and of establishing points by predetermined angular and linear measurements.

Incidental to the actual measurements of surveying are mathematical calculations. Distances, angles, directions, positions, areas, and volumes are thus determined from data of the survey. Also, much of the information of the survey is portrayed graphically by the construction of maps, profiles, cross-sections, and diagrams.

Thus the process of surveying may be divided into the *field work* of taking measurements and the *office work* of computing and drawing necessary to the purpose of the survey.

2. Uses of Surveys.—The earliest surveys known were for the purpose of establishing the boundaries of land, and such surveys are still the important work of many surveyors. With the increase in the population of the country has come the subdivision of the public lands, a decrease in the size of the unit owned by the individual land holder, and in many localities a large increase in land values. It is evident that, while the *principles* employed in making surveys of rural land worth a few dollars an acre might be the same as those employed in making surveys for valuable city property, the required *precision* would necessarily increase with the land value, and that methods and instruments sufficiently refined for the former case might prove totally inadequate in the latter.

Every construction project of any magnitude is based to a greater or less degree upon measurements taken during the progress of a

survey, and is constructed about lines and points established by the surveyor. Aside from surveys made primarily for fixing the boundaries of landed property, practically all surveys of a private nature and most of those conducted by public agencies are of assistance in the conception, design, and execution of engineering works.

For many years the government, and in some instances the individual states, have conducted surveys over large areas for a variety of purposes. The principal work so far accomplished consists of the fixing of national and state boundaries, the charting of coast lines and navigable streams and lakes, the precise location of definite reference points throughout the country, the collection of valuable facts concerning the earth's magnetism at widely scattered stations, and the mapping of certain portions of the interior, particularly near the seacoasts, along the principal rivers and lakes, in the localities of valuable mineral deposits, and in the older and more thickly settled territories. In general, these government surveys are of a complex nature requiring a high degree of skill and much experience on the part of those prosecuting the work. The groundwork for these extensive surveys is of a high precision requiring instruments, methods, and refinements unnecessary in ordinary surveying.

Summing up, it is seen that surveys have been divided into three classes: (1) those for the primary purpose of establishing the boundaries of landed properties, (2) those forming the basis of a study for or necessary to the construction of public or private works, and (3) those of large extent and high precision conducted by the government and to some extent by the states.

There is no hard and fast line of demarcation between surveys of one class and those of another, either as regards methods employed, results obtained, or use of the data of the survey.

3. The Earth a Spheroid.—The earth is an oblate spheroid of revolution, the length of its polar axis being somewhat less than that of its equatorial axis. The lengths of these axes are variously computed, as follows:

	Polar axis, ft.	Equatorial axis, ft.
Clarke (1866)	41,710,242	41,852,124
Hayford (1909)	41,711,920	41,852,860
Adopted (1924) by International Geodetic and Geophysical Union	41,711,940[a]	41,852,860

[a] Computed from equatorial axis by assuming that the flattening of the earth is exactly 1 ÷ 297.

The lengths computed by Clarke have been generally accepted in the United States and have been used in government land surveys. The values of Hayford are now regarded as being more nearly correct than those of Clarke. The values adopted by the International Geodetic and Geophysical Union are published by the United States Naval Observatory.

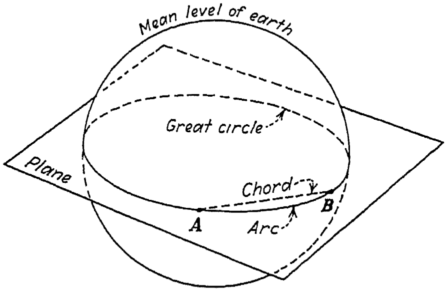


Fig. 3*a*.

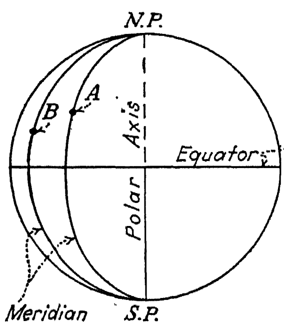


Fig. 3*b*.

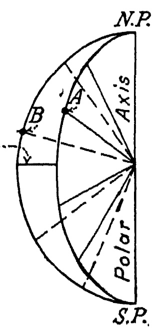


Fig. 3*c*.

It will be seen that the polar axis is shorter than the equatorial axis by about 27 miles. Relative to the diameter of the earth this is a very small quantity, less than 0.34 per cent. Imagine the earth as shrunk to the size of a billiard ball, still retaining the same shape. In this condition, it would appear to the eye as a smooth sphere, and only by precise measurements could its lack of true sphericity be detected.

Let us consider that the irregularities of the earth have been removed. The surface of this imaginary spheroid is a curved surface every element of which is normal to the plumb line. Such a surface is termed a *level surface*. The particular surface at the average sea level is termed *mean sea level*.

Imagine a plane as passing through the center of the earth, as in Fig. 3*a*. Its intersection with the level surface forms a continuous line around the earth. Any portion of such a line is termed a *level line*, and the circle defined by the intersection of such a plane with the mean level of the earth is termed a *great circle* of the earth. The distance between two points on the earth, as A and B (Fig. 3*a*), is the length of the arc of the great circle passing through the points, and is always somewhat more than the chord intercepted by this arc. The arc is a level line; the chord is a mathematically straight line.

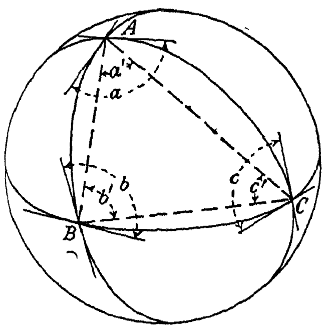


Fig. 3*d*.

If a plane is passed through the poles of the earth and any other point on the earth's surface, as A (Fig. 3*b*), the line defined by the intersection of the level surface and plane is called a *meridian*. Imagine two such planes as passing through two points as A and B (Fig. 3*b*) on the earth, and the section between the two planes removed like the slice of an orange, as in Fig. 3*c*. At the equator the two meridians are parallel; above and below the equator they converge, and the angle of convergency increases as the poles are approached. No two meridians are parallel except at the equator.

Imagine lines, normal to the meridians, drawn on the two cut surfaces of the slice. If the earth be regarded as a perfect sphere these lines converge at a point at the center of the earth. Considering the lines on either or both of the cut surfaces, no two are parallel. The radial lines may be considered as vertical or plumb lines, and hence we arrive at the deductions that all plumb lines converge at the earth's center and that no two are parallel. Strictly speaking, this is not quite true, owing to the unequal distribution of the earth mass and also owing to the fact that normals to an oblate spheroid do not all meet at a common point.

Consider three points on the mean surface of the earth. Let us make these three points the vertices of a triangle, as in Fig. 3*d*. The surface within the triangle ABC is a curved surface, and the lines forming its sides are arcs of great circles. The figure is a spherical triangle. In the figure the dotted lines represent the plane triangle whose vertices are points A, B, and C.[1] Lines drawn tangent to the

[1] Actually the "auxiliary plane triangle" of geodetic work has sides equal in length to the *arcs* of the corresponding spherical triangle.

sides of the spherical triangle at its vertices are shown. The angles a, b, and c of the spherical triangle are seen to be greater than the corresponding angles a', b', and c' of the plane triangle. The amount of this excess would be small if the points were near together, and the surface forming the triangle would not depart far from a plane passing through the three points. If the points were far apart the difference would be considerable. Evidently the same conditions would obtain for a figure of any number of sides. Hence we see that angles on the surface of the earth are spherical angles.

In everyday life we are not concerned with these facts. We think of a line passing along the surface of the earth directly between two points as being a straight line, we think of plumb lines as being parallel, we think of a level surface as a flat surface, and we think of angles between lines in such a surface as being plane angles.

As to whether the surveyor must regard the earth's surface as curved or may regard it as plane (a much simpler premise) depends upon the character and magnitude of the survey and upon the precision required.

4. Plane Surveying.—That type of surveying in which the mean surface of the earth is considered as a plane, or in which its spheroidal shape is neglected, is generally defined as *plane surveying*. From this assumption it follows that a level line or a line lying within a level surface is considered as mathematically straight, that the direction of the plumb line at any point within the limits of the survey is considered as parallel to the direction of the plumb line at any other point, and that angles of polygons are considered as plane angles.

By far the greater number of all surveys are of this type. When it is considered that the length of an arc 11½ miles long lying in the earth's surface is only 0.05 ft. greater than the subtended chord, and further that the difference between the sum of the angles in a plane triangle and the sum of those in a spherical triangle is only one second for a triangle at the earth's surface having an area of 75.5 square miles, it will be appreciated that the shape of the earth need be taken into consideration only in surveys of precision covering large areas.

Surveys for the location and construction of railroads, highways, canals, and, in general, the surveys necessary for the works of man are plane surveys, as are also the surveys made for the purpose of establishing boundaries, except state and national.

The operation of determining *elevation* is usually considered as a division of plane surveying. Elevations are referred to a spheroidal surface, a tangent at any point in the surface being normal to the

plumb line at that point (most commonly this imaginary surface of reference is mean sea level). Curiously, this surface is called a "datum plane." The procedure ordinarily used in determining elevations automatically takes into account the curvature of the earth, and elevations referred to the curved surface of reference or "datum plane" are secured without extra effort on the part of the surveyor. In fact it would be much more difficult for him to refer elevations to a true plane than to the imaginary spheroidal surface which he has chosen. Imagine a true plane, tangent to the surface of mean sea level at a given point. At a horizontal distance of 10 miles from the point of tangency the vertical distance (or elevation) of the plane above the surface represented by mean sea level is 67 ft., and at a distance of 100 miles from the point of tangency the elevation of the plane is 6,670 ft. above mean sea level. Evidently the curvature of the earth's surface is a factor which can not be neglected in obtaining even very rough values of elevations.

The United States system of subdividing the public lands employs the methods of plane surveying, but takes into account the shape of the earth in the location of certain of the primary lines of division.

This book deals chiefly with the methods of plane surveying.

5. Geodetic Surveying.—That type of surveying which takes into account the shape of the earth is defined as *geodetic surveying*. All surveys employing the principles of geodesy are of high precision and generally extend over large areas. Where the area involved is not great, as for a state, the required precision may be obtained by assuming that the earth is a perfect sphere. Where the area is large, as for a country, the true spheroidal shape of the earth is considered. Surveys of the latter character have been conducted only through the agencies of governments. In the United States such surveys have been conducted principally by the U. S. Coast and Geodetic Survey, a highly specialized branch of the Department of Commerce, but also by the Great Lakes Survey, the Mississippi River Commission, the several boundary commissions, and others. Surveys conducted under the assumption that the earth is a perfect sphere have been completed by such large cities as Washington, Baltimore, Cincinnati, and Chicago.

Though only a few engineers and surveyors are employed in geodetic work, the data of the various geodetic surveys are of great importance in that they furnish precise points of reference to which the multitude of surveys of less precision may be tied.

6. Definitions.—A *level surface* is one parallel with the mean spheroidal surface of the earth. A body of still water provides the best example.

A *horizontal plane* is a plane tangent to a level surface.

A *horizontal line* is a line tangent to a level surface.

A *horizontal angle* is an angle formed by the intersection of two lines in a horizontal plane.

A *vertical line* is a line normal to a level surface. A plumb line is an example.

A *vertical plane* is a plane of which a vertical line is an element.

A *vertical angle* is an angle between two intersecting lines in a vertical plane. In surveying it is commonly understood that one of these lines is horizontal and a vertical angle to a point is understood to be the angle in a vertical plane between a line to that point and the horizontal plane.

At any point there can be but one horizontal plane and but one vertical line, but there may be an infinite number of horizontal lines and vertical planes.

In surveying, measured angles are either vertical or horizontal.

In plane surveying, distances measured along a level line are termed *horizontal distances*. The distance between two points is commonly understood to be the horizontal distance from the plumb line through one point to the plumb line through the other. Measured distances may be either horizontal or inclined, but in most cases the inclined distances are reduced to equivalent horizontal lengths.

The *elevation* of a point is its vertical distance above (or below) some arbitrarily assumed level surface. Such a surface is called a "*datum*" or a "*datum plane*." The datum to which most elevations are referred is *mean sea level*, the normal level of the ocean at mean tide.

The vertical distance between two points is termed *the difference in elevation*. It is the distance between an imaginary level surface containing the high point and a similar surface containing the low point. The operation of measuring difference in elevation is called *leveling*.

7. Units of Measurement.—It has been stated that the operations of surveying entail both angular and linear measurements.

The units of angular measure are the *degree*, *minute*, and *second*. In the greater number of surveys, measurement to the nearest minute is sufficiently exact. On very precise surveys, angles are frequently determined to tenths of seconds.

In the United States, as in all English-speaking countries, the common units of linear measurement are the *yard*, *foot*, and *inch*. On most surveys in these countries, distances are measured in feet, tenths, and hundredths, and surveyor's tapes are usually graduated in these units. In laying out construction work for men of the

building trades, the surveyor will often find it necessary to employ the foot, the inch, and the eighth of an inch. Most measurements in surveying need not be taken closer than hundredths of a foot, and very often distances to the nearest foot or even to the nearest 10 ft. are sufficient for the purpose of the survey. Formerly the *rod* and the *Gunter's chain* were units much used in land surveying. In old deeds, distances will frequently be found expressed in one or the other of these units. The Gunter's chain is 66 ft. long and is divided into 100 links each 7.92 in. long. The Gunter's chain as a unit of length is employed in the subdivision of the United States public lands. A convenient relation exists between the rod, the Gunter's chain, the acre, and the mile: 1 mile = 80 chains = 320 rods.

Many other civilized countries of the world employ the *meter* as the unit of length. 1 meter = 39.37 in. = 3.2808 ft. = 1.0936 yd. The meter is the unit of length employed by the U. S. Coast and Geodetic Survey.

The *vara* is a Spanish unit of measure used in Mexico and several other countries falling under early Spanish influence. In portions of the United States formerly belonging to Spain or to Mexico, the surveyor will frequently have occasion to rerun property lines from old deeds in which lengths are given in terms of the vara.

1 vara = 32.9931 in. (Mexico), 33 in. (California), or $33\frac{1}{3}$ in. (Texas).

In the United States the units of area commonly used are the *square foot* and the *acre*. Formerly the *square rod* and the *square Gunter's chain* were also used.

43,560 sq. ft. = 160 sq. rods = 10 sq. Gunter's chains = 1 acre.

The units of volumetric measurement are the *cubic foot* and the *cubic yard*.

8. Kinds and Operations of Surveying.—The nature of the measurements made by the surveyor has been indicated in preceding articles.

In *land surveying* his work consists in

1. Rerunning old land lines to determine their length and direction.
2. Reestablishing obliterated land lines from recorded lengths and directions and such other information as it is possible to secure.
3. Subdividing lands into parcels of predetermined shape and size.
4. Setting monuments to preserve the location of land lines.
5. Locating the position of such monuments with respect to permanent landmarks.
6. Calculating areas, distances, and angles or directions.
7. Portraying the data of the survey on a *land map*.
8. Writing descriptions for deeds.

A *topographic survey* is a survey made to secure data from which may be made a map indicating the relief or elevations and inequalities of the land surface. The result of such a survey is the *topographic map*. The work consists in:

1. Establishing by angular and linear measurements the horizontal location of certain points for the skeleton of the survey, termed the *horizontal control.*
2. Determining the elevation of control points by the operation of leveling, termed the *vertical control.*
3. Determining the horizontal location and elevation of a sufficient number of ground points to provide data for the map.
4. Locating such other natural or artificial details as the requirements of the survey demand.
5. Calculating angles, distances, and elevations.
6. Plotting and finishing the topographic map.

Route surveying as the term is here used has reference to those surveys necessary for the location and construction of lines of transportation or communication, such as highways, canals, railroads, transmission lines, and pipe lines. The preliminary work usually consists of a topographic survey. The location and construction surveys may further consist in:

1. Locating the center line by stakes at short intervals.
2. Running levels to determine the profile of the ground along the center line.
3. Plotting such profile, and fixing grades.
4. Taking cross-sections.
5. Calculating earthwork.
6. Measuring drainage areas.
7. Laying out structures, such as culverts and bridges.
8. Locating right-of-way boundaries.

Hydrographic surveying has reference to surveying bodies of water. Broadly speaking, the operations of hydrographic surveying may consist in:

1. Making a topographic survey of shores and banks.
2. Taking soundings to determine the depth of water and the character of the bottom.
3. Locating such soundings by angular and linear measurements.
4. Plotting the hydrographic map showing the topography of the shores and banks, the depths of soundings, and other desirable details.
5. Observing the fluctuation of the ocean tide or of the change in level of lakes and rivers.
6. Measuring the discharge of streams.

In a sense, the surveys for drainage and for irrigation are hydrographic in character, but the principal work is essentially either topographic or route surveying.

Mine surveying makes use of the principles of land, topographic, and route surveying, with modifications in practice made necessary by altered conditions. Both surface and underground surveys are required. The work of the mine surveyor consists in:

1. Establishing (on the surface) the boundaries of claims for mineral patent (on the order of the Surveyor General of the state in which the claim is located) and fixing reference monuments.
2. Locating (on the surface) shafts, adits, bore-holes, railroads, tramways, mills, and other details.
3. Making a topographic survey of the mine property.
4. Constructing the surface map.
5. Making underground surveys necessary to delineate fully the mine workings.
6. Constructing the underground plans showing the workings in plan, longitudinal section, and transverse section.
7. Constructing the geological plan.
8. Calculating volumes removed.

City surveying is the term frequently applied to the operation of laying out lots and to the municipal surveys made in connection with the construction of streets, water-supply systems, and sewers. There is no distinction between such surveys and those just described except that the degree of refinement observed in making measurements is made proportional to the value of the land with which the survey is concerned.

Cadastral surveying, as the term has come to be used in the United States, has particular reference to extensive surveys of some of our large cities, made for the purposes of fixing with great precision a system of reference monuments and of locating property lines and improvements in great detail. In connection with these surveys, topographic surveys have been made. The results are available in the form of large-scale maps, which are of great value in planning city improvements.

In a broad sense a cadastral survey is one made for the purpose of accurately locating property lines in horizontal projection, and in addition, locating the natural and artificial features of the terrain.

Photographic surveying may be employed either in topographic work or in reconnaissance. Photographs made with specially designed cameras are taken either from stations located on the ground

or from airplanes and sometimes from balloons. Photographic surveying from ground stations has been found a useful adjunct to other methods in the small-scale topographic mapping of mountainous areas. The work consists in taking photographs from two or more control stations and in utilizing the photographs for the projection of details of the terrain in plan and elevation.

Aerial photography furnishes a rapid means of securing a plan view of a given territory. When the necessary adjustments and projections are made, the photographs may be employed to furnish the detail for ground surveys. The advantages of aerial photography in connection with accurate ground surveys are the speed with which the field work is accomplished and the wealth of detail secured. Attempts to utilize such photographs alone for portraying the relief of the ground have so far not been successful, but in connection with ground surveys made for the purpose of accurately establishing visible control points, aerial photography is coming to be used on extensive topographic surveys as a part of the field work of topographic mapping.

9. Precision of Measurements.—In dealing with abstract quantieies, we have become accustomed to thinking largely in terms of exact values. At the start, the student of surveying ought to appreciate that he is dealing with physical measurements which are correct only within certain limits, owing to errors that can not be completely eliminated. The degree of precision of a given measurement depends upon the methods and instruments employed and upon other conditions surrounding the survey. It is desirable that all measurements be made with high precision, but unfortunately a given increase in precision is accompanied by more than a directly proportionate increase in the time and labor of the surveyor. It therefore becomes his duty to maintain a degree of precision as high as justified by the purpose of the survey, but not higher. It is important, then, that he have a thorough knowledge of the sources and kinds of errors, of their effect upon field measurements, and of methods to be followed in keeping the magnitude of the errors within allowable values. It follows that he must understand the intended use of the survey data.

Before beginning work, the surveyor ought to consider the following questions:

1. What is the purpose of the survey?
2. What degree of accuracy is required for that purpose?
3. What are the sources of error?

4. What methods must be employed to reduce these errors within allowable limits?

5. How is the correctness of the work to be verified?

6. What instruments should be used to facilitate the work?

7. How may the work be organized to reduce the labor to a minimum?

10. Principles Involved.—The underlying principles of plane surveying are not difficult. They involve a thorough knowledge of geometry and plane trigonometry, and to a less degree a knowledge of physics, of astronomy, and of the theory and methods of adjustment of errors. Such portions of the latter three subjects as are necessary to the understanding of the text will be given in succeeding chapters as the need arises. Geodetic surveying requires an expert knowledge of all of the above subjects.

11. Practice of Surveying.—Like other arts based upon the sciences, the practice of surveying is complex, and no amount of theory will make a good surveyor unless he has the requisite skill in the art of observing and is versed in field and office practice. The student should realize the importance of a knowledge of the practical phases of the subject and seek to become as well grounded in the practice as possible.

Often surveying is one of the first professional subjects studied by the engineering student. He may not expect to become a surveyor, but he ought to understand that the training he will receive in the art of observing and computing, in the study of errors and their causes and effects, and in the practice of mapping will directly contribute to success in other subjects, regardless of the branch of engineering in which he may be interested.

12. Requisites of a Good Surveyor.—As the term "surveyor" is here used it has reference not only to that individual who makes his chief livelihood from surveying and expects so to continue in the remote future, but also to that individual of a large army of engineers to whom surveying is merely one of the arts of his profession, to whom the survey is perhaps the work of today and the adaptation of the results to the engineering problem is the work of tomorrow.

A thorough knowledge of the theory of surveying and skill in its practice are the first requisites of the surveyor; but, upon the evidence of employers themselves, it is also true that traits of character are far more potent factors in the success of the surveyor or engineer than is technical knowledge or skill. Therefore, it should be stated with all emphasis, that while mastering the theory and practice of surveying, the student will do himself a great benefit if he also develop traits of character and habits of mind which will be advan-

tageous to him whatever may be his later work. This can be accomplished only by diligent application of the laws of habit formation, which are fairly well known. Some definiteness may be given to this suggestion by the mention of a few of the traits which should be possessed by the surveyor.

He should maintain the attitude of the scientist, that no result is trustworthy until every reasonable test of its accuracy has been applied.

He should be reliable.

He should be of sound judgment.

He should possess initiative and should attack a problem with resourcefulness and energy.

He should be thorough, not content with his work until it has been finished in a workmanlike fashion.

He should be able to think without confusion, and to reason logically without prejudice.

He should be of good temper, thoughtful of those coming under his direction, commanding the respect of his associates, and watchful of the interests of his employer.

CHAPTER II

FIELD WORK

13. General.—The nature of surveying measurements has already been indicated. Field work consists in:

1. Adjusting instruments and caring for field equipment.
2. Determining the position of or establishing stones, stakes, or other more or less permanent monuments for the control of the survey or for other purposes.
3. Fixing the horizontal position of objects or points by horizontal angles and distances.
4. Determining the elevations of objects or points by one of the methods of leveling.
5. Making a record of the field measurements, usually in the form of field notes in the field notebook, but sometimes directly in the form of a map drawn to scale.

On all surveys the field work is of primary importance. To become skilled in surveying operations requires a certain amount of experience in the field. The study of a text may serve to enlighten one as regards the underlying theory, the instruments and their uses, and the methods; but in surveying, as in other arts, mastery depends in a large degree upon the length of time and upon the extent and the variety of actual experience.

14. Student Field Practice.—In most courses in surveying a certain amount of field practice is given in connection with the study of the text. Field problems designed to give the student practice in the elementary operations of surveying are outlined in later pages of this book.

It is not possible, in the ordinary field course in surveying, to develop the student into an expert instrumentman; it is expected, however, that the course will give the student a working knowledge of surveying instruments and their uses. In elementary field work no long surveys are attempted, but a number of short problems are taken up which in practice might become parts of extended surveys.

Members of the student field parties should from day to day alternately assume the various duties involved in the field work. The ability to hold the rod properly is as essential as the knowledge of how to manipulate the level, for a thorough understanding of details is necessary for intelligent direction.

15. Study the Problem.—Before going into the field, the student should understand exactly what he is to do and why he is to do it. This can be accomplished only by a thorough study of the problem, noting first its object and then conducting a critical examination of the course of procedure. In his mind he should go through the various steps involved so that while in the field he may spend his time and attention in putting into practice that of which he has already learned the theory. After a thorough study of the problem the student should prepare a list of the equipment necessary for its performance.

16. Field Instruments.—The principal surveying instruments and accessories and their uses are given below:

Engineer's Transit.—The universal instrument. Used principally for measuring horizontal and vertical angles and for prolonging straight lines. Has a telescope which may be revolved about either a horizontal or a vertical axis. Usually equipped with a magnetic needle and mounted on a tripod. The operator is called a transitman. (See Fig. 189*a*, p. 255.)

Surveyor's Compass.—Mounted on a tripod and equipped with sight vanes. Used for determining the direction of lines by means of the magnetic needle. The instrument is nearly obsolete. (See Fig. 184*d*, p. 243.)

Engineer's Level.—A telescope to which is attached a spirit-level tube, all revolving about a vertical axis. Employed for determining difference in elevation. Its use is termed leveling, and the operator is called a leveler or levelman. (See Fig. 106, p. 129.)

Flag, Flag Pole, or Range Pole.—A pole, either steel or wood, shod with a steel point, painted with bands of alternating red and white. Used as a sighting rod in connection with either angular or linear measurements. (See Fig. 82, p. 85.)

Tape.—A graduated flexible ribbon used for measuring distances. (See p. 82.)

Chaining Pins.—Steel pins about 1 ft. long, for temporarily marking the position of the ends of the tape as distances are measured. (See Fig. 81, p. 84.)

Level Rod.—A graduated wooden rod which, in conjunction with the level, is used in determining difference in elevation. Graduations usually in hundredths of feet. May be in single piece or jointed. Common length when extended, 13 ft. (See Fig. 109, p. 134.)

17. Some Surveying Terms.—For a better understanding of the following articles brief definitions of a few of the terms of surveying are appropriate.

Chaining.—The operation of measuring horizontal or inclined distances with a tape. The persons who make such measurements are called chainmen.

Flagman.—A person whose duty it is to hold the flag or range pole at selected points, as directed by the transitman or other person in charge.

Rodman.—A person whose duty it is to hold the rod and otherwise to assist the leveler or topographer.

Backsight.—(1) A sight taken with the level to a point of known elevation. (2) A sight or observation taken with the transit along a line of known direction to a reference point, generally in the rear.

Foresight.—(1) A sight taken with the level to a point the elevation of which is to be determined. (2) A sight taken with the transit to a point (generally in advance), along a line whose direction is to be determined.

Hub.—A point over which the transit is set, usually a heavy stake set nearly flush with the ground, with a tack in the top marking the point.

Line.—The path or route between points of control along which measurements are taken to determine distance or angle. To *give line* is to direct the placing of a flag pole pin, or other object on line.

Turning Point.—A fixed point or object, often temporary in character, used in leveling where the rod is held first for a foresight, then for a backsight.

Bench Mark.—A fixed reference point or object, more or less permanent in character, the elevation of which is known. A bench mark may also be utilized as a turning point.

18. Habit of Correctness.—No measurement should be regarded as correct until verified. So far as it is practicable, methods of verification should differ from the methods used in original measurements. All persons are liable to mistakes, but a mistake in field work becomes discreditable to the maker if he allows any other than himself to discover the discrepancy. Nothing, unless it be wilful dishonesty, is so injurious to the reputation as habitual carelessness.

19. Consistent Accuracy.—The accuracy of the measurements should be consistent with the purposes of the survey. Beginners often fail to comprehend the different degrees of precision necessary for the different kinds of work, or fail to maintain a consistent degree of accuracy throughout any one survey. There can be no fixed rules for the relative accuracy of different classes of surveys, for the objects and conditions are too many and too complicated, but one can always resort to common sense. Each survey is a problem in itself, for which the surveyor must establish the limit of error, using his own judgment and the experience of others to guide him. The best surveyor is not the one who is extremely accurate, but the one who makes a survey with sufficient accuracy to serve its purpose without waste of time or money.

20. Relation between Angles and Distances.—If measurements are to be consistent, it is evident that the precision of angles should

correspond to the precision of related distances—or in other words, the error in position of a point on account of error in angle should equal its error in position on account of error in distance. Thus in Fig. 20, the position of the point B with respect to the line AC is represented by the point B, its true position, and B', its erroneous position resulting from the error e_d in the measured distance AB, and the displacement e_a due to the error e_α in the measured angle.

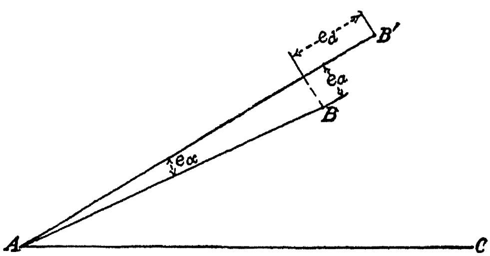


Fig. 20.

Evidently a consistent relation between errors in angle and errors in distance would require that the distances e_a and e_d be equal or nearly so. The error in distance is expressed as a ratio, as say $\frac{1}{2,000}$, *i.e.*, if the distance AB were 2,000 ft. then the distance e_a would be 1 ft. Similarly the distance e_a should equal 1 ft. and the tangent (or sine) of e_α (error in angle) would be $\frac{1}{2,000}$. Accordingly, it may be stated that a consistent relation between angles and distances will be maintained if the tangent of the allowable error in angles e_α equals the allowable error (expressed as a ratio) in the distances.

It is impossible to maintain an exact equality between these two percentages; but with one or two exceptions, which will be considered presently, surveys should be so conducted that the difference between precision of angle and precision of distance will not be large. Since the accuracy of a survey is often judged by the number of significant places in the recorded distance, it is always best to measure angles with a precision at least equal to the precision of the distances. It is common practice before beginning a survey or any distinct portion of a survey to fix the permissible error of linear measurement, this being expressed as a ratio, as $\frac{1}{1,000}$; sometimes it is desired to locate a given point within a specified distance of its true position. The following table shows for various angular errors the

corresponding ratios of precision and the errors for a length of 1,000 ft. For a length other than 1,000 ft. the linear error is in direct proportion.

Angular error	Linear error in 1,000 ft.	Ratio of precision
10′	2.9089	$\frac{1}{344}$
5′	1.4544	$\frac{1}{688}$
1′	0.2909	$\frac{1}{3,440}$
30″	0.1454	$\frac{1}{6,880}$
20″	0.0970	$\frac{1}{10,300}$
10″	0.0485	$\frac{1}{20,600}$
5″	0.0242	$\frac{1}{41,200}$

To illustrate the use of the preceding table, suppose distances are to be chained with a precision of $\frac{1}{10,000}$; the corresponding permissible angular error is 20″. Again, the distance from the instrument to a desired point is determined as 660 ft. with a probable error of 2 ft. For an angular error of 10′ the corresponding linear error is $0.66 \times 2.9 = 1.91$ ft. Therefore the angle need be determined only to the nearest 10′.

The *exceptions* referred to in the first part of this article are surveys in which the distances are roughly determined and the angles are measured with more than the required accuracy without increased effort or loss of time. For example, in rough chaining the ratio of precision might be $\frac{1}{1,000}$, corresponding to an angular error of 03′. But with the ordinary transit, the angles could be determined to the nearest 01′ as quickly as to the nearest 03′.

21. Angles Used in Trigonometric Computations.—Very often field measurements are made the basis of computations involving the trigonometric functions, and it is necessary that the computed results be of a required precision. If the values of these functions were exactly proportional to the size of the angles—or in other words, if any increase in the size of an angle were accompanied by a

proportional increase or decrease in the value of a function—the problem of determining the precision of angular measurements would resolve itself into that explained in the preceding article; but since the ratios of the rates of change to the values of the sines of small angles, the cosines of angles near 90°, and the tangents and cotangents of small and large angles are relatively large, it is evident that the

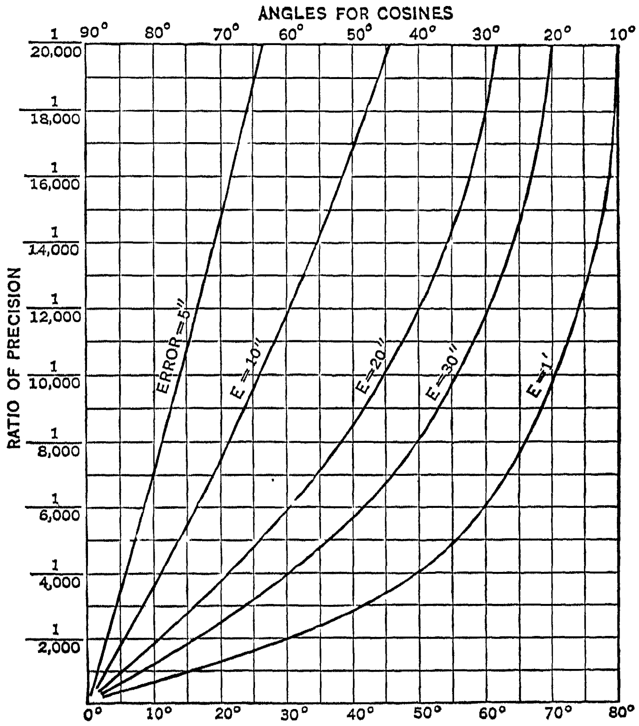


FIG. 21a.—Ratios of precision for sines or cosines.

precision with which an angle is determined should be made to depend upon the size of the angle and upon the function to be used in the computations. Usually too little attention is paid to this important phase of the precision of measurements, even by experienced surveyors, and as a consequence computed results are very often assumed to be more precise than they really are. It is not practicable to measure each angle with exactly the precision necessary to insure sufficiently accurate computed values, but at least the surveyor should have a sufficiently comprehensive knowledge of the

purpose of the survey and of the properties of the trigonometric functions to keep the angles within the required precision.

The curves of Figs. 21*a* and 21*b* show the ratios of precision corresponding to various angular errors from 05″ to 01′ for sines, cosines, tangents, and cotangents. For the function under consideration these curves may be utilized (1) to determine the ratio of precision corresponding to a given angular error and angle, (2) to

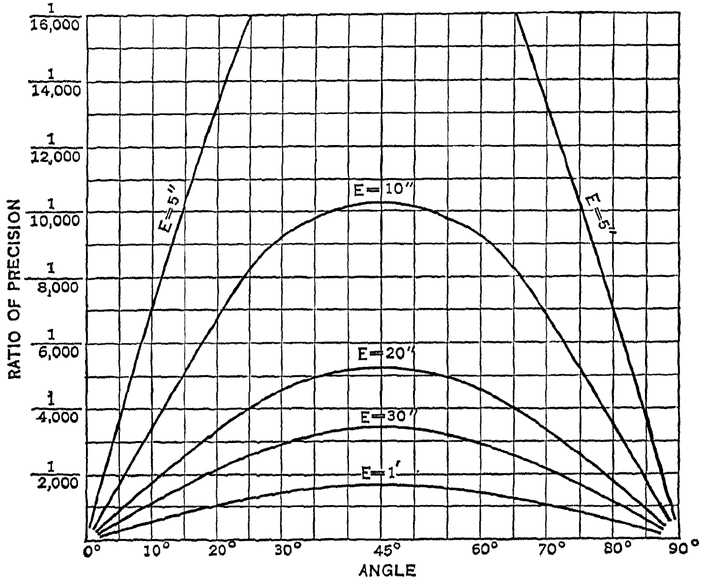


FIG. 21*b*.—Ratios of precision for tangents and cotangents.

determine the maximum or minimum angle that for a given angular error will furnish the required ratio of precision, or (3) to determine the precision with which angles of a given size must be measured to maintain a required ratio of precision in computations.

The following examples illustrate the use of the curves:

1. An angle measured with a 1-min. transit is recorded as 32°00′. The maximum error of the angle is 30″. It is desired to know the ratio of precision of a computation involving the tangent of the angle. On the diagram of Fig. 21*b* it will be seen that the ratio of precision opposite the intersection of the curve $E = 30''$ and a line corresponding to 32° is $\frac{1}{3,000}$.

2. In a triangulation system the angles can be measured with an error not exceeding 05″. Computations involving the use of sines must maintain a ratio of precision of not less than $\frac{1}{20,000}$. It is desired to

determine the minimum allowable angle. On the diagram of Fig. 21*a* (sines) the angle corresponding to a ratio of precision of $\frac{1}{20,000}$ and an angular error of 05″, is 26°.

3. In computations involving the use of cosines a ratio of precision of $\frac{1}{10,000}$ is to be maintained. It is desired to know with what precision angles must be measured. On the diagram of Fig. 21*a* (cosines) opposite $\frac{1}{10,000}$ it will be seen that for angles of about 76° the angular error cannot exceed 05″, for angles of about 64° the angular error cannot exceed 10″, and so on.

It should here be noted that linear as well as angular measurements must always be taken with the required precision of the computed results, for no result can be more nearly correct than the data from which it was obtained.

22. Speed.—Speed in field work depends to a large extent upon practice in handling instruments; but no amount of practice will secure rapid work and at the same time secure satisfactory results unless the work be carefully planned, systematized, and carried out with consistent accuracy.

23. Signals.—Except for short distances a good system of hand signals between different members of the party makes a more efficient means of communication than is possible by word of mouth. A few of the more common signals are given below:

"Right" or *"Left."*—The arm is extended in the direction of the desired movement, the right arm being extended for a movement to the right and the left arm for a movement to the left. A long, slow, sweeping motion of the hand indicates a long movement; a short, quick motion indicates a short movement. This signal may be given by the transitman in directing the chainman on line, by the leveler in directing the rodman for a turning point, by the chief of the party to any member, or by one chainman to another chainman.

"Up" or *"Down."*—The arm is extended upward or downward, with wrist straight. When the desired movement is nearly completed, the arm is moved towards the horizontal. The signal is given by the leveler.

"All Right."—Both arms are extended horizontally and the forearms waved vertically. The signal may be given by any member of any party.

"Plumb the Flag" or *"Plumb the Rod."*—The arm is held vertically and moved in the direction that the flag or rod is to be plumbed. The signal is given by the transitman or leveler.

"Give a Foresight."—The instrumentman holds one arm vertically above his head.

"*Establish a Turning Point*" or "*Set a Hub.*"—The instrumentman holds one arm above his head and waves it in a circle.

"*Turning Point*" or "*Bench Mark.*"—In profile leveling the rodman holds the rod horizontally above his head and then brings it down on the point.

"*Give Line.*"—The flagman holds the flag horizontally in both hands above his head and brings it down and turns it to a vertical position. If he desires to set a hub, he waves the flag with one end in the ground, from side to side.

"*Wave the Rod.*"—The leveler holds one arm above his head and moves it from side to side.

"*Pick Up the Instrument.*"—Both arms are extended outward and downward, then inward and up, as one would do in grasping the legs of the tripod and shouldering the instrument. It is given by the chief of the party or by the head chainman when the transit is to be moved to another point.

24. Care of Surveying Instruments.—As the use of the various surveying instruments is discussed in the following chapters, suggestions for the care and manipulation of these instruments are given. In performing the field problems these should be heeded, not only because the student is responsible for the equipment he is using, but also because he will establish the foundation of a very desirable qualification—that of carefulness.

Transit and Level.—(1) Handle the instrument carefully when removing it from its box. (2) See that it is securely fastened to the tripod head. (3) Avoid carrying the instrument on the shoulder while passing through doorways or beneath low-hanging branches. Before climbing over a fence or similar obstacle place the instrument on the other side with the tripod legs well spread. (4) Never leave the instrument while it is set up in the street, on the sidewalk, in the vicinity of buildings in the process of construction, in fields where stock is grazing, or in any other place where there is the slightest possibility of accident. (5) Do not set the tripod legs too close together; and see that they are firmly planted. (6) Tighten the leveling screws until they come to a firm bearing, but not enough to injure the screws. *The general tendency is to tighten these screws far more than is necessary.* (7) Always put on the sun shade. Always attach or remove the sun shade by a screw motion in a clockwise direction, thus doing away with any possibility of loosening the objective lens. (8) *When the magnetic needle is not in use, see that it is raised off the pivot.* When the needle rests on the pivot, impact is apt to blunt the point of the pivot or to chip the jewel, thus causing the needle to be sluggish. (10) Dust caps for both the objective and the eyepiece should be put in place and the instrument wiped

clean before returning it to the box. (11) If the instrument is not to be returned to its box, see that it is protected from dust with a cloth hood.

Small Equipment.—(1) Keep the tape straight when in use. (2) Do not use the flag pole as a bar to loosen stakes or stones. Such usage bends the steel point and soon renders the flag pole unfit for lining purposes. (3) The brass or steel shoe on the foot of the leveling rod should not be struck against hard objects as this, if continued, will round off the foot of the rod, thus introducing a possible error in leveling work. (4) Long rods when not in use should be placed in an upright position or supported their entire length, for otherwise they are likely to warp. Jointed leveling rods should have all clamps loosened when not in use, to allow for possible expansion of the wood. (5) To avoid losing pins, tie a piece of colored cloth (preferably bright red) through the ring of each. (6) Remember that hard knots in the plumb-bob string indicate a slovenly or inexperienced instrumentman; learn to make a sliding bowknot that can be easily undone.

25. Adjustment of Instruments.—The ability to perform the adjustments of the ordinary surveying instruments is an important qualification of the surveyor. While it is a fact that the effect of instrumental errors may largely be eliminated by proper field methods, it is also true that instruments in good adjustment greatly expedite the field work. The operations of making the adjustments are not laborious, nor are the principles upon which the adjustments are based difficult to understand, yet strange to say a considerable number of surveyors regard the making of adjustments as something requiring the skill of an instrument maker. It is important that the surveyor:

1. Understand the principles upon which the adjustments are based.
2. Learn the method by which nonadjustment is discovered.
3. Know how to make the adjustments.
4. Appreciate the effect of one adjustment upon another.
5. Know the effect of each adjustment upon the use of the instrument.
6. Learn the order in which adjustments may most expeditiously be performed.

The frequency with which adjustments are required depends upon the instrument, its care, and the precision with which measurements are to be taken. Often in a good instrument, well cared for, the adjustments will be maintained with sufficient accuracy for ordinary surveys over a period of months or even years. On the other hand,

blows that may pass unnoticed are likely to disarrange the adjustments at any time. On ordinary surveys, it is good practice to test the critical adjustments once each day, especially on long surveys where frequent checks on the accuracy of the field data are impossible. Not infrequently failure to observe this simple practice results in the necessity of retracing lines perhaps representing the work of several days. Testing the adjustments with reasonable frequency lends confidence to the work and is a practice to be strongly commended. The instrumentman should, if possible, make the necessary tests at a time that will not interfere with the general progress of the survey party.

The adjustments are made by tightening or loosening certain screws. Usually these screws have capstan heads which may be turned by a pin called an adjusting pin. Following are some general suggestions:

1. The adjusting pin should be carried in the pocket and not left in the instrument box. Disregard of this rule frequently leads to loss of valuable time.
2. The adjusting pin should fit the hole in the capstan head. If the pin is too small, the screw is soon ruined.
3. Preferably make the adjustments with the instrument protected from the rays of the sun.
4. When several interrelated adjustments are necessary, time will be saved by first making an approximate or rough adjustment of each part and then by making finer adjustments through a repetition of the series. In this way, the several disarranged parts are gradually brought to their correct position. This practice does not refer to those adjustments which are in no way influenced by others.
5. Bring the capstan-headed screws to a firm bearing, but do not tighten them sufficiently to strain the metal. The general tendency of the novice is to tighten the screws far more than is necessary. This often results in stripping the threads or in other damage which cannot be repaired without considerable delay.

26. Field Notes.—No part of the operations of surveying is of greater importance, yet no part is more often neglected, than the field notes. In fact, the competency of the surveyor is reflected with much greater fidelity in the character of his field notes than in his use of the instrument. These notes should constitute a permanent record of the survey with data in such form as to be interpreted with ease by anyone having a knowledge of surveying. Unfortunately, this is often not the case. Many surveyors seem to think that their work is well done if the field record, reinforced by their own memories, is sufficiently comprehensive to make the field data of immediate use for whatever purpose the survey may have. On most surveys

it is impossible to predict to what extent the information gathered may become of value in the remote future. Not infrequently court proceedings involve surveys made long before. Often it is desirable to rerun, to extend, or otherwise to make use of surveys made years previously. In such cases it is quite likely that the old field notes will be the only visible evidence, and their value will depend largely upon the clearness and completeness with which they are recorded.

The notes consist of numerical data, explanatory notes, and sketches. In addition to the above, the record of every survey should include the date, the weather conditions, the names of the surveyor and his assistants, and a title indicating the location of the survey and its nature or purpose.

No matter what the character of the survey, all field notes should be recorded in the field at the time the work is being done. Notes made later, from memory or copied from other field notes, may be useful but they are not field notes. Notes should be neat. They are generally recorded in pencil, but they should be regarded as a permanent record and not as memoranda to be used only in the immediate future.

It is not easy to take good notes. The recorder should realize that the notes will very likely be used by other persons not familiar with the locality, who must rely entirely upon what he has recorded. For this reason not only should the notebook contain all necessary information, but data should be recorded in a form which will admit of only one interpretation, and that the correct one. A good sketch will perhaps help more than anything else to convey a correct impression to others, and for this reason sketches should be used freely. The use to be made of the notes will guide the recorder in deciding what data are necessary and what are not. To make the notes clear, he should put himself in the place of one who is not on the ground at the time the survey is made. Before making any survey, the necessary data to be collected should be carefully considered; and when doing the field work, all such data should be obtained, but no more.

While a few convenient forms of notes are in common use, it will generally be necessary to supplement these, and in many cases it will be necessary for the surveyor to devise his own form of record for his field data.

26a. Notebook.—In actual practice the notebook should be of good quality paper, with stiff board or leather cover, made to withstand hard usage, and of a size convenient to slip into the coat pocket. There are several special field notebooks sold by engineering supply companies which are intended for particular kinds of notes. For

general surveying or for students in field work where the problems to be done are general in character, an excellent form of notebook has the right-hand page divided into small rectangles with a red line running up the middle, and has the left-hand page divided into several columns. In general, tabulated numerical values are written on the left-hand page; sketches and explanatory notes on the right.

26b. Suggestions for Keeping Student Field Notes.—Use a 4H pencil and keep it well pointed. Make a neat title either on the flyleaf or on the cover, showing the owner of the notebook, the number and name of the course, and the year in which the notes are taken. Leave several pages in the front of the book for an index. Show in the index the number, name, date, and pages of each problem done in the field. Also show whether or not the work has been accepted by the instructor. Always keep the index up to date.

Take all notes in the notebook while in the field. Leave nothing to memory or guesswork.

For each problem make a title at the top of the page, showing name and number of problem, date, weather, names of the members of the party and the duty of each, and time occupied in doing the field work.

If a page of notes is abandoned for any reason, do not obliterate the notes or cut out the page, but write diagonally across the page in large letters "*abandoned*" and designate the page number of the continuation of the notes. Notes may be abandoned because they are illegible or because they contain such erroneous or useless data that another survey is necessary.

Do not make the notes appear either more accurate or less accurate than they really are.

In general *do not erase* numerical data. If a number is in error, draw a line through it and write the corrected value above. Portions of sketches and explanatory notes may be erased when there is a good reason for doing so. Make the sketches freehand and of liberal size.

Because of its simplicity Reinhardt's style of slope lettering (Fig. 46*a*, p. 50) is generally conceded to be the best form of lettering for taking notes rapidly and neatly.

26c. Explanatory Notes.—The purpose of explanatory notes should be to make clear that which the numerical data and sketches fail to do. In some surveys explanatory notes entirely take the place of sketches, in which case they are placed on the right-hand page in the same line with the numerical data they explain. In other surveys they are used in conjunction with sketches and numerical values, and are placed in such position as not to interfere with other data. Sketches themselves frequently contain features which are only

intelligible through some note. When used in this capacity, notes should be placed in such position on the page with the sketch as not to interfere with other figures or letters. At the same time they should be placed in close proximity to that which they explain.

26d. Numerical Data.—The figures used should be plain; one figure should never be written over another. If the numerical data are to be tabulated in columns on the left-hand page, place the numbers so that all figures in the tens column will be on a straight line; thus at a glance one can tell what numbers are in tens or in hundreds or in thousands. Where decimals are used, the decimal point should never be omitted; and the number should always show with what degree of precision the measurement was taken. Thus an angle of 64° when measured to the nearest minute should be recorded as 64°00′, a distance measured to the nearest 0.1 ft. should be recorded not as 142.20 ft. but as 142.2 ft., and a rod reading taken to the nearest 0.01 ft. should be recorded not as 7.4 ft. but as 7.40 ft. Numbers placed on sketches should always be in such position as to clearly indicate to what they refer. Frequently this is impossible except by the aid of dimension lines or arrows.

26e. Sketches.—Sketches are rarely made to exact scale, but in most cases they are made approximately to scale. What the sketch should show must be determined by the recorder, for the most part before a line is drawn. A sketch crowded with unnecessary data is often very puzzling even though all necessary features are included. Many features may be most readily shown by conventional signs (Figs. 269*a–d*, pp. 385–388).

CHAPTER III

COMPUTATIONS

27. General.—Calculations of one kind or another form a large part of the work of surveying, and the ability to compute with speed and accuracy is an important qualification of the surveyor. Computing is an art not to be acquired without practice, but no amount of experience will make an efficient computer unless he possesses a knowledge of the precision of measurements and the effect of errors in given data upon the precision of values calculated therefrom, and is familiar with the algebraic and graphical processes and mechanical devices by means of which the labor of computing may be reduced to a minimum.

Calculations are made *algebraically* through the use of the simple arithmetical processes, logarithms, and the trigonometric functions; *graphically* by accurately scaled drawings; or *mechanically* by devices such as the slide rule and computing machines. A knowledge of the short methods of arithmetic is of value. The ability to make mental calculations quickly is desirable.

Before making calculations of any great importance or extent, the computer should carefully plan a clear and orderly arrangement. This is necessary (1) to save time, (2) to prevent mistakes, (3) to make the calculations legible to others, (4) to afford proper checks, and (5) to facilitate the work of the checker.

28. Office Computations.—All computations should be preserved in a notebook for that purpose. This book may be the same as that used for field notes, but preferably it should be larger, say about 8 by 10 in. This size will give considerable space for a problem without turning to a new page, and the columns of tabulated values will not have to be crowded as will often be the case with the smaller book. The pages should be cross-ruled, as with such an arrangement columns can be kept straight without additional ruling and sketches can easily be made.

In general, computations in the office are a continuation of some field work. They are required for purposes of obtaining areas, plotting maps, profiles, and cross-sections, calculating dimensions to be laid off in the field, or ascertaining other desired information concerning the survey. It is desirable that these computations be

easily accessible for future reference, and for this reason the pages of the computation book should be numbered and the contents of the book should be shown by a complete index. Parts of problems separated by other computations should be cross-referenced. Each problem should have a clear heading which should include the name of the survey, the kind of computations, the field book and number of page of the original notes, the name of the computer, the name of the checker, and the dates of computing and checking. Enough of the field notes should be transcribed to make computations possible without further reference to the field book.

29. Checking.—In practice, *no confidence is placed in results that have not been checked*, and important results are checked by more than one method. The student should see the necessity for this, for from experience he knows that a computation of any considerable length is rarely made without some mistake, and he should form the habit of checking his own work until he is certain that his results are correct. *Each student should depend upon himself.* It is true that students may check results by comparing work, but to do the work together comparing each step as it is completed is not checking in the true sense of the word. Such methods would not be countenanced in practice.

Most problems can be solved by more than one exact method. Since by using the same method in checking the same error is likely to occur, results should be checked by a different method when this is feasible. Approximate checks may be obtained in many cases through such mechanical devices as the slide rule, the planimeter, and the protractor. The slide rule is a valuable means of checking approximately each step. Large arithmetical mistakes are almost sure to be discovered in this way, though of course mistakes due to confusion of numbers or to wrong methods will not be shown. Graphical methods may often be used as an approximate check, to good advantage. They generally take less time than arithmetical or logarithmic solutions, and possible incorrect assumptions in the precise solution will be avoided.

30. Significant Figures.—The term *significant figures* is used to refer to those digits in a number which have meaning; *i.e.*, those digits whose values are known. Confusion in the matter of computations involving measured quantities arises from the failure of the novice to distinguish between exact numbers and numbers which carry with them the inevitable errors of measured quantities. If we measure roughly a given distance with a steel tape we may find it to be 732 ft., but if we measure it more carefully it may be found to be 732.4 ft., or by still greater refinements it may be determined as 732.38 ft. But we have not yet reached an exact number, nor can

we, for whatever refinement may be used, there will remain an error of indeterminate amount. The number of significant figures in these three results is 3, 4, and 5, respectively.

Obviously then, the number of digits that will have meaning and that may be used to indicate the length of this line is strictly limited by the precision with which the measurement has been made.

This precision is not always apparent in measurements, as will now be explained. Measurements are of two kinds, either *direct* or *indirect*. A direct measurement is made when the observed quantity is compared with the scale directly, as for example, when a carpenter measures the width of a board with his rule. An indirect measurement is made when the observed quantity is determined by several related and dependent observations.

When direct measurements are taken, the number of significant figures in the result is evident. Thus, suppose the surveyor measures a distance with the steel tape (graduated to hundredths of a foot) and finds it to be 37.42 ft. There is no question but that the number of significant figures in this result is four. Suppose the distance is much greater, however, so that the tape must be stretched several times over rough ground and the total distance is determined as 623.58 ft. It is now very doubtful if the last digit is correct, even though it can be read with certainty on the tape, because the indirect measurement has introduced a number of sources of error (as marking the ends of the tape, keeping the tape level, etc.) which renders the accuracy of the last digit, and possibly the last two digits, uncertain. Hence, it cannot be said, offhand, how many significant figures there are in any measured quantity until the character and magnitude of the errors have been examined.

It is not always easy to determine just the degree of uncertainty with which a measurement has been made, but in some cases it can be estimated or calculated with some precision and is expressed by a number called the *probable error* (Art. 69). We say, in such cases, that each digit is a significant figure until we reach that one for which the probable error equals or exceeds ten units. Thus we say that the number 623.58 ± .02 ft. has five significant figures, but that the number 623.58 ± .10 has four significant figures; and the latter number should properly be written 623.6. Obviously if the last digit is uncertain by as many as ten units, then the next to the last digit becomes uncertain and it would be absurd to assign values to any digits beyond one which itself is uncertain.

In a decimal the number of significant figures is not necessarily the number of decimal places, as the following examples will illustrate:

0.0000065 contains two significant figures.
0.00000650 contains three significant figures.
10.00000650 contains ten significant figures.
0.08000650 contains seven significant figures.

31. Precision of Computations.—A proper regard for consistency between measured values and the computed results based upon them requires an understanding of the effects of the errors of measurement when combined in the operations of arithmetical computations.

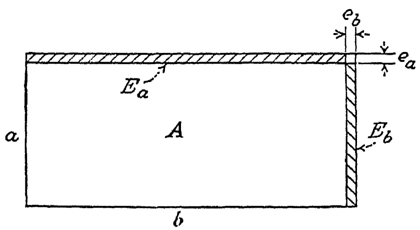


FIG. 31*a*.

Addition.—Let it be supposed that a considerable number of earthwork quantities are to be added, as below, and that the probable error in each quantity is known to be ± 0.3 cu. yd. The sum is 501.7$\pm$ cu. yd., but this number is affected by the probable error of each quantity of which it is composed. These separate probable errors, assumed here to be ± 0.3 cu. yd. are accidental in nature (Art. 65). Hence they will probably combine in the sum, as the square root of the number of times which they occur. In this example then since there are ten numbers, the total probable error will be ± 0.3 cu. yd. $\times \sqrt{10}$ or about ± 1.0 cu. yd. But when the last digit is in doubt by more than ten units, it has ceased to be a significant figure and the result should more properly be written as 502 cu. yd. And whether it is so written or as 501.7 cu. yd., it has three significant figures and no more.

37.2
45.6
53.1
63.2
43.7
45.2
63.8
72.1
36.4
41.4
501.7

Multiplication.—The amount of the total error in a product resulting from errors in the factors, may be shown by supposing the case of a rectangular field where the area is the product of two factors, namely the length and width. Thus in Fig. 31*a* let

b = length of the field
a = width of the field
A = ab = area
e_a = error in length of side a
e_b = error in length of side b

E_a = error in area due to e_a
E_b = error in area due to e_b
$E_A = E_{ab}$ = total error in area.
Evidently

$$\frac{e_b}{b} = \frac{E_b}{A} \text{ and } \frac{e_a}{a} = \frac{E_a}{A}$$

but

$$\frac{e_b}{b} \text{ is the relative error in the side } b$$

and

$$\frac{E_b}{A} \text{ is the relative error in the area due to } e_b$$

also

$E_A = E_b + E_a$ and the relative error in A is given by

$$\frac{E_A}{A} = \frac{E_b}{A} + \frac{E_a}{A} = \frac{e_b}{b} + \frac{e_a}{a},$$

or the relative error in the area is equal to the sum of the relative errors in the length and width.[1]

Hence, the relative error in a product is equal to the sum of the relative errors in the factors. And from this fact the important principle follows, that on a relative basis the probable error of a product can not be less than that of the least accurate factor.

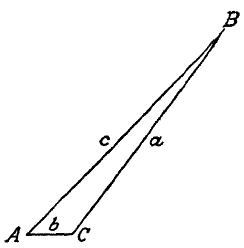


Fig. 31b.

This principle must be kept in mind by the surveyor in the field if his computed results are to have the accuracy requisite to his purpose. For example, suppose he wishes to determine the area of a triangle (Fig. 31b) whose sides $a = 680.8$ ft., $b = 75.3$ ft., and $\angle C$ are given.[2] If his purpose requires four significant figures in his result, say, a permissible error of about $\frac{1}{4,000}$, then each of these sides must be measured with such accuracy as to yield four significant figures; and it is seen that side b must be measured to hundredths, whereas the others need be measured to tenths only.

The accuracy required in the measurement of the angles is governed by the same considerations, and the ratio-of-precision curves (Figs.

[1] To simplify this demonstration the negligible error $e_a e_b$ at the corner has been omitted.

[2] Area $= \frac{1}{2}ab \sin C$.

21a and 21b) will aid in determining the precision with which the angles should be measured. Thus, in the example cited above, if the area is to be calculated by the formula, Area = $\frac{1}{2}ab \sin C$, then the angle C (which in this case is 132°02′) must be measured in the field with such accuracy that the use of sin C will introduce an error not greater than $\frac{1}{4{,}000}$. From Fig. 21a this allowable error is found to be slightly less than 1′.

Another consideration often misunderstood by computers, is the fact that the accuracy of a result is entirely independent of the unit in which it is expressed. Thus, in the case of the triangle mentioned above, the area is 19,060 sq. ft., which is properly expressed in acres not as 0.44 acres but as 0.4376 acres. There are four significant figures in each value.

The following example illustrates the value of examining the given data and of knowing the purpose of the computation:

Notes of a land survey indicate that the distances were measured with a precision of $\frac{1}{10{,}000}$, while angles were measured with a probable error of 01′. It is desired that computations for area (involving sines and cosines) be as precise as the data warrant. Referring to the ratio-of-precision curves for sines and cosines (Fig. 21a), it will be seen that such an angular error for angles of average size (near 45°) corresponds to a precision of about $\frac{1}{3{,}000}$. It is this precision which will govern the accuracy of the computed area, and the computed area should contain four significant figures. Had it been presumed that the precision of angles was consistent with that of distances, doubtless five significant figures would have been shown in the result.

32. Computations for Angles and Distances.—When, as is often the case, computations for angles or distances may be made in one of several ways, each of which depends upon data of like precision, it is best to compute angles by using functions which change rapidly, *i.e.*, by using tangents or cotangents; and to compute distances by using functions which change slowly, *i.e.*, by using sines or cosines.

For example the ratio-of-precision curves for tangents and cotangents (Fig. 21b) show that if the precision of data is $\frac{1}{10{,}000}$ the maximum error of angles computed by tangents or cotangents is about 10″; the ratio-of-precision curves for sines and cosines (Fig. 21a) show that with data of a similar precision, angles between 27° and 63° could not have been computed by either sines or cosines with a precision as great as 10″.

33. Trigonometric Tables.—The number of places to be used in trigonometric tables will depend upon the ratio of precision of the particular value of the function involved rather than upon the angular error. The number of significant figures for a particular angle, angular error, and function may be readily determined either by inspecting Figs. 21*a* and 21*b* or from trigonometric tables. *Assuming that the precision of angle is the governing precision*, for tangents and cotangents three places will be sufficient for angles in error more than 02′; four places, for angles in error less than 02′ but more than 10″; and five places, for angles in error less than 10″ but more than 01″.

For sines and cosines of angles of *average size* (near 45°) it will be seen that four places are sufficient for angles in error not less than 20″; five places for angles in error less than 20″ but more than say 05″; and six places for angles in error less than 05″ but more than ½″. Since angles are likely to be other than average size, the preceding limits should, in general, be doubled, though for sines of very large angles and for cosines of very small angles the necessary number of places should be ascertained by first determining the ratio of precision corresponding to the given angular error. For example the ratio of the tabular difference for 10″ to sin 80° (or cos 10°) is about $\frac{1}{120,000}$, therefore six places will be necessary for angles as small as 10° when the error is 10″; or from the ratio-of-precision curves for sines it will be seen that for angles greater than 70° when the angular error is 01′, five places will be required.

For very small angles, the number of places required is affected by the number of significant figures in the function. For example, the sine of 02′ to five places is 0.00058, which contains but two significant figures. In such cases, the use of logarithms is desirable.

34. Logarithms *vs.* **Natural Functions.**—Whether logarithms should be used depends upon the computations under consideration. It takes less time to multiply two numbers of three digits each by arithmetic than by logarithms; possibly these numbers might be extended to four digits each, if it were only for a single computation. But for multiplying, dividing, squaring, cubing, or taking the roots of numbers, it is doubtful if arithmetic can ever be used to advantage beyond four significant figures. Where there are a number of similar computations, as is quite often the case in surveying, there is a decided advantage in using logarithms for even four significant figures, for not only will less time be consumed in computing by logarithms but also the liability of mistakes will be lessened and the mental strain on the computer will be decidedly decreased. Short

arithmetical methods of multiplication and division are valuable, but often numbers that have enough significant figures to make these methods economical are large enough to make the use of logarithms more so.

The use of logarithms is described in Art. 37.

35. Graphical and Mechanical Methods.—These methods are of particular value in approximately checking more precise computations. In general, results may be obtained graphically with less labor than by arithmetic, and mistakes are less likely to occur. Frequently, combined graphical and mechanical methods may be utilized in conjunction with algebraic processes.

For example, earthwork cross-sections may be plotted to scale (graphical), the area of the cross-sections may be measured with the planimeter (mechanical), and the volume of earthwork may be determined by arithmetic.

Again, the area of a field may be determined by logarithmic computations of its partial areas, and by adding its partial areas on the adding machine. The result may be checked against large mistakes by use of the slide rule.

Similarly, unknown lengths and angles which have been algebraically or mechanically computed may be approximately checked by plotting the known data, measuring the unknown length with a scale, and measuring the unknown angle with a protractor.

The use of the common mechanical devices is described in a later chapter. It is pertinent here to draw attention to the advantages of some of these.

The most common mechanical aid available is the ordinary 10-in. *slide rule*. This rule greatly facilitates calculations involving no more than three significant figures and is in every way the equivalent of a three-place table of logarithmic functions. Probably no other calculating device yet invented has as wide a range of usefulness, and certainly none other can compare with it in the rapidity with which computations can be made.

For results of more than three significant figures the *calculating machine* is coming largely to replace other methods of computing in many offices. As opportunity presents itself the student should familiarize himself with the operations of a calculating machine in all the steps of arithmetic. Its use relieves the computer of the mental fatigue accompanying arithmetical or logarithmic calculations. The chances of mistakes are greatly decreased. With the improved types the operations of multiplication, division, squaring, and taking the square root are instantly proven, so that further checking is not required, and for most calculations involving more than three

significant figures the result may be obtained more quickly by the calculating machine than by any other method.

Another device frequently utilized by the surveyor is the *polar planimeter* (Art. 157, p. 213). It is of great value in finding the areas of figures plotted to scale. The precision with which results may be obtained depends upon a number of factors but principally depends upon the skill of the operator in traversing the lines of the drawing with the tracing point. In general, results may be determined to three significant figures, a precision in keeping with much of the field data upon which calculations of area are based. It is a simple instrument to operate and furnishes the most efficient means of determining the areas of figures with irregular or curved boundaries.

36. Arithmetical Short Cuts.—In multiplying two numbers which are not exact quantities (for example, the measured lengths of the sides of a rectangular field) the figures in the product beyond the number of significant figures in the multiplicand or in the multiplier (whichever has the lesser number of significant figures) are of no particular significance, as has already been shown.

Although the slide rule alone cannot be employed in computations involving more than three significant figures, it may frequently be advantageously used as an aid in the multiplication of numbers containing a larger number of places. This is best illustrated by an example:

Example: Two numbers 1,231.5 and 1,628.7 are to be multiplied and the product is to contain five significant figures. By arithmetic multiply 1,600 by 1,231.5. With the slide rule multiply $28.7 \times 1{,}231 = 35{,}300$. Add the partial products as shown.

$$\begin{array}{r}
1{,}231.5 \\
1{,}628.7 \\
\hline
738900 \\
1231500 \\
35300 \\
\hline
2{,}005{,}700
\end{array}$$

If all the partial products in the above example had been determined by arithmetic the time required to solve the problem would have been more than doubled.

In general the slide rule may be used to good advantage to find with certainty the last two places in any quotient, and if the operator is skillful the error in three figures need not exceed $\frac{1}{400}$ or 0.25 per cent.

36a.—An approximate method of finding the square root of a number is given by the following rule:

Rule.—*Divide the number by a quantity whose square is known and is roughly equal to the number. The arithmetical mean of the quotient and the divisor is approximately the square root of the given number.*

Example 1: $\sqrt{99} = \frac{1}{2}(\frac{99}{10} + 10) = 9.950$ (true value, 9.9499)

Example 2: $\sqrt{6146.56} = \frac{1}{2}\left(\frac{6146.56}{80} + 80\right) = 78.416$ (true value, 78.400)

The degree of approximation depends upon the closeness of agreement between the true root and the quantity chosen as a divisor, as the preceding examples show.

37. Use of Logarithms.—The logarithm of a number is the power to which some base must be raised to produce the number. In computations made by the surveyor the *common* system of logarithms is employed, for which the base is 10. Hence

$$\log 10 = \log 10^1 = 1; \log 100 = \log 10^2 = 2;$$
$$\log 1{,}000{,}000 = \log 10^6 = 6; \log 1 = \log 10^0 = 0;$$
$$\log 0.1 = \log \frac{1}{10} = \log\left(\frac{10^0}{10^1}\right) = 0 - 1 = -1 = 9 - 10$$

For any number (except 1) which is not a power of 10, the logarithm is a fractional quantity. For example

$$\log 1.5 = \log 10^{0.17609} = 0.17609; \quad \log 15 = \log 10^{1.17609} = 1.17609;$$
$$\log 0.015 = \log\left(\frac{15}{1000}\right) = \log\left(\frac{10^{1.17609}}{10^3}\right) = 1.17609 - 3 = 8.17609 - 10$$
$$= \bar{2}.17609$$

The whole number of the logarithm is called the *characteristic;* the decimal is called the *mantissa.* For a number greater than one the logarithm is a positive quantity and the value of the characteristic is one less than the number of places in the integer of the number.

For a number less than one the logarithm is a negative quantity, and to determine the characteristic the common practice is to deduct from 10 a number equal to one more than the number of ciphers to the right of the decimal point. When a logarithm is written in this manner it is *understood* that 10 is to be deducted from it. Instead of writing the logarithm in this manner the characteristic is often shown as a quantity one greater than the number of ciphers to the right of the decimal point, and a negative sign is placed over it. The logarithm of 0.0435 may therefore appear either as 8.63849 or as $\bar{2}.63849$.

For the same sequence of figures the mantissa remains unchanged regardless of the position of the decimal point. Thus the logarithm of 4,350 is 3.63849 and the logarithm of 0.00435 is 7.63849 or $\bar{3}.63849$.

The same considerations govern the number of places to be used in logarithmic computations as govern those of arithmetic. The number of significant figures in the final result should be consistent with its purpose or with the precision of the given data, as discussed in Arts. 31 to 33. Generally if the computation is of some length the practice is to use a number of places of logarithms one greater than the number of places desired in the final result.

In the ordinary work of the surveyor five places are usually sufficient, but not infrequently six places are required. On the more precise surveys seven places and occasionally eight places are necessary. Tables XVIII and XIX give logarithms of numbers and of the functions of angles to six places, but in using these tables the last figure should be dropped if only five places are required. In tables of logarithms of numbers (see Table XVIII) only the mantissa is shown and the characteristic must be supplied by the computer. Tables of the logarithmic functions of angles (see Table XIX) show both the characteristic and the mantissa.

37a. The process of finding the logarithm of a number from tables is best illustrated by an example.

Example 1: Find the logarithm of 6,458.6 correct to the sixth place.

In Table XVIII opposite the number 645 the mantissa in the column headed 8 is .810098, and in the column headed 9 is .810165. The difference between the two is 67. The last figure in the given number is 6 and hence the desired mantissa is 810,098 + 0.6 × 67. To facilitate the multiplication the table of proportional parts at the bottom of the page is given. Opposite 67 the quantity in the 6 column is 40.2. Hence 0.6 × 67 = 40.2. The mantissa is therefore .810098 + 40 = .810138. The number has four digits to the left of the decimal point, and hence the characteristic is 3 and the logarithm is 3.810138. Note that all logarithms between two adjacent horizontal broken lines have the same figures in their first two places, these figures being shown in the column headed 0.

The process of finding the number when its logarithm is known is the reverse of that given above.

Example 2: Find the number whose logarithm is 2.688544. The number is to have five significant figures.

By Table XVIII it is seen that the logarithm of 4,881 is 688,509 and the logarithm of 4,882 is 688,598. The difference between these two logarithms is 688,598 − 688,509 = 89; the difference between the given logarithm and that for 4,881 is 688,544 − 688,509 = 35. The required number is therefore 4,881 + $^{35}\!/_{89}$. By the table of proportional parts opposite 89 find the number nearest 35 (it is 35.6). At the head of the column the corresponding number is seen to be 4. The characteristic is 2, and the number is therefore 488.14.

If six places were required it would be necessary to interpolate between values given in the table of proportional parts.

Thus, for the above example, from the table of proportional parts the difference between 35 and 26.7 is 8.3, the difference between 35.6 and 26.7 is 8.9. But $\frac{8.3}{8.9} = 0.9$ (approximately). Therefore the sixth place is 9 and the number is 488.139.

This interpolation may also be made by using the table of proportional parts directly, merely moving the decimal point one place.

37b. The product of two numbers is determined by adding their logarithms.

Example 1: $15 \times 12 = 10^{1.18} \times 10^{1.08} = 10^{2.26}$ = number whose log is $2.26 = 180$.

One number is divided by another by subtracting the logarithm of the divisor from that of the dividend.

Example 2: $\frac{180}{12} = \frac{10^{2.26}}{10^{1.08}} = 10^{1.18}$ = number whose log is $1.18 = 15$.

A number is raised to a power by multiplying its logarithm by that power.

Example 3: $12^4 = 10^{1.08 \times 4}$ = number whose log is $4.32 = 21{,}000$ (about).

Following is an example of a logarithmic process of raising a number less than one to a fractional power:

Example 4: $0.6324^{1.718} = 10^{(9.8010-10)1.718}$ = number whose log is $(9.8010 - 10)1.718$

$\text{Log } 9.8010 + \log 1.718 = 0.991270 + 0.235023 = 1.226293$

$9.8010 \times 1.718 = 16.838$

$0.6324^{1.718}$ = number whose log is $16.838 - (10 \times 1.718)$

= number whose log is 9.658

= 0.455

In the above example the logarithm of a logarithm has been determined; for short this is called the *log log*. Similarly for brevity, the number corresponding to a logarithm is frequently termed the *antilog*.

38. Use of the Slide Rule.—Special books of instruction for each of the several varieties of slide rules are issued by the manufacturers. Space will not permit a detailed discussion of the use of each of these rules, but some of the more frequent calculations which may be performed on all rules of the Mannheim type will be described.

In Fig. 38 is shown one style of 10-in. slide rule, on the face of which are four graduated scales. The two scales on the body of the rule are lettered *A* and *D* and those on the slide are lettered *B* and *C*.

The rectangular glass runner may be moved to any position along the rule, its setting being indicated by a fine line etched on the glass at right angles to the axis of the rule.

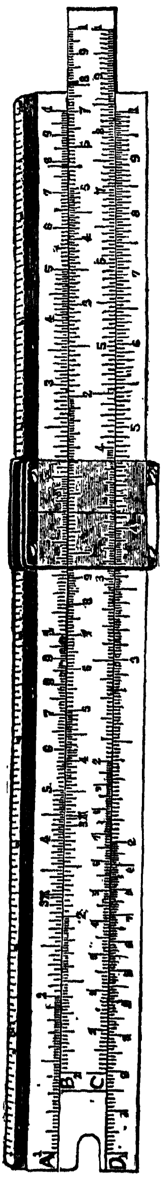


Fig. 38.—Slide rule (Mannheim type).

The C and D scales are exactly alike and are graduated with numbers from 1 to 10 within the 10-in. length. The A and B scales are similar to the C and D scales but the corresponding graduations are only one-half as great. All four of the scales are logarithmic, and if the properties and use of logarithms are borne in mind, facility in calculating with the slide rule will be more quickly acquired. On the C and D scales logarithms are to the scale of 1 = 25 cm. The figures shown are for the numbers, not the logarithms. The distance from a graduation corresponding to a given number to the "1" at the left of the C or D scale (left index) represents to the scale of 1 = 25 cm. the mantissa of the logarithm of that number. If the distance from the "2" graduation to the left index were measured with a 25-cm. scale it would be found to be 0.301 of the length of the scale, which is the value of the mantissa of the logarithm of 0.2, 2.0, 2,000, or any number having the same sequence of figures.

38a. In logarithmic computations two numbers are multiplied together by adding their logarithms. This operation may be mechanically performed by using the slide rule as illustrated by the following examples:

Example 1: Multiply 4 by 2.

Set the runner at 2 on the D scale, move the slide until the left index is at the runner. (Graphically the mantissa of 2 has now been laid off.) Move the runner to 4 on the C scale. (Graphically the mantissa of 4 has been added to that of 2.) On the D scale read 8.

Example 2: Multiply 8.2 by 7.3.

Set runner to 8.2 on D scale; right index to runner; runner to 7.3 on C scale; read answer 59.9 on D scale. The position of the decimal point is determined by mental calculation.

Had the initial setting been made with the left index at 8.2 the result would have been off the rule, at a *logarithmic* distance of 1 to the right of the final setting as determined in the example. For a logarithmic change of 1, the mantissa, and hence the sequence of numbers, is the same. Therefore settings may be made with either index, that one being chosen which will bring the final result within the length of the D scale.

Example 3: Find the product of 8.2, 7.3, 9.1, and 0.151.
Set runner at 8.2 on D scale; right index to runner; runner to 7.3 on C scale; right index to runner; runner to 9.1 on C scale; left index to runner; runner to 151 on C scale; read answer 82.3 on D scale.

38b. Division of one number by another is accomplished by finding the difference between their logarithms. Recalling the manner in which the logarithmic scales are represented on the slide rule, the operation of division becomes self-evident.

Example 4: Divide 8 by 4.
Set the runner at 8 on the D scale. (The distance from the left index on the D scale to 8 on the D scale represents the mantissa of the logarithm of 8.) Set 4 on the C scale to the runner. (The distance from 4 to the left index of the C scale represents the mantissa of the logarithm of 4.) Set the runner to the left index on the C scale. Read the answer 2 on the D scale. (The difference between the mantissa of the logarithms of the two numbers is represented by the distance from 2 on the D scale to the left index on the D scale.)

In the further examples the following abbreviations will be used:
A, B, C, and D refer to corresponding scales.
R is the runner.
LI is left index.
MI is middle index.
RI is right index.

Example 5: $\dfrac{48 \times 63}{97 \times 15}$.
R to 48 D; 97 C to R; R to 63 C; 15 C to R; R to LI C; Answer, 2.08 D.

38c. Scales A and B may be used for solving multiplications and divisions in exactly the same manner as are the C and D scales, but are more generally employed in conjunction with the C and D scales for finding the squares and square roots of numbers.

The *logarithmic* scale employed in the construction of the A and B scales (1 = 12½ cm.) is one-half of that of the C and D scales (1 = 25 cm.). Hence if the runner is set to a given number on the D scale, the square of the number is given by the runner reading on the A

scale, for the effect has been graphically to multiply the mantissa by two.

Example 6: Square 6.23.
Set R to 6.23 D; read answer 38.8 A. Or
Set R to 6.23 C; read answer 38.8 B.

Square root is performed by setting the runner to the number on the A (or B) scale and reading the root on the D (or C) scale. If the *integer* of the number contains an odd number of places (as 4.83; 125; 17,536) the runner is set on the left scale; if it contains an even number of places (as 16; 42.8; 1,174) the runner is set on the right scale. If the number is a decimal without ciphers between the decimal point and the first finite figure (as 0.428; 0.87) or with an even number of ciphers between the decimal point and the first finite figure (as 0.0087; 0.000064) the runner is set to the number on the right scale; if the number is a decimal with an odd number of ciphers between the decimal point and the first finite figure (as 0.0426; 0.0000065) the runner is set to the number on the left scale. If in doubt, the correct scale to use may be found by rough trial, using a round number which is a perfect square and which is near to the number under consideration.

Example 7: Find the square root of 16.4.
Set R to 16.4 right A; read answer 4.05 D.

Example 8: $\dfrac{0.53(14.3)^2}{0.035\sqrt{1{,}178}}$.

Set R to 0.53 D; 0.035 C to R; R to 14.3 C; LI to R; R to 14.3 C; 1,178 right B to R; R to RI; read answer 90.4 D.

Example 9: $(12.2)^{5/2}$.
Set R to 12.2 D; LI to R; R to 12.2 C; LI to R; R to 12.2 on right B; read answer 520 D.

38d. On the back of the slide is a scale for sines (S) to the *logarithmic* scale of 1 = 12½ cm., and one for tangents (T) for which the logarithmic scale is 1 = 25 cm. If the slide is removed, turned over, and replaced so that the indices of the S and T scales coincide with those of the A and D scales, the values of the natural sines of angles between 0°34′ and 90° and of natural tangents between 0°34′ and 45° are read directly from the A and D scales respectively. Thus by readings on the A scale the sine of 0°34′ is seen to be 0.0100, the sine of 5°44′ is seen to be 0.100, the sine of 30° is seen to be 0.500, etc., and by reading on the D scale the tangent of 5°43′ is seen to be 0.100, the tangent of 30° is seen to be 0.577, etc. The tangent of any angle less than 5°43′ may be considered to equal the sine, within the precision of the rule. Sines of angles less than 0°34′ may be assumed to

be proportional to the angle. On this assumption, since the sine of 0°34′ is 0.0100, the sine of 0°05′ would be $\frac{05}{34} \times 0.0100 = 0.00147$. The true value is 0.00145.

The values of other trigonometric functions may be obtained by the simple relationships of trigonometry, as will be shown in the following examples.

The trigonometric scales are used principally with the slide in its normal position (*i.e.*, with the *S* and *T* scales underneath so that they are read by the back index at the right end of the rule). In the following examples the slide is assumed to be in this position. For the sake of brevity the letters *S* and *T* will refer to the sine and tangent scales respectively, and the reading of either of these scales will refer to the reading shown by the index on the back and at the right end of the rule.

Example 10: sin 20°45′; cos 69°15′.
Set *S* to 20°45′; set *R* to *RI* on *A*; read answer 0.354 *B*.

Example 11: sec 69°15′; cosec 20°45′

$$\sec 69°15' = \frac{1}{\cos 69°15'} = \frac{1}{\sin 20°45'}$$

Set *S* to 20°45′; set *R* to *LI* on *B*; read answer 2.82 *A*.

Example 12: tan 20°45′; cot 69° 15′.
Set 20°45′ on *T*; set *R* to *RI* on *D*; read answer 0.379 *C*.

Example 13: tan 69°15′; cot 20°45′.

$$\tan 69°15' = \cot 20°45' = \frac{1}{\tan 20°45'}$$

Set 20°45′ on *T*; set *R* to *LI* on *C*; read answer 2.64 on *D*.

38e. Almost every slide rule has a scale divided into 100 equal parts. On the rule shown in Fig. 38, this scale is on the back of the slide and its graduations are numbered from the right end. From this scale the logarithm of a number may be found as follows: Set the runner on the number on the *D* scale. Move the slide until the left index is at the number, as indicated by the runner. Read the mantissa from the scale of equal parts. This amounts to scaling the distance from the left-hand index to the number.

CHAPTER IV

DRAFTING

39. The Drawings of Surveying.—It is assumed that the student is familiar with the use of the ordinary drafting instruments and with the elements of mechanical drawing. Much of the drafting with which the student is here concerned calls for a degree of skill and precision of execution quite unnecessary on dimensional plans. The beginner is likely to be ignorant of the importance of this fact, and he should realize from the start that a consistent relation between the field measurements and the map requires great care in plotting.

The drawings of surveying consist of maps, profiles, cross-sections, and, to some extent, graphical calculations, the usefulness of which is largely dependent upon the accuracy with which points and lines are projected upon paper. For the most part, few dimensions are shown, and the person who makes use of the drawings must rely either upon distances as measured with a scale or upon angles as measured with a protractor. Moreover, the drawings of surveying are so irregular and the data upon which the drawings are based are in such form that the use of the T-square and triangles (as in mechanical drawing) for the construction of parallel and right lines is the exception rather than the rule.

40. Maps.—Maps may be divided into two classes: those that become a part of public records of land division, and those that form the basis of a study. The best examples of the former are the plats filed as parts of deeds in the county registry of deeds (in most states); and good examples of the latter are the preliminary maps along the proposed route of a railroad. It is evident that the dividing line between these two classes is indistinct, since many maps might serve both purposes.

The requirements of a map will depend upon its purpose. In general the information that should appear on a map that is to become a part of a public record includes:

1. The lengths of all lines shown.
2. The bearings of all lines shown, or the angles between intersecting lines.
3. The location of the tract either with reference to principal meridian and base line, if the tract is within the boundaries of the United States

public-land surveys; or with reference to coordinate axes, if within a city having a coordinate system.

4. The number of each block and lot, if the plat is of a city, town, subdivision, or addition.

5. The position and kind of each monument set, with distances to reference marks.

6. The positions and names of all roads, streams, landmarks, etc.

7. The names of adjacent property owners, and the bearings of all property lines intersecting lines of the tract mapped.

8. The direction of the meridian (true or magnetic or both).

9. A legend or key to symbols shown on the map.

10. A graphic scale used with a corresponding note stating the scale. (The latter is usually shown in the title.)

11. A full and continuous description of the boundaries of the tract by bearing and length of sides; and the area of the tract.

12. The witnessed signatures of those possessing title to the tract mapped; and if the tract is an addition to town or city, a dedication of all streets and alleys to the use of the public.

13. A certification by the surveyor that the plat is correct to the best of his knowledge.

14. A neat and explicit title showing the name of the tract, or its owner's name, its location, the scale of the drawing, the surveyor's name, the draftsman's name, and the date.

Of maps made the basis of studies, there are so many varieties and the requirements are so varied that a definite statement of all that each should include would be impossible. In general, maps of this kind show very few dimensions (often not any), the value of the map depending upon the correct representation of the position of features of the land rather than directly upon field measurements or computed values. Maps of this class may be divided into two kinds:

1. Those that graphically represent in plan such natural and artificial features as streams, lakes, boundaries, condition and culture of land, and public and private works; and

2. Those that include some or all of the preceding features, but also represent the relief or contour of the ground.

Other things which should *always* appear on these two kinds of maps are:

1. The direction of the meridian.

2. A legend or key to symbols used, if they are other than the common conventional signs (Figs. 269*a–d*, pp. 385–388).

3. A graphic scale of the map with a corresponding note stating the scale.

4. A neat and appropriate title generally stating the kind or purpose of the map, the name of the tract mapped or the name of the project

for which the map is to be used, the location of the tract, the scale of the drawing, the contour interval, the name of the engineer or draftsman or both, and the date.

40a. Maps of large areas, as of a state or country, which show the location of cities and towns, streams and lakes, and the boundary lines of the principal civil divisions are called *geographic maps.* Maps of this character which show also the general location of some kind of the works of man are designated by the name of the works represented. Thus we may have a *railroad map of the United States,* showing the names of railroads and the principal cities through which they pass; an *irrigation map of the state of California,* showing the location of tracts which have been placed under irrigation; or a *highway map of the state of New York,* indicating the location of highways under state control.

Cadastral maps are large-scale maps which show the natural features of the land but also show the position of land lines and reference monuments, the names of streets, the ownership of property, and in more or less detail the nature and extent of public and private improvements. Such maps usually represent relatively small areas, as a city or portion thereof.

Topographic maps indicate the relief of the ground in such manner that elevations may be determined by inspection. The relief is usually shown by irregular lines, called contour lines, drawn through points of equal elevation (Art. 431, p. 634). Maps of this kind not made for a specific purpose are called general topographic maps. In addition to the topographic and geographic features, such maps usually represent public and, to some extent, private works. Usually the general topographic map is drawn to a small scale. The quadrangle maps of the U. S. Geological Survey are good examples.

Hydrographic maps show the shore lines, the location and depth of soundings or lines of equal depth, and often the topographic and other features of lands adjacent to the shores. General hydrographic maps are those the use of which is intended to be general in character, as for example, the charts of the U. S. Coast and Geodetic Survey.

Many maps are constructed for the specific purpose of aiding in determining the feasibility of engineering projects or of facilitating the location and construction of the works of man. The features included on such maps are entirely dependent upon the purpose for which the maps are made and may or may not represent the topography or hydrography of the area mapped. The maps are usually designated according to the purpose for which they are made. For example, the map made the basis for preliminary studies to determine the location of a railroad is termed the "preliminary map," the one

showing the alinement of the located line is called the "location map," the one showing the boundaries of rights of way and intersecting land lines is designated as the "right-of-way map," etc.

41. Map Projection.—A map shows graphically the location of certain features on or close to the surface of the earth. Since the surface of the earth is curved and the surface of the map is a plane, no map can be made to represent a given territory without some distortion. If the area is small, the earth's surface may be regarded as plane, and a map constructed by orthographic projection, as in mechanical drawing, will represent the relative location of objects without measurable distortion. The maps of plane surveying are constructed in this manner, points being plotted either by rectangular coordinates or by horizontal angles and distances.

As the size of the territory increases, this method becomes inadequate and various forms of projection are employed to minimize the effect of map distortion. The points of control are plotted by spherical coordinates through the use of elaborate geographic tables. Since the spherical coordinates of a point are its latitude and longitude, it is customary to show meridians and parallels on the finished map. The maps of states and countries, as well as those of much smaller areas, are constructed in this manner. The various methods of map projection are discussed in Chap. XXIX.

42. Scales.—The common map scales are of three types:

1. One unit of length on the drawing, the *map distance*, represents a given number of units of length on the ground, the *ground distance*, the ratio of the former to the latter being called the *representative fraction*. For example, some of the maps of the U. S. Geological Survey have a scale of $\frac{1}{62,500}$ which indicates that 1 ft. (or 1 in.) on the drawing is the equivalent of 62,500 ft. (or in.) on the ground.

2. A given number of inches on the drawing represents 1 mi. on the ground. Geographical, military, and land maps frequently exhibit the inch-mile relation.

3. One inch on the drawing represents some whole number of tens, hundreds, or thousands of feet on the ground. Most maps directly employed by engineers in the design and construction of the works of man are drawn to a scale of this type. It is called the *engineer's scale*.

The magnitude of the scale to which a map should be drawn, or in other words, the relation between the size of the map and the size of the area represented, depends upon the purpose of the map, and to some extent upon the character and extent of the tract shown. In general, it may be taken as a rule that the scale of a map should

be no larger than is necessary to represent the location of details with the requisite precision. Maps constructed for engineering projects have scales generally ranging from 1 in. = 20 ft. to 1 in. =

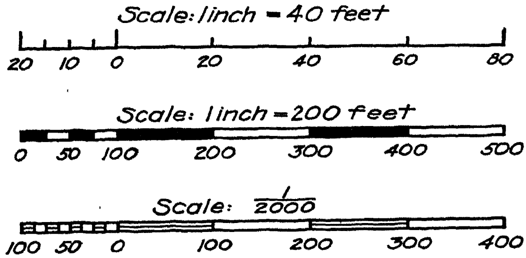


FIG. 42.—Scales.

800 ft. Maps of land subdivisions have scales ranging from 6 in. = 1 mile to 1 in. = 1 mile. Cadastral maps have scales ranging from 6 in. = 1 mile to 1 in. = 100 ft. General topographic maps have scales ranging from $\frac{1}{10,560}$ to $\frac{1}{250,000}$. Geographic maps have scales of 1 in. = 1 mile to 1 in. = 20 miles or more.

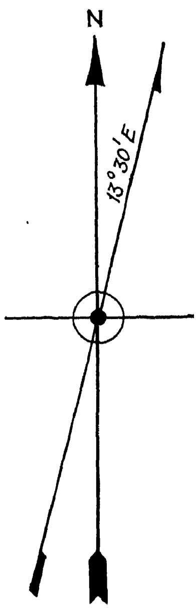


FIG. 43.—Meridian arrows.

For convenience in later discussions maps will be arbitrarily divided into those of

1. Large scale: 1 in. = 100 ft. or less.
2. Intermediate scale: 1 in. = 100 ft. to 1 in. = 1,000 ft.
3. Small scale: 1 in. = 1,000 ft. or more.

The scale should be shown near the title of the map so that it will catch the eye readily. It should be expressed both numerically, as for example, 1 in. = 200 ft. and graphically in the form of a line subdivided into convenient units of length as illustrated in Fig. 42. Drawing papers shrink and swell more or less with changes in climatic conditions. The graphical scale makes it possible to determine map distances accurately either from drawings which have been thus affected or from photographs of the drawings.

43. Meridian Arrows.—The direction of the meridian is indicated by a needle or feathered arrow pointing north, of sufficient length to be transferred with reasonable accuracy to any part of the map. The true meridian is usually represented by an arrow with *full* head; the magnetic meridian by an arrow with *half* head. When both are

shown the angle between them should be indicated. The general tendency is to make needles and arrows too large, blunt, and heavy. A simple design is shown in Fig. 43.

44. Profiles.—Longitudinal sections made by projecting the ground line upon a vertical surface are known as *profiles*. In conjunction with maps, they are of assistance to the engineer in fixing the grades and alinement of such works as sewers, railroads, highways, and canals. They are also of value in estimating volumes of earthwork. The data from which profiles are plotted consist of ground elevations at known distances apart along some line, as for example, the center line of a highway. The ground profile is formed by a continuous line drawn through the plotted points. In addition to the ground profile there is also shown one or more grade profiles and other pertinent information. For example, the profile along the center line of a canal would show the ground line and the canal bed and probably also the water surface and top of bank.

Profiles are usually made on *profile paper*, obtainable in standard rulings. For each of these rulings, every fifth horizontal line and every tenth vertical line is accentuated by making it heavier than the intervening lines.

45. Cross-sections.—The calculations of volumes of earthwork are frequently facilitated by plotting cross-sections of the earthwork to scale. The area of a cross-section is then determined either by means of a planimeter or by dividing the figure into triangles and rectangles and arithmetically calculating the partial areas. The data for plotting consist of elevations and distances either calculated or measured in the field. For example, the earth for the construction of a railroad fill may be obtained from an adjacent borrow pit. Elevations of ground points along lines transverse to the axis of the borrow pit, taken prior to and again after the removal of the earth, furnish measurements from which cross-sectional areas can be calculated. If the borrow pit is irregular in shape it will perhaps facilitate calculations if the sections are plotted.

Cross-sections may be shown on ordinary paper, are sometimes drawn on profile paper, but more often are plotted on cross-ruled paper called *cross-section paper*. The divisions are the same horizontally as vertically. The lines marking the half-inches or inches are made heavier than the rest.

46. Lettering.—Inasmuch as a drawing is likely to be judged by the quality of its lettering, it is important that the draftsman be able to form letters with at least a fair degree of skill and to assemble them in such form, size, and position as to make the drawing clear and of pleasing appearance. In machine and structural drawing,

simplicity and clearness are of primary importance; a considerable portion of the drawings made by the map draftsman require, in

ABCDEFGHI
JKLMNOPQ
RSTUVWXYZ
abcdefghijklmn
opqrstuvwxyz
1234567890
NORMAL
Hickory Tree 10 ft.
COMPRESSED
WASHINGTON Sta. 71+43.8
EXTENDED
NEVADA

Fig. 46*a*.—Reinhardt letters, slope form.

ABCDEFGHI
JKLMNOPQR
STUVWXYZ
abcdefghijklmn
opqrstuvwxyz
1234567890
NORMAL
Excavation 23 cu. yd.
COMPRESSED
RICHARDSON ESTATE 300 Ac.
EXTENDED
RED RIVER

Fig. 46*b*.—Reinhardt letters, vertical form.

addition, a certain artistic quality. This is particularly true of maps which are to be largely used by the public, and on such work the

GOTHIC
ABCDEFGHI
JKLMNOPQR
STUVWXYZ
abcdefghijklmn
opqrstuvwxyz
1234567890
HAIRLINE ANTIQUE
ABCDEFGHI
JKLMNOPQR
STUVWXYZ
abcdefghijklmn
opqrstuvwxyz
1234567890

Fig. 46*c*.—Gothic and hair-line antique lettering.

ROMAN
ABCDEFGHI
JKLMNOPQR
STUVWXYZ
abcdefghijklmn
opqrstuvwxyz
1234567890
ITALIC
ABCDEFGHI
JKLMNOPQR
STUVWXYZ
abcdefghijklmno
pqrstuvwxyz
1234567890

Fig. 46*d*.—Roman and italic lettering.

draftsman is often justified in expending considerable time in adding the quality of beauty to that of utility. This should not be con-

strued to mean that he is to employ the complex forms of letters with scrolls and flourishes to be seen on many old drawings.

In general, lettering should be freehand, and it should be of a style in keeping with the purpose of the drawing.

For *office drawings*, or drawings which are not to be used by the general public, the *Reinhardt* style of single-stroke lettering is almost universally employed in this country. The ease with which the letters may be read and the rapidity with which they may be constructed make this style a very practical one for the great bulk of all engineering drawings. Reinhardt letters are made in two forms, inclined and vertical, as illustrated in Figs. 46*a* and 46*b*.

Variations of these forms in the matter of increasing or decreasing the horizontal dimensions of the letters (relative to the vertical dimensions) are frequently desirable. When the horizontal dimensions are reduced and the letters are close together, the lettering is said to be *compressed*. When the horizontal dimensions are elongated and the letters are some distance apart, the lettering is said to be *extended*.

Frequently the two forms with their variations may be advantageously combined on a single drawing. Thus the names of streams might be shown in the extended slope capitals, the names of streets in extended vertical capitals, the names of property owners in normal lower-case vertical letters, and notes in compressed lower-case inclined letters.

On drawings where their use is justified, *gothic*, *roman*, and *italic* letters may be employed if the draftsman is sufficiently skilled in the execution of these styles.

The *gothic* alphabet in vertical form is shown in Fig. 46*c*. Gothic letters with a few exceptions are similar to Reinhardt letters, but their lines are heavier. To one skilled in the execution of Reinhardt letters the gothic style offers no particular difficulties. It is a style which may be employed when it is desired that the lettering stand out from the body of the drawing to catch the eye quickly.

Hair-line antique lettering (Fig. 46*c*), a modification of the gothic style, may be used when the letters are to be subdued so as not to interfere with the general clearness of the drawing. As the name indicates, the letters are composed of very fine lines, which with the forms of the capitals, makes it a style rather more difficult to execute than is the gothic style. Generally only the capitals are employed.

Roman and *italic* letters (Fig. 46*d*) are shaded, and for this reason are difficult to construct. Possibly on account of our familiarity with the appearance of the perfect form through the printed page, slight deviations in roman lettering at once catch the eye, and rela-

tively few draftsmen possess the skill to make roman letters that look well on a drawing. *Italics* are very little different from roman letters except that they are inclined, but slight deviations in their form are not as noticeable. Lettering in either the roman or italic styles is a relatively slow process, and unless the draftsman is thoroughly familiar with these styles and is a fairly good letterer it is better to keep to the simpler forms.

In map drafting, where the details to be shown are many and are varied in character, it often renders the map clearer if a particular style of lettering is employed for each class of objects shown. Not infrequently the several styles just described might be employed on a single map. For example, the topographic maps of the U. S. Geological Survey show the names of civil divisions in roman letters, the names of streams and lakes and other hydrographic features in italics, the names of mountains, valleys, and other land forms in vertical gothics, public works in inclined gothics, and marginal lettering in hair-line antique.

Detailed information concerning lettering is to be found in textbooks on drawing, but it is perhaps appropriate here to offer a few suggestions to the beginner.

1. Always use guide lines.
2. Be sufficiently familiar with the construction of each letter so that its form will always appear the same.
3. It is very important that the slope of letters in a word, sentence, or paragraph be uniform. If this is accomplished, a good effect will be secured even though the separate letters may be faulty. The inclination of slope letters should not be excessive.
4. Never seek to improve lettering by making straight portions of freehand letters by mechanical means.
5. Three common defects in the lettering of the beginner are: (1) letters of varying shape, (2) excessive spacing, and (3) the unequal or apparently unequal spacing of letters as they appear in words.
6. Avoid sharp angles in the rounded portions of letters. The curves should be smooth.
7. If spacing is important, do all lettering in pencil as neatly as possible before inking in.
8. Make all the elements of Reinhardt letters by single strokes. Use a pen that will produce a line of the required weight at the first stroke.
9. Follow the same procedure for gothic letters, unless they be unusually large.
10. Do not attempt to make the shaded portions of italic and roman letters by single strokes. Outline the letters with a fine pen and fill in the shaded portions.

11. Make the letters of a size in keeping with their purpose. The names of the larger or more important objects should catch the eye quickly; notations concerning relatively unimportant details should be inconspicuous.

12. In lettering drawings which are to be reproduced to a reduced scale, make the size and weight of the letters conform to the requirements of the process of reproduction.

13. Leave a generous interval between the letters forming the names of elongated or large objects as streams, streets, lakes, mountain ranges, counties, railroads, etc.

47. Titles.—Titles should be so constructed that they will readily catch the eye, and the sizes of different parts should be in proportion

TOPOGRAPHIC MAP

OF

DON PEDRO RESERVOIR AREA

MERCED IRRIGATION DISTRICT

SCALE — 1" = 4000'

1000 0 1000 2000 3000 4000

MERCED, CALIF. JUNE 15, 1922

M. M. STARR-CH'F ENG'R - A. J. WILLIAMS-RES. ENG'R

DRAWN BY C. J. CALLEN

Fig. 47.—Title for map.

to their importance. The best position for the title is the lower right-hand corner of the sheet. In general each line should be centered and the distance between lines should be such that the title as a whole will appear well balanced. The space occupied by the title should be in proportion to the size of the map. The general tendency is to make the title too large. The different parts of the title should be in straight lines, and only the common styles of letters should be used. It is best to construct the title of all gothic or all roman letters, but a change of style of lettering between different parts is permissible when such change will serve to accentuate the important parts of the title. Thus the principal part may be constructed in roman letters, while the remainder is in gothics. Slope and vertical letters should not be included in the same title. In general, titles should be constructed freehand, as well as the letters on the body of the map. The use of Reinhardt letters for the important parts of titles, even those of office plans, is not prevalent, though they may

be appropriately used for the less important parts where small letters are used. The use of gothic capitals is most common, for not only are they effective but also they are easily made.

Thus a complete title might be as shown in Fig. 47. The parts ought to be weighted in order of their importance. Since the purpose of a title is to distinguish a given map from others of a similar character, the thing which should catch the eye first is the name of the area or the principal object of the drawing. The title given shows how portions may be accentuated by variations in size and style of lettering.

48. Notes and Legends.—Explanatory notes or legends are usually of assistance in interpreting a drawing. They should be as brief as circumstances will allow, but at the same time should include sufficient information as to leave no doubt in the mind of the person using the drawing. A key to the symbols representing various details ought to be shown even though the symbols be conventional in character. The nature and source of data upon which the drawing is based ought usually to be made known. For example the data for a map may be obtained from several sources, perhaps partly from old maps, partly from old survey notes, and partly from new surveys; the surveys have been made with a certain precision; the direction of the meridian has been determined by astronomical observation; elevations are referred to a certain datum as indicated by a certain bench mark of a previous survey. These facts in the form of notes explaining such items are frequently of inestimable value to the person who later has occasion to use the drawing.

Notes ought to be in such a position on the drawing as to catch the eye readily, conditions allowing. A favorable position is the lower right-hand corner just to the left of the title.

49. Drawing Papers.—Drawings of a temporary nature or pencil drawings are sometimes made on a smooth manila *detail* paper, of which there are several grades and weights. The manila papers will not take ink nor stand erasures as well as a regular drawing paper nor are they as durable.

Regular drawing papers are obtainable in a variety of grades, weights, and finishes. For general map work a fairly smooth, tough paper of uniform texture is desirable. It should stand erasures without its surface becoming fibrous, and should take ink well. For permanent drawings a paper should be chosen which will not become brittle nor discolor with age.

Large drawings or those that must withstand hard usage should be constructed on *mounted* paper. Any of the better grades of drawing papers may be obtained mounted on muslin, either in sheets or in rolls.

In consideration of the importance of the map, and of the slight expense of the paper in relation to the survey as a whole, it may be considered false economy to use any but high-grade papers for mapping purposes.

Most papers have watermarks which are legible from that side of the paper intended for the drawing, or the "right" side.

50. Tracings.—A tracing is a drawing copied in ink or pencil on a transparent sheet of paper or cloth, for the purpose of reproduction.

Tracing paper comes in several grades, all suitable for pencil drawings. The better grades, usually processed, are also suitable for ink drawings. Tracing papers will not stand repeated erasures well and will become torn and cracked unless they are handled carefully. However, they are economical and are entirely satisfactory for either preliminary or rough drawings.

Tracing cloth is made from fine linen cloth specially treated to render it firm, transparent, and smooth. It is used for drawings of a permanent character or for drawings which will be subject to considerable handling. The glazed or *smooth* side is seldom used although it will take ink and it stands erasures well. The unglazed or *rough* side is preferred by most draftsmen for ink drawings. Pencil drawings are made on the rough side; however, pencil drawings on tracing cloth become smudged easily and are seldom made. Good grades of tracing cloth will not deteriorate with age, but all tracing cloths turn white and wrinkle where touched by water.

Preparatory to making a tracing in ink on cloth, the surface is dusted with powdered talc or chalk and is rubbed with a dry cloth; any excess powder is removed.

Erasures of ink on cloth are made with least damage to the surface by rubbing lightly with a soft pencil eraser, using an erasing shield. Ordinary ink erasers are too abrasive and produce a fibrous surface which does not take ink well. Some draftsmen employ a sharp knife to gently scrape or pick off the ink on the surface, then use the eraser to remove the ink impregnating the fibers of the cloth. Pencil lines are removed and the tracing is cleaned by rubbing either with art gum or with a cloth saturated in gasoline, cleaner's naphtha, benzine, or carbon tetrachloride. The use of gasoline is said to cause tracing cloth to deteriorate more rapidly.

Ordinarily it is difficult to trace from blueprints, but good results are obtained by drawing over glass strongly illuminated from below. This method is also used to transfer drawings on to drawing paper which otherwise would not be transparent. Making one tracing from another is facilitated by having a sheet of white paper underneath the lower tracing.

51. Reproduction of Drawings.—A drawing may be reproduced at the same scale by making a contact print from a tracing on processed paper, resulting either in a *blueprint*, a *vandyke print* (brown), or a *direct blackline print*. By direct printing or by reprinting from a vandyke contact negative, either of these prints may be made with white lines on a dark background or with dark lines on a white background.

A drawing may be reproduced either at the same scale or at different scales (enlarged or reduced), from either drawings or tracings, by various photographic processes such as the *photostat* process, the *photo-offset* methods, and methods employing *duplicate tracings* from which contact prints are made.

Maps and other drawings for general distribution are usually either lithographed or printed from etchings.

52. Blueprints.—The most common and economical method of reproduction at the same scale is that of making from the tracing a blueprint, in which white lines appear on a blue background. To produce blue lines on a white background, the method herein described is followed, except that a vandyke negative (see Art. 53) is employed instead of the tracing. Blueprints are made by placing a tracing against the processed side of the sheet, exposing this side to light, and developing the exposed sheet in a bath of water.

Most blueprints are made on paper, but those likely to be subjected to rough handling are often made on a sized cloth. Both blueprint paper and blueprint cloth are available in rolls covered with light- and moisture-proof wrappers. What is said herein regarding paper applies also to cloth. In appearance, the fresh, unexposed paper is a pale greenish yellow. In a moist climate, unless carefully protected the paper soon takes on a bluish tinge and becomes unfit for use even though unexposed to light. In handling the paper, it should be protected from exposure to light except during the period of actual printing.

Although manufactured blueprint paper is so economical that ordinarily no one would consider making his own, emergencies may arise when a knowledge of the process of manufacture is of value. A satisfactory blueprint paper may be prepared by applying to paper a mixture of equal parts of the following solutions:

1 part (by weight) of prussiate of potash to 5 parts of water.

1 part (by weight) of citrate of iron and ammonia to 5 parts of water.

Either of the two solutions may be prepared in sunlight, but they should be combined and applied to the paper in a dark room or in a subdued light. The paper is placed on a flat surface and the sensitizing solution is spread with a sponge, cloth, or camel's-hair brush, employing long strokes first lengthwise and then crosswise of the sheet. Only

enough of the solution is applied to wet the surface of the sheet. The paper is dried in a dark place, and is ready for use.

Blueprints are made by exposure either to sunlight or to artificial light. The electric blueprinting machine is a part of the equipment of many large offices. In most cities there are firms who specialize in the making of blueprints, and most engineers and surveyors having a limited amount of work find it more economical and more satisfactory to have their blueprints made by commercial firms.

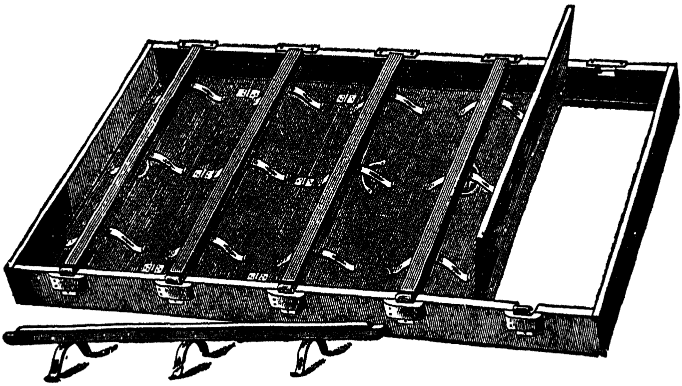


Fig. 52.—Blueprint frame.

A common method of blueprinting is to expose the paper to sunlight. On cloudy days, the same procedure is followed, except that the time of exposure is longer. The inked side of the tracing is placed next to the glass in a blueprint frame, and the sensitized surface of the paper is placed next to the tracing. The proper time of exposure depends upon the intensity of the light and upon the quality of the paper, and is best determined by trial, using small pieces of the paper. The rapid papers which are commonly used require an exposure of ½ to 1 min. in strong sunlight; others require an exposure of 2 or 3 min. The sensitizing formula given herein will produce a slow paper, for which the necessary time of exposure is 5 or 6 min. A larger proportion of citrate will make a faster paper. If underexposed, the body of the print when washed will be a pale blue; if overexposed or *burned*, it will be a dark mottled blue and the fine lines of the drawing will be indistinct or missing. During the period of exposure the sheet should be square to the sun's rays.

An exposed print is developed by washing it in water. To avoid streaks and indistinct lines, the print is quickly submerged and is washed until the greenish tinge has disappeared. The print is then soaked for 10 to 30 minutes. The blue of the print will be

intensified by the addition to the bath of a small quantity of potassium dichromate; this solution also tends to overcome the results of overexposure. A commercial "no potash" paper is available which produces a deep blue color when washed in water only.

After being thoroughly washed and soaked, the print is removed from the bath and is hung to dry in a subdued light. Wrinkles may be removed by ironing.

52a.—A blueprint frame of common design is shown in Fig. 52. The glass should be clear, preferably a fairly heavy plate. A thick felt or pneumatic pad is placed next to the paper. The spring clamps force the hinged back against the felt and insure perfect contact between tracing and paper.

For methods of making alterations on blueprints, see Art. 60.

53. Vandyke Prints.—A print made on processed *vandyke* paper has white lines on a dark brown background, which is nearly impervious to light. The color of the fresh paper is light yellow. The operation of exposing the paper is the same as in blueprinting, but the necessary time of exposure is considerably greater, usually about 5 min. in strong sunlight or until the paper protruding beyond the tracing takes on a rich brown color. The exposed sheet is washed for about 5 min. in water, becoming lighter in color. It is then transferred to a fixing bath consisting of 2 oz. of hyposulphate of soda to 1 gal. of water. When the print takes on a deep brown it is again washed thoroughly in clear water and is left to soak for 20 to 30 minutes.

The principal use of the vandyke process is in the making of a negative from which blueline positive prints or other contact prints may be made. To render the white lines more nearly transparent, the vandyke negative may be sponged with a banana-oil compound, but this compound should be thoroughly evaporated from the surface before blueprinting is attempted. Blueline positive prints are then made by placing the brown surface of the vandyke negative next to the sensitized sheet of blueprint paper, and by exposing and washing in the usual manner. Brownline vandyke prints and blackline prints are similarly made.

Prints with a white background are clearer than those with a dark background, and additional notations stand out well. As in the case of blueprints, alterations are evident because erasure of the lines damages the paper.

54. Blackline Prints.—A blackline contact print on transparent paper, called a *blackline tracing*, is made from a vandyke negative by a process similar to that for blueline prints. Blackline tracings shrink during the process of manufacture, and hence the scale is

altered. However, they are economical and are useful for preliminary working drawings from which prints are to be obtained.

A *direct blackline print* is made from the tracing by contact printing in sunlight or artificial light, using a special sensitized paper. The print is developed by applying a chemical furnished by the manufacturer, after which it is thoroughly washed in water and is dried. This type of print is increasing in use, as it has the advantages of printing without a negative, a white background, freedom from excessive shrinkage, and difficulty of alteration without detection.

Blackline reproductions by other processes are described in Arts. 56 to 58.

55. Ozalid Prints.—*Ozalid prints* with red, blue, or brown lines are made, from tracings, on a special sensitized paper by direct contact printing in the usual manner. They are developed by placing in a tight container, which is then filled with ammonia fumes; no washing is required. This process is not widely used.

56. Photostat Process.—A drawing on any kind of paper or cloth may be reproduced to any scale by the photostat process, provided that the lines are of a color which photographs well. The process is widely used, especially in the reproduction of pages from books.

The photostat machine is a modified form of camera. The drawing is strongly illuminated by artificial light, and a negative to the desired scale is made, in which black lines of the original appear white. By rephotographing, a positive is produced in which black lines of the original appear black on a white or gray background.

Large reproductions are considerably distorted near the edges, but in the usual sizes the distortion is not great. Reproductions up to 40 by 60 inches may be made by this process.

Photostat reproduction of a blueprint may be made in one operation, using the blueprint as the negative.

57. Photo-offset Process.—The photo-offset process, known by various trade names, is useful when many prints of a drawing are required. This process consists in making a negative to the desired scale by photographing the original; from this negative a plate is prepared and mounted for use in a printing press. The prints may be made on any good grade of bond paper and have distinct black lines. For large quantities the cost is low.

58. Duplicate Tracings.—Duplicate tracings may be made to any desired scale on transparent cloth or paper. The lines are black, and the reproduction is identical in appearance with the

original copy. Additions or alterations may be made as readily as on the original.

59. Pencils.—For drawings on hard, fine-grained papers the 6H pencil is quite widely used. For very thin lines and for permanent work the 8H is occasionally employed. Many drawings on smooth papers are made with the 4H pencil. On the soft profile and cross-section papers, lines made with a 2H pencil show up well. For coarse tracings made on tracing paper a pencil as soft as 2B is sometimes necessary. Drawings made directly on the rough side of tracing cloth will show up sufficiently well for inking if a 2H pencil is used. Lines that are to be traced should in any case be made sufficiently heavy to show clearly through the tracing cloth. Many draftsmen sharpen one end of the pencil to a wedge point for use in drawing straight lines, and the other end to a conical point for sketching irregular lines and for lettering. For any but simple sketches of a few lines, care should be taken to choose a pencil which will not smudge readily.

60. Inks and Colors.—The bottled inks commonly used by the map draftsman are black, brown, orange, blue, green, and vermilion. For line drawings, the waterproof inks are satisfactory. Lines made with them will not smudge when rubbed with the moist hand and are not affected by the application of liquid tints or washes. The stick India inks are not much used but are preferred by some draftsmen for intricate drawings. India inks are prepared for use by grinding and mixing with water. Inks may be thinned by adding distilled water or a dilute solution of ammonia.

Drawings may be tinted with water colors which are obtainable in a variety of colors either in tubes or pans. The colors are mixed with water until the desired tint is produced and are then applied with a camel's-hair brush. Tracings may be tinted by rubbing the rough side with colored crayons, but cannot be tinted with a wash.

Inks other than black are frequently made from water colors. Few bottled colored inks possess sufficient body to be used on tracings from which blueprints are to be made. For such work nearly all of the darker water colors can be mixed sufficiently thick to make lines which will show up well on blueprints, and at the same time will flow readily from the pen. Generally bottled inks are not regarded as entirely satisfactory for the blue and brown lines of colored maps. Better colors are secured through the use of prussian blue and burnt sienna water colors mixed with sufficient water to produce the desired tints.

Alterations on blueprints may be made with a weak solution of caustic soda. This is used as an ink to produce white lines. It

removes the blue color by chemical action. Being a thin liquid, it is absorbed by the fibers of the paper, and unless applied very sparingly will produce a wide ragged line. If a colored line is desired the solution may be mixed with ink. Several solutions of this nature are on the market in bottled form and are known as erasing fluids. Alterations on blueprints may also be made by using "salts of sorrel" (potassium acid oxalate, a poison) as an ink. A sharp, white line is thus produced which does not spread over the sheet in damp weather and which does not turn yellow.

61. Drawing Instruments; Scales.—Besides the equipment commonly used in mechanical drawing there are several instruments

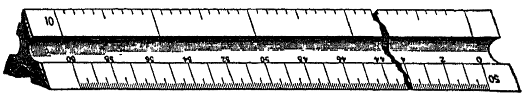


FIG. 61*a*.—Engineer's scale, triangular.

and devices which are generally useful in the work of the surveyor and with which the student should be familiar.

Engineer's scales are divided into 10, 20, 30, 40, 50, 60, 80, or 100 parts to the inch. Rules thus divided are flat or triangular in shape and are obtainable in lengths of 3, 6, 12, 18, and 24 in. The 12-in. triangular boxwood rule with 10-, 20-, 30-, 40-, 50-, and 60-ft. scales on its three faces (Fig. 61*a*) is most commonly used. It has the advantage of compactness.

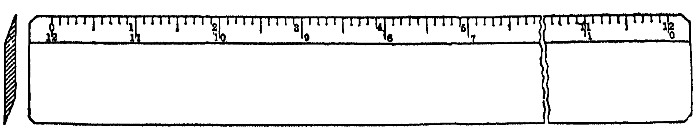


FIG. 61*b*.—Engineer's scale, flat with opposite bevels.

The flat rule with two scales on edges of opposite bevel (Fig. 61*b*) is most satisfactory to use. Mistakes in plotting or in scaling distances through using the wrong scale are much less likely to occur than with the triangular rule.

The 10-ft. scale is intended for measuring to the scale of 10, 100, or 1,000 ft. to the inch, but the divisions are so large that it cannot be used for accurate plotting. For such work it is better to use the scale divided into 50 parts to the inch. Probably $1/100$ in. is as close as distances can be plotted by the ordinary methods of drafting.

For accurate drawings, points should be pricked with a needle point, and distances should be plotted with the eye directly above the graduation to which distance is measured.

61a. Protractors.—The protractor is a device for laying off and measuring angles on drawings. In its usual form it consists of a full circle or semicircular arc of metal, celluloid, or paper graduated in degrees or fractions of a degree. Protractors are obtainable in sizes from 3 in. to 14 in. in diameter and in a variety of designs. The smaller protractors are usually graduated to degrees or ½ degrees; the larger sizes are frequently graduated to ¼ degrees. Some are equipped with verniers reading to five minutes or to single minutes but this refinement adds little to either the precision with which angles can be laid off or to the facility with which the protractor can be used. Others have radial scales by means of which a distance and angle may be plotted at one operation.

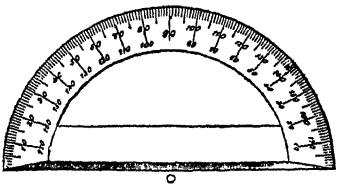


FIG. 61*c*.—Semicircular protractor.

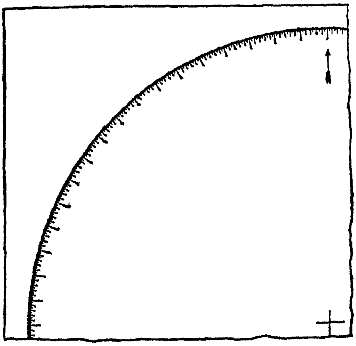


FIG. 61*d*.—Paper protractor.

Figure 61*c* illustrates the most common form of semicircular metal or celluloid protractor. To lay off an angle, the center *O* is placed at the vertex with the lower edge of the diametrical bar coinciding with the line to which the angle is referred. A mark is then made on the drawing at the proper graduation of the arc, the protractor is removed, and a line joining this mark with the vertex *O* is drawn.

Protractors with full circles are also in common use. Figure 61*d* shows part of a full-circle paper protractor. These are usually printed in 8 and 14-in. sizes on rectangular sheets of tough paper or bristol board, without the graduations being numbered. The 14-in. size is graduated to ¼ degrees and the 8-in. size to ½ degrees. To prepare such a protractor for use, its graduations are numbered as desired and it is cut either on a circle passing through the outer ends of the graduations or on a circle of somewhat smaller radius. In the former case the outer portion is discarded and angles are laid off by using the inner portion in the manner described for the semicircular metal protractor. In the latter case the inner portion

is discarded, and angles are laid off by means of a straightedge passing through the center of the protractor. The protractor is centered and oriented by means of two intersecting lines on the drawing, one line being at right angles to the other.

Another form of paper protractor which is frequently employed for plotting details is described in Art. 268, p. 383.

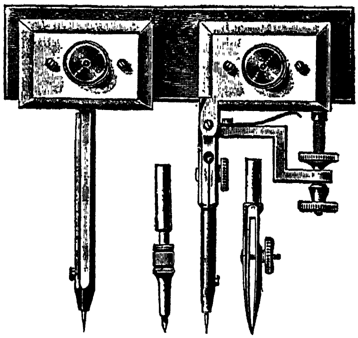


FIG. 61*e*.—Beam compass.

61b. Beam Compass.—This compass, illustrated in Fig. 61*e*, is used for drawing the arcs of large circles. The rigidity of the beam compass makes it a more reliable instrument than the ordinary compass of the drawing set equipped with its extension arm. For accurate drawings the beam compass preferably should be used for circles having radii greater than 6 in.

A beam compass may be improvised from any thin strip of wood by driving a needle through the strip near one end and cutting a V-shaped notch in the edge of the strip at the required distance from the needle point. The point of a pencil or ruling pen is held in the notch and the arc is drawn as with the regular beam compass.

61c. Railroad Curves.—These are thin strips of cardboard, wood, metal, hard rubber, or celluloid, the edges of which are arcs of circles. A number on each curve indicates its radius in inches, and sometimes also an additional number indicates the degree of curvature for a given scale. With these curves, arcs of circles may be drawn without determining the center and also arcs may be drawn with much larger radii than could be used with a beam compass.

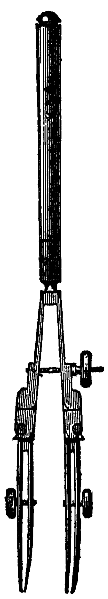


FIG. 61*f*.—Railroad pen.

61d. Railroad Pen (Fig. 61*f*).—This pen, sometimes also called the *road pen*, is used principally for drawing two parallel lines either freehand or by means of a straightedge or curve. It consists of two ruling pens with spring shanks attached to a handle, the distance between the two pens being controlled by a screw passing through the shanks. Its use greatly facilitates the drawing of parallel lines which are curved or irregular.

61e. Contour Pen (Fig. 61*g*).—This pen is useful for drawing contours or other freehand curves. The pen is rigidly connected to a shaft which freely turns in the handle. The point of the pen is eccentric with the axis of the shaft so that the pen will turn in whatever direction it is drawn on the paper. In use, the handle is held vertically as illustrated in the figure, the fingernails of the third and fourth fingers of the right hand being in contact with the paper.

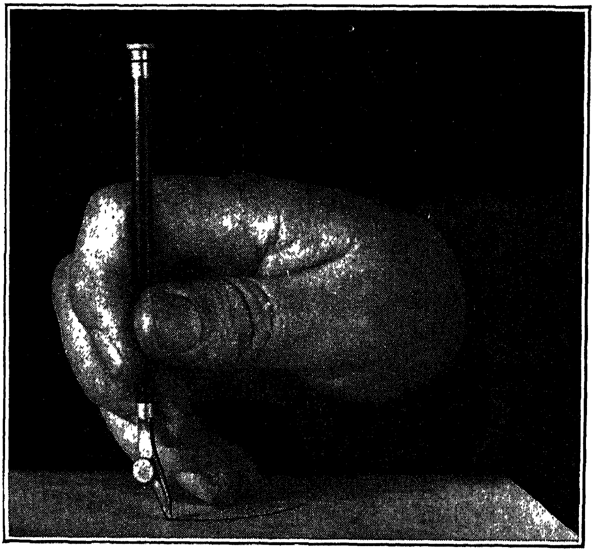


FIG. 61*g*.—Drawing line with contour pen.

61f. Straightedge.—For the drawings with which the surveyor is chiefly concerned, the T-square will not produce parallel lines with sufficient precision. Moreover, in map drafting the lines are seldom perpendicular or parallel to one another. The most satisfactory form of straightedge for general office use is of nickel-plated or rustless steel, with one edge beveled. Such a straightedge will lie flat on the drawing, its weight makes it less easily displaced than are those of wood, and it will not warp. For all-round use, the 42-in. length is satisfactory.

CHAPTER V

ERRORS

62. General.—In an earlier article reference was made to the necessity of the surveyor's appreciating the errors involved in measurements. Every observed or measured quantity contains errors of unknown magnitude due to a variety of causes, and hence a measurement is never exact. One of the important functions of the surveyor is to secure measurements which are correct within certain limits of error prescribed by the nature and purpose of the survey. This requires that he know the sources of errors, understand the effect of the various errors upon the observed quantities, and be familiar with the procedure necessary to maintain a required precision. Numerous instances could be cited where surveyors of considerable experience have displayed an ignorance of this phase of their work which was both ludicrous and lamentable.

63. Kinds of Errors.—The *resultant error* in a given quantity is the difference between the measurement and its true value. If the measurement is too large the error is said to be positive; if too small the error is said to be negative. The resultant error in a measurement is made up of individual errors from a variety of sources, some of the individual errors tending perhaps to make the measurement too large, and others to make it too small. For a single quantity which has been determined by observation, neither the resultant error nor any of its individual parts can ever be determined exactly, but can be fixed within certain probable limits.

A *mistake* is an unintentional error of conduct arising from faulty judgment or from confusion in the mind of the observer. It is quite distinct from the mathematical or physical meaning of error. Throughout this text this distinction will be observed. Mistakes have no place in a discussion of the theory of errors. They are detected and eliminated by checking all work.

A *systematic error* is one that, so long as conditions remain unchanged, always has the same magnitude and the same algebraic sign (which may be either positive or negative). When conditions do not change, it is termed a *constant* systematic error. When conditions change during the series of measurements, resulting in a

corresponding change in the magnitude of the error, it is termed a *variable* systematic error. A systematic error always follows some definite mathematical or physical law.

An *accidental error* is an error due to a combination of causes beyond the ability of the observer to control and for which it is impossible to make correction; for each observation the magnitude and algebraic sign of the accidental error are matters of chance and hence cannot be calculated as can the magnitude and algebraic sign of a systematic error. However, accidental errors taken collectively obey the law of probability (see Art. 67). Since each accidental error is as likely to be positive as negative, a certain compensative effect exists, and accidental errors are sometimes incorrectly called "compensating" errors. As an example of the occurrence of an accidental error, in chaining it is impossible to set the chaining pin exactly at the proper graduation on the tape. Accidental errors remain after mistakes have been eliminated by checking and systematic errors have been eliminated by correction.

64. Sources of Error.—Errors arise from three sources:

1. From imperfections in the instruments or devices with which measurements are taken. For example, a tape may be too long, or a level may be out of adjustment. Such errors are termed *instrumental errors.*

2. From the limitation of the human senses of sight and touch. For example, an error may be made in reading the angle on the graduated circle of a transit or in estimating the tension in a steel tape. Such errors are called *personal errors.*

3. From variations in the phenomena of nature, such as temperature, humidity, wind, gravity, refraction, and magnetic declination. For example, the length of tape will become greater or smaller according as the temperature increases or decreases, and readings of the magnetic needle are affected by variations in the magnetic declination. Such errors are called *natural errors.*

65. Systematic and Accidental Errors Compared.—The total systematic error in any given number of measurements is the algebraic sum of the individual errors of each of the individual measurements. Thus if a distance is measured with a tape which is too short, the systematic error due to the tape's not being of the standard length would be directly proportional to the length of the line.

Example 1: The length of a line as measured with a 100-ft. tape at 60°F. is 1,000.00 ft. Later the tape is compared with the standard length and is found to be 100.021 ft. long. The error in the recorded length of the line is $-0.021 \times 10 = -0.21$ ft., and the true length of the line is 1,000.21 ft.

The example above illustrates the manner in which a *constant* systematic error increases with the number of observations. The example below illustrates the effect of a *variable* systematic error.

Example 2. A line measured with a 300-ft. tape is found to be 1,200.00 ft. long. Calculations based upon observations of temperature of the tape indicate that its probable length was 299.998 ft. for the first tape length, 300.001 ft. for the second tape length, 300.008 ft. for the third tape length, and 300.004 ft. for the fourth tape length. The total systematic error due to variation in temperature would therefore be the sum of the above errors, or, $+0.002 - 0.001 - 0.008 - 0.004 = -0.011$ ft., and the length of the line would be $1{,}200.00 + 0.01 = 1{,}200.01$ ft.

Quite often a systematic error from one source may be of opposite sign to that from another source so that the resultant systematic error is perhaps smaller than any of the errors from individual sources. Thus under a given tension and at a given temperature a tape might be of the standard length. Suppose that for the conditions under which the measurement of a line were made a variation from standard in tension in the tape produced an error of -0.022 ft. per tape length and a variation from standard in temperature produced an error of $+0.018$ ft. per tape length; the resultant unit error due to variations in temperature and tension would be -0.004 ft.

Systematic errors may be either instrumental, personal, or natural in character. Their relative importance as compared to accidental errors depends upon the nature of the observations, the care exercised by the observer, and the instruments and methods of procedure employed. Generally speaking, the rougher the methods used, the larger the systematic errors as compared with the accidental errors. For many observations the order of procedure is such that systematic errors are eliminated, or at least reduced to a negligible quantity. Thus in chaining, the error due to temperature change in a steel tape may be nearly eliminated by observing the temperature and making correction; and errors in leveling due to faulty adjustment of the level may be eliminated by balancing backsight and foresight distances.

Accidental errors, as the name signifies, are purely accidental in character, and there is no way of determining or eliminating them in the sense that we may determine or eliminate most systematic errors. Thus, while the effect of change of temperature upon the length of a tape can be approximately eliminated by calculations based upon physical measurements, there is no corresponding method of eliminating the accidental error due to marking the ends of the tape on the ground or due to reading the rod in leveling. While accidental

errors are as likely to be positive as negative, the error for one observation of a quantity is not likely to be the same as for the second observation.

According to the mathematical theory of probability, *accidental errors tend to increase in proportion to the square root of the number of opportunities for error.* Thus if the accidental error in measuring one tape length were ± 0.02 ft. the chances would be even that the total accidental error due to measuring 100 tape lengths would not exceed $\pm \sqrt{100} \times 0.02 = \pm 0.20$ ft. A systematic error of the same magnitude would produce a total error of $100 \times 0.02 = 2.0$ ft. It is thus seen that for any connected series of observations of independent but related quantities the accidental errors, as compared with systematic errors of the same probable magnitude, are of relatively small importance. Though accidental errors cannot be eliminated, they may be reduced to a small quantity through the use of proper instruments and methods. By taking a series of like observations of a single quantity, an estimate of the accidental error may be made, as will later be shown; but its true magnitude can never be determined.

66. Discrepancies.—If a given quantity is measured twice, the difference between the two measurements is termed the *discrepancy.* Frequently quantities measured by the operations of surveying are "checked" by a second measurement. If the discrepancy between two such measurements is small it is a good indication that no mistake has been made and also that the accidental errors are small. But attention is drawn to the fact that the magnitude of the discrepancy is in no way indicative of the magnitude of the systematic errors. For example, two measurements of a line a mile long made with a steel tape might show a discrepancy of 0.3 ft., but the systematic errors due to such causes as temperature, sag of tape, and slope of tape might perhaps be 3 ft.

67. Theory of Probability.—It has been stated that by employing proper methods, systematic errors may be largely eliminated. While this is true, it is also true that for certain kinds of surveys, particularly those of low precision, it is unnecessary and impracticable even approximately to eliminate such errors. For the surveys of higher precision special effort is made to eliminate systematic errors, and the accuracy of a measured quantity is governed by the accidental error which it contains. To form a judgment of the probable value or the probable precision of a quantity, *from which systematic errors have been eliminated,* it is necessary to rely upon the theory of probability, which deals with accidental errors of a series of like or related measurements. It is assumed that:

1. Small errors are more frequent than large ones.
2. Errors are as likely to be positive as negative.
3. Very large errors do not occur.
4. The mean of an infinite number of like observations of a quantity is its true value.

A thorough understanding of the law of probability may only be obtained by the study of a text on *Least Squares*, but a few of the rules for simpler cases of the adjustment of observations and determination of probable values and probable errors will here be given. It should be clearly understood that the theory of probability applies to accidental errors only; the theory is useful in indicating the accuracy of results only in so far as they are affected by accidental errors and does not in any way determine the magnitude of systematic errors which may be present.

68. Probable Values.—In practice it is neither possible entirely to eliminate systematic errors nor to take an infinite number of observations, hence the value of a quantity is never exactly known. Assuming that systematic errors are so far eliminated as to be a negligible factor, the *most probable value* is a mathematical term used to designate that adjusted value of a quantity which, according to the principles of least squares, has more chances of being correct than has any other.

For a series of measurements of the same quantity made under identical conditions, the most probable value is the mean of the measurements.

Example 1: After all systematic errors have been eliminated the several measured lengths of a line are 1,012.36, 1,012.35, 1.012.38, 1,012.32, 1,012.33 and 1,012.30 ft. The most probable value is the mean of the measurements, or 1,012.34 ft.

For related measurements (all taken under like conditions) the sum of which should equal a mathematically exact quantity, the most probable values are the observed values corrected by part of the error between the sum of the observed values and the true quantity. This correction is in proportion to the *number* of related measurements and not to the magnitude of the individual measurements.

Example 2: The angles about a point have the following observed values: 130°15′20″, 142°37′30″, and 87°07′40″. The sum of the measurements is 360° 00′30″; therefore the total error is 30″. Since there are three angles, the error is assumed to be 10″ for each measurement. The most probable values are:

$$
\begin{array}{rcl}
130°15'20'' - 10'' &=& 130°15'10'' \\
142°37'30'' - 10'' &=& 142°37'20'' \\
\underline{87°\ 07'40''} - \underline{10''} &=& \underline{87°07'30''} \\
360°00'30'' - 30'' &=& 360°00'00''
\end{array}
$$

In the example the discrepancy is divided equally among the several measured angles regardless of the relative sizes of the angles.

For related measurements (taken under identical conditions), the sum of which should equal a single measurement (also taken under like conditions), the most probable values of the several related quantities and also of the measured quantity representing their sum are obtained by dividing the discrepancy equally between all the measurements. If the correction is added to each of the related measurements, it is subtracted from the measurement representing their sum; and *vice versa.*

Example 3: Measurements of three angles about a point O are $AOB = 12°31'50''$, $BOC = 37°29'20''$, and $COD = 47°36'30''$. The measurement of the single angle AOD is $97°37'00''$. The discrepancy between the sum of the three measured angles and the measurement of the angle representing their sum is $40''$. Since the size of the errors is independent of the size of the angle, the discrepancy is divided into equal parts: $40/4 = 10''$. This correction is to be subtracted from measurements of each of the angles AOB, BOC, and COD and is to be added to the measurement of the angle AOD. The most probable values are:

$$\begin{aligned} AOB &= 12°31'50'' - 10'' = 12°31'40'' \\ BOC &= 37°29'20'' - 10'' = 37°29'10'' \\ COD &= 47°36'30'' - 10'' = \underline{47°36'20''} \\ & \qquad\qquad\qquad\qquad\quad\; 97°37'10'' \\ AOD &= 97°37'00'' + 10'' = 97°37'10'' \text{ check} \end{aligned}$$

It is desired to emphasize the fact that the methods illustrated by the three preceding examples apply only to measurements each of which in a given example is made under the same conditions as are all the others, with the same instrument, observer, weather conditions, etc. In example 2, the probabilities are that the error of measuring any one of the angles is the same as that of any other. It is true that the chances of the error in one measurement being the same as in the others is very small, but it is more probable that the errors will each be of the same magnitude than that they will be of any other assignable value.

In adjusting several measurements the sum of which should equal a single measurement it should be noted that there is a distinction between observations of the character of those cited in example 3 and those for which several operations are involved in the measurement of a single quantity. For example, if distances between adjacent points along a line of considerable length were measured with a tape and then if the full length of the line were measured in the same manner, it is evident that the corrections should be made to depend upon some function of the number of applications of the tape, being

greater for the full distance than for any of its parts, and being greater for long segments of the line than for short ones.

Suppose that each of the values of the angles of example 3 represented the mean of several measurements, but that for angle *AOB* more measurements were taken than for the others. The probabilities are that the error in *AOB* would be less than the errors in the other given values. In adjusting observations of this and similar character one needs to determine what are termed the "probable errors" of each series of observations and then to correct each value in proportion to the square of its probable error. In the following pages the operation of determining probable error will be explained and a few of the simpler cases of its use in the adjustment of observations will be given.

69. Probable Error of Single Measured Quantity and Mean.—If all systematic errors are assumed to have been eliminated from a given measurement, there is left an accidental error which can neither be eliminated nor be exactly determined. If a series of like or related observations of a single quantity is made, a number of values of the quantity are obtained. The variations between these values furnish data from which the *probable error* can be determined. The probable error of a measurement is a mathematical quantity giving an indication of precision and does not signify the true error nor the error most likely to occur. It is a valid measure of the precision of observations only after systematic errors have been reduced to a negligible quantity.

Probable error is a plus or minus quantity within which limits the actual error is as likely as not to fall. In other words if the probable error of a measurement is both added to and subtracted from the observed value, the chances are even that the true value of the measured quantity lies inside (or outside) the limits thus set. Thus if 6.23 represents the mean of several measurements and 0.11 represents the probable error of the mean value, the chances are even that the true value lies between the limits $6.23-0.11=6.12$ and $6.23+0.11=6.34$. In writing the probable error, its numerical value is usually preceded by the sign $\pm$ and is placed after the number to which it refers. For the example just cited the quantity would be written 6.23 ± 0.11.

As has already been stated, when a series of observations of a single quantity is made under uniform conditions and in such manner as to eliminate systematic errors, the mean of the series is the most probable value of the quantity sought. For the purpose of determining the probable error this mean value is mathematically regarded as being the true value, and the difference between each of the individ-

ual measurements and the mean value is determined. These differences are termed the *residuals*. The theory of least squares demonstrates that the probable error is a function of the square root of the sum of the squares of the residuals.

The probable error of a single observation is calculated by the expression

$$E = 0.6745\sqrt{\frac{\Sigma v^2}{n-1}} \tag{1}$$

in which Σv^2 is the sum of the squares of the residuals, and n is the number of observations.

The probable error of the mean of a number of observations is determined by the equation

$$E_m = 0.6745\sqrt{\frac{\Sigma v^2}{n(n-1)}} = \frac{E_s}{\sqrt{n}} \tag{2}$$

The probable error of a single observation may be determined approximately by the expression

$$E = \frac{0.8453\Sigma v}{\sqrt{n(n-1)}} \text{ (approximate)} \tag{3}$$

in which Σv is the sum of the residuals without regard to signs.

The probable error of a single observation may be calculated with about the same degree of approximation by the formula

$$E = 0.8453\ \bar{v} \text{ (approximate)} \tag{4}$$

in which $\bar{v}$ is the mean value of the residuals without regard to signs.

Equations (3) and (4) are more convenient to apply than is Eq. (1). Whether or not they may be properly used will depend upon the number of observations, the distribution of the residuals, and the desired number of places in the probable error.

While no attempt to deduce the above expressions will here be made, it is well to state that they are based upon the hypothesis that a large number of measurements of a single quantity has been taken. The results of experiment indicate, however, that they may be applied to a limited number of observations with good results. It seems doubtful if they can be consistently applied to a series of observations containing less than ten measurements.

The following example illustrates the methods of applying the preceding equations and also indicates the degree of approximation arising through the use of Eqs. (3) and (4).

Example: Following is a series of 10 rod readings which were taken with a 20-in. wye level. The day was calm and cloudy. The instrument

was set up, and the target rod was held on a point 600 ft. away. Before each reading the target was moved and the instrument was leveled.

Rod reading	v, feet	v^2
2.467	0.002	0.000004
2.460	0.005	0.000025
2.469	0.004	0.000016
2.465	0.000	0.000000
2.471	0.006	0.000036
2.461	0.004	0.000016
2.463	0.002	0.000004
2.466	0.001	0.000001
2.460	0.005	0.000025
2.468	0.003	0.000009
Mean value	Σv	Σv^2
2.465	0.032	0.000136

Using the sums in the above tabulation, the probable error of a single observation is

By Eq. (1) $E_s = \pm 0.6745\sqrt{\frac{0.000136}{9}} = \pm 0.00262$ ft.

By Eq. (3) $E_s = \pm\frac{0.8453 \times 0.032}{\sqrt{10 \times 9}} \doteq \pm 0.00284$ ft.

By Eq. (4) $E_s = \pm 0.8453 \times 0.0032 = \pm 0.00271$ ft.

Using the value of E_s determined by Eq. (1)

$$E_m = \pm\frac{0.00262}{\sqrt{10}} = \pm 0.00083 \text{ ft.}$$

The preceding example is illuminating not only as showing the steps made in calculating probable errors but also as indicating in a measure the degree of approximation introduced by using the approximate expressions for the probable error of a single observation (Eqs. (3) and (4)).

In work involving many calculations of probable errors a decided saving in labor will be accomplished if one or another of the approximate formulas is used. The results of the preceding example have purposely been extended to more places than are consistent for the given data (see Art. 30), in order to make the above comparison. Except for the observations taken on surveys of high precision, the approximate formulas are always sufficiently exact.

70. Weighted Observations.—It has been repeatedly stated that all observations were assumed to have been taken under the same conditions and consequently were equally reliable. Quite

frequently in surveying, however, it is required to combine the results of measurements which were not made under similar conditions and which therefore, have different degrees of reliability.

For example, suppose that the angles about a point have been measured under identical conditions with results as follows: $A = 121°46'00''$ (one measurement); $B = 179°14'27''$ (average of four measurements); and $C = 58°58'53''$ (average of nine measurements). The sum of these values is 359°59'20'', which shows a total error of 40''. Obviously, this error should not be equally divided among the three angles, and hence we must determine what proportion of the total error shall be assigned to each of the measurements.

If it is assumed that each single reading was made with equal care, then it is a logical assumption that angle B, the average result of four readings, has four times the reliability of angle A; and that angle C, the average of nine readings, has nine times the reliability of angle A. The numbers four and nine are called *weights* since they represent degrees of reliability. The weight of A is, of course, unity. These numbers are not absolute but merely relative. Thus the numbers 2, 8, and 18 would represent the weights as well as the numbers 1, 4, and 9.

The mean or average value of measurements which have different weights is called the *weighted mean.*

For example, suppose that an angle has been measured at different times by different observers, with results as follows: 47°37'30'' (one measurement); 47°37'22'' (average of four measurements); and 47°37'40'' (average of nine measurements). To obtain the average value of this angle as determined by these three observers (assuming that all measurements were made with equal care) we must multiply each value by its weight, add the products, and divide by the sum of the weights, thus

$$\begin{array}{rcr}
47°37'30'' \times 1 & = & 47°37'30'' \\
22'' \times 4 & = & 88'' \\
40'' \times 9 & = & \underline{360''} \\
\underline{14}| & & \underline{478} \\
 & & 47°37'34''
\end{array}$$

The resultant value 47°37'34'' is the weighted mean.

Often weights will be assigned to observations, not according to the number of observations, but arbitrarily according to the best judgment of the observer.

Thus, by one line of levels the elevation of a point of reference might be determined as 537.64 ft. and by another line of levels run over the same route as 537.58 ft. But the levelman might assign a weight of 3 to the first elevation and a weight of 1 to the second, because the first line was run on a calm, temperate day and the second line was run on a very windy, cold day. In other words he might judge the results secured

on the first day as being equal in value to the average of 3 lines of levels run on the second day. Under these assumptions the weighted and accepted elevation would be $\frac{537.64 \times 3 + 537.58 \times 1}{4} = 537.62$ ft.

70a. Adjustment of Weighted Observations.—Since field measurements are frequently made under dissimilar conditions as to observers, instruments, and weather, it becomes necessary in the adjustment of related measurements to take into account the weights (as nearly as they may be determined) which apply to each of the separate measurements.

In the preceding article it was assumed that weights vary directly with the number of observations. It has been stated that probable errors vary inversely with the square root of the number of observations. It follows that, for observations made with equal care, weights are inversely proportional to the squares of the corresponding probable errors or if W_1 and W_2 are the weights to be assigned given measurements and E_1 and E_2 the corresponding probable errors, then $W_1 : W_2 :: \frac{1}{E_1^2} : \frac{1}{E_2^2}$. Accordingly, if the probable errors of given measured quantities are computed, the relative weights of these quantities can be determined, which weights may be used in any adjustment of values that may be required.

Applications of the principles stated above are shown by the following examples:

Example 1: Three angles about a point are each measured by a series of observations. The mean values with their probable errors are tabulated below. Their sum should equal 360°. It is desired to determine the most probable values of the angles.

a	121°46′37″	± 02″
b	179°14′35″	± 04″
c	58°58′34″	± 06″
Sum	359°59′46″	

The total error in the sum of the above angles is seen to be 14″. This is to be distributed among the three angles. Let W_a, W_b, and W_c be the weights of the angles a, b, and c respectively.

$$\text{Then } \frac{W_a}{\left(\frac{1}{2}\right)^2} = \frac{W_b}{\left(\frac{1}{4}\right)^2} = \frac{W_c}{\left(\frac{1}{6}\right)^2}$$

Let C_a, C_b, and C_c be respectively the corrections to be applied to angles a, b, and c. Since the corrections are inversely proportional to the weights

$$\frac{C_a}{(2)^2} = \frac{C_b}{(4)^2} = \frac{C_c}{(6)^2} \text{ or}$$
$$C_a = \frac{C_b}{4} = \frac{C_c}{9}$$

Also $$C_a + C_b + C_c = 14$$

Therefore $$C_a = \frac{1}{14} \times 14 = 01''$$

$$C_b = 4/14 \times 14 = 04''$$
$$C_c = 9/14 \times 14 = 09''$$

The most probable values are therefore

a	121°46′37″	+ 01″	=	121°46′38″
b	179°14′35″	+ 04″	=	179°14′39″
c	58°58′34″	+ 09″	=	58°58′43″
	359°59′46″	+ 14″	=	360°00′00″ *check*

A somewhat similar case arises when a quantity is determined by several independent series of measurements, as illustrated by the example below.

Example 2: *Solution a.*—Lines of levels to establish the elevation of a point are run over four different routes. The observed elevations of the point with probable errors are given below.

Line	Observed elevation, feet
a	721.05 ± 0.12
b	721.37 ± 0.24
c	720.62 ± 0.36
d	721.67 ± 0.48

Remembering that weights vary inversely as the square of the probable error

$$\frac{W_a}{(1/12)^2} = \frac{W_b}{(1/24)^2} = \frac{W_c}{(1/36)^2} = \frac{W_d}{(1/48)^2} \text{ or}$$
$$W_a = 4W_b = 9W_c = 16W_d$$

Let $W_a = 1$; then $W_b = 1/4$; $W_c = 1/9$; and $W_d = 1/16$.
Multiplying the weights by each of the observed elevations, the weighted means are obtained as shown in the tabulation at the top of page 77. The most probable value of the elevation is the sum of the weighted observations divided by the sum of the weights, or

$$\frac{1{,}026.56 \times 144}{205} = 721.10 \text{ ft.}$$

Line	Observed elevation	Weight	Weighted observation
a	721.05	1	721.05
b	721.37	$\frac{1}{4}$	180.34
c	720.62	$\frac{1}{9}$	80.07
d	721.67	$\frac{1}{16}$	45.10
Total................	......	$\frac{205}{144}$	1,026.56

Solution b.—For problems in which the quantities are large, as in the solution above, it reduces the labor considerably if *differences* are weighted, rather than the observed values themselves. To illustrate, let us work example 2 by weighting the differences between 721.00 and the observed elevations. Also instead of recording the weights as fractions, let us assign them in whole numbers.

Line	Observed elevation	Less 721.00	Weight	Weighted difference
a	721.05	+0.05	144	+ 7.2
b	721.37	+0.37	36	+13.4
c	720.62	−0.38	16	− 6.1
d	721.67	+0.67	9	+ 6.0
Total...............	......	......	205	+20.5

The most probable difference between 721.00 and the most probable value of the elevation is $\frac{+20.5}{205} = +0.10$ ft. Hence the most probable value of the elevation is $721.00 + 0.10 = 721.10$ ft.

In the above example, solution a requires multiplications to five places, while solution b requires multiplications only to three places. If the slide rule is used, the problem may be solved by considering differences in about one-third the time that it takes for the solution in which the observed values are weighted directly.

This example also illustrates another principle which is worthy of notice. Let us suppose that the various probable errors ± 0.12, ± 0.24, ± 0.36, and ± 0.48, resulted not because of dissimilar conditions attending the work, but only because of the lengths of the routes over which each of the four lines was run. If we suppose that line a was 1 mile long, then by the law that accidental errors vary as the square root of the number of opportunities (*i.e.*, the number of miles, in this case) we find

that the lengths of the lines which would yield these probable errors are given by the expression:

$$0.12:0.24:0.36:0.48::1:2:3:4::\sqrt{L_a}:\sqrt{L_b}:\sqrt{L_c}:\sqrt{L_d}$$

in which L_a is the length of line a in miles, L_b is the length of line b, etc. or squaring, $L_a:L_b:L_c:L_d::1:4:9:16$

Hence, the lengths of the lines would be: $a = 1$ mile, $b = 4$ miles, $c = 9$ miles, and $d = 16$ miles.

But these values are seen to correspond to the inverse ratios of the weights and hence, directly to the corrections found above. Therefore, we have the important principle in the adjustment of independent observations where the accidental errors vary with the square root of the distance, that the corrections to be applied are directly proportional to the distance.

71. Interrelation of Errors.—If E_1 and E_2 represent respectively the probable errors of lengths L_1 and L_2 then the probable error of the area representing the product of these two lengths is

$$E_a = \sqrt{L_1^2E_2^2 + L_2^2E_1^2} \tag{5}$$

The probable error of the product of a constant or known quantity C and a measured quantity Q for which the probable error is E, is

$$E_p = CE \tag{6}$$

The probable error of the sum of independent measurements Q_1, Q_2, . . ., Q_n for which the probable errors are E_1, E_2 . . ., E_n, is

$$E_s = \sqrt{E_1^2 + E_2^2 \cdots + E_n^2} \tag{7}$$

The probable error of the difference between two independent measurements Q_1 and Q_2 for which the probable errors are E_1 and E_2 is

$$E_d = \sqrt{E_1^2 + E_2^2} \tag{8}$$

SECTION II
ELEMENTS OF SURVEYING

CHAPTER VI

MEASUREMENT OF DISTANCE

GENERAL METHODS

72. Distance.—When the distance between two points is referred to in surveying, it is always understood to mean the *horizontal* distance, regardless of the relative elevation of the points. In geodetic surveying horizontal distance is reduced to the equivalent at sea level, but in plane surveying such reductions are unnecessary. Though slope distances are frequently measured, they are subsequently reduced to their equivalent on the horizontal projection, and in the latter form are used in plotting maps, calculating land areas, etc.

Distances may be measured by a variety of methods each of which will be found useful under certain conditions, depending upon the degree of precision required and upon other variables. On rough reconnaissance, for example, a precision of $1/100$ or less may be sufficient for the purpose of the survey; on the other hand, certain base lines established by the U. S. Coast and Geodetic Survey have been measured with a probable error of about $\frac{1}{2,000,000}$.

73. Pacing.—The method of pacing furnishes a rapid means of approximately checking more accurate measurements of distance. It is extensively employed in small-scale mapping not only for locating details but also in traversing with the plane table. It is also used on exploratory or reconnaissance surveys, the paces of saddle animals frequently forming the basis of measurement.

The precision of the man-pace depends largely upon the experience of the individual and upon the character of the terrain over which he is passing. In numerous instances, pacing over rough country has furnished a precision of $1/200$. Under average conditions a person of experience will have little difficulty in pacing with a precision of $1/100$.

Many land surveyors estimate distances by the 3-ft. pace, which is somewhat longer than the natural pace of the average person. Mili-

tary sketchers and many topographers of government surveys maintain a pace that is the natural length or a little shorter. The authors favor the $2\frac{1}{2}$-ft. pace; since it is a little less than the natural step, allowance can be made for uneven ground by lengthening the pace without tiring; and a convenient relation exists between the pace and the foot, *i.e.*, 40 paces = 100 ft. Each two paces or double step is sometimes called a *stride*. Thus for the above length of pace the stride would be 5 ft., or there would be roughly 1,000 strides per mile.

Paces or strides are usually counted by pressing a tally register, or by a pedometer or passomoter which registers mechanically. The passometer is a device about the size of a watch which is attached either to the body or to one leg and which registers the number of paces or strides. The pedometer is a similar device except that it registers the distance usually in miles and fractions thereof.

The student should standardize his pace by walking over known distances both on level ground and on uneven and sloping ground. For further suggestions see field problem 1, Art. 95.

74. Stadia.—This method offers a rapid means of determining distances. Two additional horizontal hairs are mounted on the cross-hair ring in the telescope of the transit, level, or plane-table alidade. The distance from the instrument to a given point is indicated by the intercept between the stadia hairs as shown on a rod held vertically at the point. The accuracy of the stadia depends upon the instrument, the observer, the atmospheric conditions, and the length of sights. Under average conditions it will yield a precision between $\frac{1}{300}$ and $\frac{1}{1{,}000}$. It is particularly useful in topographic surveying. The stadia method is described in detail in Chap. XIV.

75. Direct Measurement.—The most accurate and most common method of determining distance is by direct measurement. Formerly on surveys of ordinary precision it was the practice to measure the length of lines with the engineer's chain or the Gunter's chain; for measurements of the highest precision special bars were used. At the present time practically all direct linear measurements, as they occur on surveys, are made with tapes.

The *engineer's chain* is 100 ft. long and is composed of 100 links each 1 ft. long. At every 10 links brass tags are fastened, notches on the tags indicating the number of 10-link segments between the tag and the end of the tape. Distances measured with the engineer's chain are recorded in feet and decimals.

The *surveyor's* or *Gunter's chain* is 66 ft. long and is divided into 100 links each 7.92 in. long. It was formerly much used in land sur-

veying on account of the convenient relation between its length and the units of land measure.

1 (Gunter's) chain	= 100 links = 4 rods
80 (Gunter's) chains	= 1 mile
10 square (Gunter's) chains	= 1 acre

Distances were recorded in chains and links.

Measuring with chains was called "chaining." The term has survived and is now generally used also to refer to the operation of measuring lines with tapes.

The precision of distance measured with tapes depends upon the degree of refinement with which meaurements are taken. On the one hand, rough chaining through broken country may be less accurate than the stadia. On the other hand, when extreme care is taken to eliminate all possible errors, measurements have been taken with a probable error of less than $\frac{1}{1{,}000{,}000}$.

76. Other Methods.—The revolutions of the wheel of a vehicle furnish a means of measuring distance along highways. The mileage recorder attached to the ordinary automobile speedometer registers distance to 0.1 mile. By driving over a course of known length, the mileage recorder may be standardized so that long distances can be determined with an accuracy considerably greater than by pacing.

The odometer, a device which registers directly the number of revolutions of the wheel, may be readily attached to any vehicle. By measuring the circumference of the wheel with a tape, the relation between distance and revolutions is fixed. On smooth roads the precision may be as great as that obtained with the stadia. The distance indicated by either the mileage recorder or the odometer is of course, somewhat greater than the true horizontal distance, but under the conditions for which they are used, neither requires correction except in hilly country. The odometer is extensively used on plane-table traverses for small-scale maps.

Distances are sometimes roughly estimated by time interval of travel, and this method is quite satisfactory for very rough reconnaissance. The average time per mile for person at walk, saddle animal at walk, or saddle animal at gallop is usually established for several characters of terrain.

By graphical or algebraic methods, unknown distances may be determined through their relation to one or more known distances. Problems of this nature will be considered in later articles.

77. Choice of Methods.—Practically all important lines, including land boundaries, main traverses of horizontal control for extensive

maps, and the lines for the location and construction of the works of man, are measured with tapes because no other practicable method furnishes the required precision. It is certain, however, that much time has been wasted in chaining distances that could have been measured with all necessary precision by some less laborious method. Each of the methods mentioned in the preceding articles has a field of usefulness, and may properly be employed when it will furnish measurements of the required accuracy. Within the past few years the advantages of the stadia method have come to be more fully appreciated, and now practically all linear measurements for many surveys for maps are obtained through its use. On the surveys for a single enterprise the authors have found occasion to employ almost all of the above methods to good advantage.

CHAINING

78. General.—Chaining, as the term is customarily used, refers to the operation of measuring with the chain or the tape for the purpose of obtaining the horizontal distance between points on or near the surface of the earth. The persons who manipulate the tape are generally called "chainmen." To some extent the term "taping" is succeeding the term "chaining" and not infrequently the operators are called "tapemen."

79. Tapes.—Tapes are made in a variety of materials, lengths, and weights. Those more commonly used by the surveyor are the heavy steel tape, sometimes called the surveyor's tape or the chain tape, and the metallic tape.

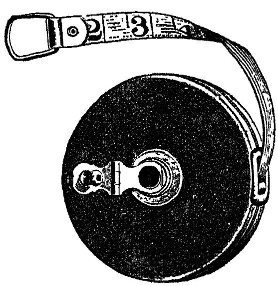


FIG. 79.—Metallic tape.

The ribbon of the metallic tape (Fig. 79) is of waterproofed fabric into which are woven small brass or bronze wires to prevent its stretching. It is usually 50 ft. in length and is graduated to feet, tenths, and half-tenths. It is used principally in earthwork cross-sectioning and in similar work where a light, flexible tape is desirable and where small errors in length are of no consequence.

A tape of phosphor bronze is rustproof and is particularly useful when working in the vicinity of salt water.

For very precise measurements the invar tape is coming into general use. Invar metal is a composition of nickel and steel with a very low coefficient of thermal expansion, sometimes as small as $\frac{1}{30}$ that of steel. It is a soft metal, and the tape must be handled very

carefully to avoid bends and kinks. This property and also its high cost make it impracticable for ordinary use.

80. Care of Steel Tapes.—The steel tape is generally employed for the direct linear measurement of all important survey lines. In the United States and Canada the length most commonly used is 100 ft., but tapes may be also obtained in lengths of 25, 50, 66, 75, 200, 300, and 500 ft. The tapes used in surveying are usually graduated in feet and decimals. An exception is the 66-ft. tape which, corresponding to the old Gunter's chain, is graduated into 100 links each 7.92 in. long.

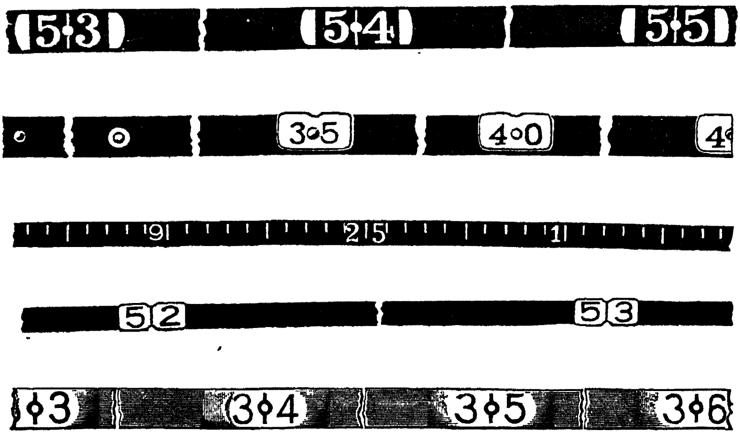


Fig. 80.—Steel tapes.

Small, lightweight box tapes usually have graduations etched every 0.01 ft. throughout their entire length. The heavy chain tapes usually have graduations with numbers every foot, but with only the end feet graduated to tenths or hundredths of feet. The graduations may be etched or may be stamped on babbitt metal or on brass sleeves. Chain tapes are also obtainable with etched graduations every 0.01 ft. throughout their length.

The chain tape may be wound on a reel, but except in city and mining surveying the reel is seldom used, the tape being done up into a figure 8 and thrown into circular form, with diameter about 10 in. (see field problem 2, Art. 95). Rawhide thongs serving as handles are ordinarily fastened to the rings at each end. The thongs are preferable to the wire handles which are sometimes used, especially when the tape must be dragged through grass or brush.

Some tapes have an extra graduated foot on one or both ends. Others have shoulders at the zero and last graduation to assist in locating these points. Usually the ribbon extends about 6 in. beyond

the graduated portion of the tape, but for some tapes the ends of the rings mark the zero and last graduation. The latter type is not well adapted to precise measurements. The tape with shoulders is objectionable when chaining through brush. The common widths of tape are 3/16, 1/4, and 5/16 in. Figure 80 shows some of the common graduations.

Steel tapes rust readily and for this reason should be wiped dry. The heavy tapes will stand considerable abuse, but any tape will break when kinked and subjected to a strong pull.

The steel tape, being elastic, stretches when a pull is applied. It also expands or contracts as the temperature changes. Tapes when received from the manufacturer are usually quite close to the standard length when subjected to a given pull and a given temperature. For the 100-ft. tape some manufacturers attempt to furnish the standard length at 68°F. under a pull of 10 lb., the tape being horizontal and supported throughout its entire length, but among manufacturers there is no uniformity of practice in this respect. It is well to have a standard of length available to which the tape can be referred occasionally. Many cities have such standards.

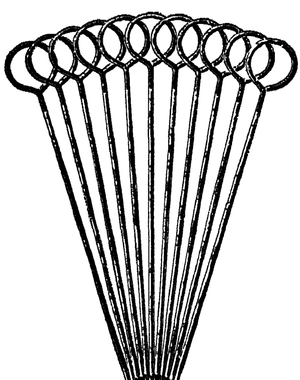


Fig. 81.—Chaining pins.

For a small fee the National Bureau of Standards, Washington, D. C., will standardize a tape for any specified pull and will issue a certificate stating its length under the conditions of the standardization test.

81. Chaining Pins.—Chaining pins, which are also called surveyor's arrows, are commonly employed to mark the ends of the tape during the process of chaining between two points more than a tape length apart. They are of steel and are usually 10 to 14 in. long. A set consists of eleven pins (see Fig. 81).

82. Range Poles.—Flags, flag poles, or lining-rods, are used as signals to indicate the position of points or the direction of lines. They are constructed of either steel or wood, are of octagonal or circular cross-section, and are pointed at the lower end. The common length is 8 ft. The wooden range poles are shod with a steel point. Usually they are painted with alternate bands of red and white, the length of band being 1 ft. (see Fig. 82).

83. Measurement with Tape; Smooth Level Ground.—The procedure followed in chaining distances with the tape depends

to some extent upon the required precision and the purpose of the survey. The following represents the usual practice when the measurements are of ordinary precision $\left(\text{say } \frac{1}{5{,}000}\right)$: The tape is supported throughout its length and the only requirement is that the distance between two fixed points (as the corners of a parcel of land) be determined. The equipment will be assumed to consist of one or more range poles, 11 chaining pins, and a 100-ft. heavy steel tape, with the intervals 0 to 1 ft. and 99 to 100 ft. graduated in tenths of feet and the remainder of the tape graduated in feet. One range pole is placed behind the distant point to indicate its location.

The rear chainman with one pin stations himself at the point of beginning. The head chainman, with the zero end of the tape and 10 pins, advances towards the distant point. When the head chainman has gone nearly 100 ft. the rear chainman calls "*chain*" or "*tape*," a signal for the head chainman to halt. The rear chainman holds the 100-ft. mark at the point of beginning and by hand signals or by speaking, lines in a chaining pin (held by the head chainman) with the range pole marking the distant point. During the lining-in process the rear chainman is in a kneeling position on the line and facing the distant point; the head chainman is in a kneeling position to one side and facing the line so that the rear chainman will have a clear view of the signal marking the distant point (Fig. 83). The head chainman with his right hand sets the pin vertically on line and a short distance to the rear of the zero mark. With his left hand he then pulls the tape taut and making sure that it is straight, brings it in contact with the pin. The rear chainman, when he observes that the 100-ft. mark is at the point of beginning, calls "*stick*" or "*all right*." The head chainman pulls the pin and sticks it at the zero mark of the tape and in a position sloping away from the line. As a check he again pulls the tape taut and notes that the zero point coincides with the pin at its intersection with the ground. He then calls "*stuck*," or "*all right*," the rear chainman releases the tape, the head chainman moves forward as before, and so the process is repeated. As the rear chainman leaves each intermediate point he pulls the pin. Thus there is always one pin in the ground, and the

FIG. 82. Range pole.

number of pins held by the rear chainman at any time indicates the number of hundreds of feet from the point of beginning to the pin in the ground.

At the end of each thousand feet the head chainman has placed his last pin in the ground. He signals for pins, the rear chainman comes forward with the ten pins which he has pulled, both chainmen count them to see that none is lost, and the tally is recorded by the head chainman. The head chainman takes the ten pins, and the procedure is repeated.

When the end of the course is reached, the head chainman halts and the rear chainman comes forward to the last pin set. The head chainman holds the zero mark at the terminal point. The

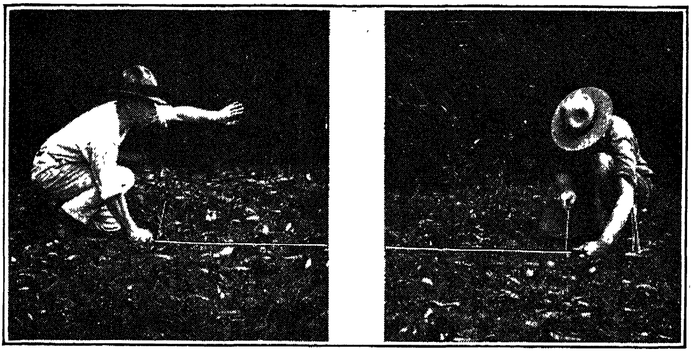


FIG. 83.—Chainmen taking a measurement with the tape and pins.

rear chainman pulls the tape taut, and observes the number of *feet* between the last pin and the end of the line. He then holds the next larger foot mark at the pin and the head chainman pulls the tape taut and reads the decimal by means of the finer graduations of the end foot. The decimal is counted from the 1-ft. mark. Thus the distance in feet between the last pin and the end of the line is one less than that indicated by the foot mark held by the rear chainman, plus the decimal read by the head chainman. For example, with the rear chainman holding at 87 ft. and the head chainman reading 0.68 ft., the distance from the last pin is $87 - 1 + 0.68 = 86.68$ ft.

When the transit is set up on the line to be measured, the transitman usually directs the head chainman in placing the pins on line. The rear chainman maintains a position that will give the transitman an unobstructed view, and the head chainman kneels or stands on line facing the transitman.

Many surveys require stakes to be set on line at short intervals, usually every 100 ft. Sometimes pins are used in chaining as already described, each stake being driven by the rear chainman after he has pulled the pin. On surveys of low precision the measurements are carried forward by using stakes instead of pins, the head chainman setting the stakes and the distance being measured between centers of stakes at their junction with the ground. On more accurate surveys, measurements are carried forward by setting a tack or small nail in the head of each stake. In setting the tack, the head chainman holds the pin (or range pole) on the head of the stake and places it on line as directed by the transitman. He pulls the tape taut, making sure that one edge is on line by bringing it in contact with the pin. He then uses the pin to mark the position of the tack at the zero point of the tape. When the tack has been driven, it is tested for line and distance.

84. Horizontal Measurements over Uneven or Sloping Ground.— The process of chaining over uneven or sloping ground is much the same as that just described for level ground. The measurements are carried forward by holding the tape in a horizontal position and the plumb line is used by either or, at times, by both chainmen for projecting from tape to pin or *vice versa*. If the course is downhill, the head chainman must plumb from the zero (or other) point on the tape to the ground; if uphill, the rear chainman must plumb from the pin to the 100-ft. (or other) point on the tape; if uneven, each chainman will find it necessary to use a plumb bob.

To secure anything like the same precision as in chaining over level ground, considerable skill is required. Some experience is necessary to determine when the tape is nearly horizontal. The tape is unsupported for most of its length, and the pull must be increased to eliminate the effect of sag.

When the slope is less than 5 or 6 ft. in a hundred, the head chainman advances a full tape length at a time, and pins are set by him and collected by the rear chainman as described in the preceding article. If the course is downhill, the head chainman estimates when the tape is horizontal. Holding the plumb line in position at the zero point of the tape and noting that the plumb bob clears the ground by a few inches, he pulls the tape taut and is directed to the line by the rear chainman. When the plumb bob comes to rest, he releases the string quickly, allowing the bob to fall and stick in the ground. He removes the bob and sets a pin in its place. As a check, the measurement is repeated. If the course is uphill, the head chainman holds the zero end of the tape firmly on the ground and on line. The rear chainman, with plumb line suspended from

the 100-ft. mark, signals the head chainman to give or take until the bob comes to rest over the pin. The head chainman sets a pin, and the measurement is repeated.

When the course is steeper and is downhill, the head chainman advances a full tape length and then returns to an intermediate point from which he can hold the tape horizontally. He suspends the plumb line at a foot mark and is lined in by the rear chainman. A pin is set at the indicated point by the head chainman. The rear chainman comes forward, gives the head chainman a pin, and holds the tape at the foot mark from which the plumb line was previously suspended. The head chainman proceeds to another point from which he can hold the tape horizontally. The rear chainman holds the foot mark to the pin. The head chainman sets a second pin.

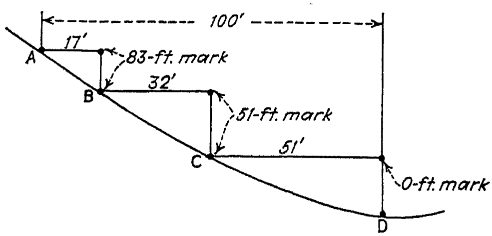


Fig. 84.—Chaining on steep slope.

And so the process is repeated until the head chainman reaches the zero mark on the tape. At each *intermediate* point of a tape length the rear chainman gives the head chainman a pin, but not at the point marking the full tape length.

To illustrate, Fig. 84 represents the profile of a line to be measured in the direction of *A* to *D*, and *A* is a pin marking the end of a 100-ft. interval from the point of beginning. The head chainman goes forward until the 100-ft. mark is at *A*, where the rear chainman is stationed. The head chainman then returns to *B* where he holds the tape horizontally and plumbs from the 83-ft. mark to set a pin at *B*. The rear chainman gives the head chainman a pin, and holds the 83-ft. mark at *B*. The head chainman plumbs from the 51-ft. mark and sets a pin at *C*. The rear chainman gives the head chainman a pin, and holds the 51-ft. mark at *C*. The head chainman plumbs from the zero mark to set a pin at *D* at the end of the full tape length. The rear chainman goes forward but retains the pin which he pulled at *C*.

In this manner the tape is always advanced a full length at a time and the number of pins held by the rear chainman at each 100-ft. point indicates the number of hundreds of feet from the last tally. The tape is usually estimated to be horizontal by eye. This quite

commonly results in the downhill end being too low, sometimes causing a large error in horizontal measurement. The safe procedure in rough country is to insist on the chainman using a hand level.

The operation of carrying the horizontal distance forward by plumbing down at an intermediate foot mark, as just described, is frequently termed "breaking chain" or "breaking tape."

85. Measurements on Slope.—Where the ground is fairly smooth, measurements on the slope may be made more accurately and quickly than horizontal measurements, but some means of determining the slope or difference in elevation between successive 100-ft. points is required. For surveys of ordinary precision the clinometer (for measuring slope) or the hand level (for measuring difference in elevation) may be used to good advantage. If all that is desired is the distance between the ends of the line, the usual procedure, so far as chaining is concerned, is exactly the same as on level ground; but a record is kept either of the difference of elevation or of the slope of each 100-ft. length. The horizontal distances are then calculated from the distances measured on the slope, and the horizontal length of the line is determined.

When stakes are to be placed every 100 ft., corrections to the slope distances may be applied as the chaining progresses, either by use of the slide rule or by mental calculation. Corrections are much more readily calculated than are the horizontal distances themselves, as will shortly be demonstrated. Unless the correction is greater than 1 ft. the 100-ft. tape with an extra foot on the zero end is particularly useful for measuring on slopes in that the head chainman can first determine the correction for slope and in one operation can then lay off the true slope distance to give a horizontal distance of 100 ft.

Measurements on the slope are preferred for the United States public-land surveys. The manual of the General Land Office states:

> The most approved method of measurement involves the use of steel ribbon tapes from 2 to 8 [Gunter's] chains in length; in its use in the public-land surveys the tape is properly alined and stretched, and the measurements are made on the slope at any convenient distance up to the length of the tape as determined by the topography. The vertical angles of the lesser slopes are determined by the use of clinometers in the hands of the chainmen, while the vertical angles of the particularly sharp slopes are determined with the transit * * * . It is not considered necessary to exhibit in the official field notes any but the true horizontal distances * * * .

86. Corrections for Slope.—For measurements of ordinary precision when the slope is not greater than about 20 in 100, the correc-

tion to slope distance to give horizontal distance may be calculated by the approximate formula developed below.

In Fig. 86*a* let s represent the slope distance, h the difference in elevation between two points, and d the horizontal distance. Then

$$h^2 = s^2 - d^2 = (s + d)(s - d)$$

When the slope is not large $s + d = 2s$ (approximate); making this substitution:

$$h^2 = 2s(s - d) \text{ (approximate)}$$

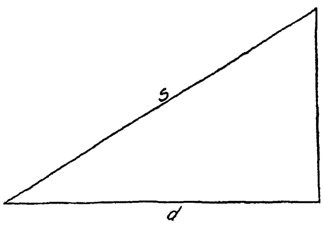


FIG. 86*a*.

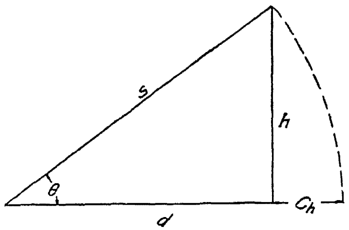


FIG. 86*b*.

and the correction is

$$C_h = s - d = \frac{h^2}{2s} \text{ (approximate)} \tag{1}$$

For the usual case, when $s = 100$ ft., the above expression may be solved mentally, and the opportunities for its use are so frequent as to make it worth remembering. The error introduced through its use is negligible for ordinary slopes. The degree of approximation is shown in the following table.

Difference in elevation per 100 ft. of slope distance, feet	Error due to using approximate formula in 100 ft. of slope distance, feet
5	0.0001
10	0.001
15	0.007
20	0.02
30	0.1
40	0.3
60	2.0

Where the angle of the slope is determined, as when using the clinometer, formula (1) may still be used quite readily if it is remembered that for angles less than 20° the difference in elevation per 100 ft. is about 1.75 ft. times the slope angle in degrees. Hence if $\theta°$ is the slope angle in degrees and S is the slope distance in 100-ft. stations,

$$C_h = \frac{(1.75S\theta°)^2}{200S} = 0.015S(\theta°)^2 \text{ (approximate)} \tag{2}$$

When vertical angles are measured with sufficient precision to so warrant, the horizontal distance may be calculated by exact trigonometric formula, the horizontal distance being most readily computed by determining the slope correction.

In Fig. 86*b*, θ is the angle of slope, s the slope distance, d the horizontal distance, and C_h the correction to be applied to the slope distance to produce the horizontal distance.

Then

$$C_h = s - d = s - s \cos \theta$$

But

$$1 - \cos \theta = \text{vers } \theta,$$

hence

$$C_h = s \text{ vers } \theta \text{ (exact)} \tag{3}$$

If a table of versed sines is not available the versed sines may be readily calculated from a table of natural cosines.

For most cases the correction can be calculated with sufficient precision by using the slide rule.

Having given the slope distance, to find its horizontal projection, the correction is *subtracted.*

For the case where it is desired to set points at a given horizontal distance (as 100 ft.) apart, the corresponding slope distance is *approximately* given by *adding* the correction, as given by Equation (3), to the required horizontal distance. For a slope of 10 in. 100 the degree of approximation is about 0.002 ft. per 100 ft. and for a slope of 20 in 100 it is about 0.04 ft. per 100 ft.

87. Errors in Chaining.—Errors in chaining may be attributed to the following causes:

1. *Tape Not Standard Length.*—This produces a systematic error which may be practically eliminated by standardizing the tape.

2. *Imperfect Alinement.*—The head chainman is likely to set the pin sometimes on one side and sometimes on the other side of the true line. This produces a variable systematic error since the horizontal angle which the tape makes with the line is not the same for one tape length as for the next. The error cannot be eliminated but can be reduced to a negligible quantity by care in lining. Generally it is the least important of the errors of chaining. The linear error when one end of the tape is off line a given amount can be calculated by Eq. (1), Art. 86. For a 100-ft. tape, the error amounts to 0.005 ft. when one end with respect to the other is off line 1 ft., and to only 0.001 ft. when the error in alinement is 0.5 ft. Many surveyors use unnecessary care in securing good alinement without paying much attention to other more important sources of error. Errors

in alinement tend to make the measured length greater than the true length.

3. *Tape Not Horizontal or Slope of Tape Not Correctly Determined.*—The effect is to produce an error similar to that due to imperfect alinement. With the eye it is difficult to estimate slopes or to tell when the tape is horizontal. Often slopes are very deceptive, even to experienced men. The authors have seen inexperienced chainmen keeping very careful alinement, yet chaining what they thought were horizontal distances on a slope of perhaps 10 per cent. The corresponding error is 0.5 ft. per 100 ft. or 26 ft. per mile. It is not uncommon to see chainmen of considerable experience chaining on slopes as steep as 4 ft. in 100 ft. without realizing that slope corrections should be made. The corresponding error per 100 ft. is 0.08 ft. In ordinary chaining this is one of the largest of contributing errors. It will not be eliminated by repeated measurements, but it may be reduced to a negligible amount by leveling the tape by means of the hand level or clinometer.

4. *Tape Not Straight.*—In chaining througn grass and brush or when the wind is blowing it is impossible to have all parts of the tape in perfect alinement with its ends. The error arising from this cause is systematic and variable, and is of the same sign as that from measuring with a tape that is too short. If the head chainman is careful to stretch the tape taut and to observe that it is straight by sighting over it, the error is of no consequence.

5. *Plumbing; Imperfections of Observing; Inability to Set Pins at Exact Distance.*—These are accidental errors, hence the probable error tends to vary as the square root of the number of tape lengths. The error due to plumbing is the only one that is of real importance. In ordinary chaining through rough country, where it is necessary to break chain frequently, the probable error per tape length may perhaps amount to ± 0.05 or ± 0.1 ft. for each tape length chained. Using the maximum of ± 0.1 ft. the probable error would be about ± 0.7 ft. per mile. The probable error of setting pins and of observing the tape graduations would perhaps be ± 0.01 ft. per tape or ± 0.07 ft. per mile. While these errors cannot be eliminated, their effect on the resultant error is usually not large. When the required precision is high, errors of plumbing may be eliminated by chaining on the slope.

6. *Variations in Temperature* (Art. 88).—If the tape is standardized at a given temperature and measurements are taken at a higher temperature, the tape is too long. For a change in temperature of 10°F., a 100-ft. steel tape will undergo a change in length of about 0.006 ft., introducing an error of about 0.3 ft. per mile. Under a

change of 50° the error per mile would be 1.5 ft. It is seen that, even for measurements of ordinary precision, the error due to thermal expansion becomes of consequence when the measurements are taken during extremely cold or extremely hot weather. A case is recalled where at 30°F. below zero, measurements were very carefully established along the track of a railroad and markers were placed at intervals permanently to establish the chainage, but no allowance was made for the extremely low temperature. Later when rechained for valuation purposes the error was found to be about 3 ft. per mile. Some tapes have a temperature scale at one end by means of which the correction for variation in temperature may be made without calculation.

7. *Variable Tension in the Tape* (Art. 89).—The tape, being of an elastic material, is elongated when tension is applied. If the pull is greater than that for which the tape is standard, the tape is too long; if the pull is less, the tape is too short. The error is systematic and is of a magnitude depending upon the methods employed and the individuals who are chaining. The error is negligible except in precise work. For the heavy chain tape a change in tension of 3 lb. changes the length of the tape about 0.001 ft.

8. *Sag in Tape* (Art. 90).—This occurs whenever the tape is supported at intervals rather than throughout its full length. If standardized flat, *i.e.*, supported for full length, then with a heavy 100-ft. tape weighing 3 lb. supported at the ends under a tension of 10 lb. the systematic error due to sag alone is about 0.35 ft. per tape length. If the pull is increased to 20 lb. the error is about 0.09 ft.; if increased to 30 lb. the error is approximately 0.04 ft.

Normal Tension (Art. 91).—The tape stretches and this partly offsets the effect of sag. For the heavy tape the resultant error for a 30-lb. pull is perhaps 0.03 ft. per 100 ft. or 1.5 ft. per mile. With the lighter tapes a pull which may be applied without undue exertion may be determined, either by calculation or by experiment, at which the effect of sag will just offset the effect of increase in tension. This is usually called the *normal* tension.

87a. It is seen that in ordinary chaining the systematic errors are likely to be of much greater magnitude than the accidental errors. Hence the resultant error varies as the number of tape lengths or as the length measured.

When every possible device is employed to detect and to eliminate these systematic errors, as in precise base-line measurement, the accidental errors of observation become of relatively great importance, and for this reason long tapes are usually employed. To make corrections for, or to eliminate, errors due to sag or to elongation of

tape by tension, the pull is observed by spring balances; to make corrections for thermal expansion, the temperature of the tape is observed by the use of thermometers.

If the tape is longer than standard the observed distance between graduations of the tape is less than the true distance; that is, the error of measurement is negative and the correction is positive. These signs are reversed if the tape is too short.

A graphical method for the correction of steel tapes is given in Ref. 14, p. 606. By the use of a general tape-correction chart developed by W. S. Weeks it is possible in one operation to correct for sag, stretch, and temperature on any slope and for any length of tape, when the pull equals 2,000 times the weight of 1 foot of tape.

88. Correction for Temperature.—The coefficient of thermal expansion of steel is about 0.0000065 per 1°F. If the tape is standard at a temperature of T_o degrees and measurements are taken at a temperature of T degrees the correction for change in length is given by the formula

$C_x = 0.0000065l(T - T_o)$ in which l is the measured length. (4)

89. Correction for Tension.—If a tension greater or less than that for which the tape is standardized is used, the tape is elongated or shortened accordingly. The correction for variation in tension in a steel tape is given by the expression:

$$C_p = \frac{(P - P_o)L}{AE} \tag{5}$$

in which

C_p is the correction per distance L, in feet,
P is the applied tension, in pounds,
P_o is the tension for which the tape is standardized, in pounds,
L is the length, in feet,
A is the cross-sectional area, in square inches, and
E is the modulus of elasticity of the steel, usually between 28,000,000 and 30,000,000 lb. per square inch.

Some idea of the effect of variation in tension may be obtained by calculating the elongation of the tapes of the following article.

Example 1: It will be assumed that both a very heavy and a medium-weight tape are standard under a tension of 10 lb.; E = 30,000,000 lb. per square inch. It is desired to determine the elongation due to an increase of tension from 10 to 30 lb. The cross-sectional area of the heavy tape is 0.01 sq. in. and of the light tape 0.005 sq. in.

For the very heavy tape

$$C_p = \frac{(30 - 10) \times 100}{30,000,000 \times 0.01} = 0.0067 \text{ ft.}$$

For the medium-weight tape

$$C_p = \frac{(30 - 10) \times 100}{30,000,000 \times 0.005} = 0.0133 \text{ ft.}$$

The results show that in ordinary chaining the error due to variation in tension is of consequence only for light tapes.

90. Correction for Sag.—When the tape sags between points of support it takes the form of a catenary. The correction to be applied is the difference in length between the arc and the subtending chord. For the purpose of determining the correction the arc may be assumed to be a parabola, and the correction is then given by the expression

$$C_s = \frac{w^2L^3}{24P^2} = \frac{W^2L}{24P^2} \tag{6}$$

in which

C_s is the correction between points of support, in feet,
w is the weight of tape, in pounds per foot,
W is the total weight of tape between supports, in pounds,
L is the distance between supports, in feet, and
P is the applied tension, in pounds.

The correction is seen to vary directly as the cube of the unsupported length and inversely as the square of the pull. An example illustrating the variation in correction due to a variation in tension, weight of tape, and distance between supports is illuminating.

TABLE 90

	Very heavy tape, weight, 3 lb.				Medium-weight tape, weight, 1½ lb.			
	Distance between supports				Distance between supports			
	100 ft.		25 ft.		100 ft.		25 ft.	
	Pull		Pull		Pull		Pull	
	10 lb.	30 lb.	10 lb.	30 lb.	10 lb.	30 lb.	10 lb.	30 lb.
C_s (correction for sag) per 100 ft. of length (feet)	0.37	0.042	0.023	0.0026	0.094	0.0104	0.0059	0.00065
Change in C_s for 1-lb. variation in P (feet)......	0.07	0.003	0.004	0.0002	0.018	0.0007	0.0011	0.00004
Elongation per 100 ft. due to change in P from 10 to 30 lb. (Art. 89) (feet)........	0.0067		0.0067		0.0133		0.0133	
Normal tension (Art. 91) (lb.) .	51.7		23.2		28.0		14.3	

Example 2: In Table 90 are results of calculations for 100-ft. tapes weighing 3 lb. and 1½ lb., for pulls of 10 and 30 lb., and for distances between supports of 100 and 25 ft. The correction for sag of the 3-lb. tape for a pull of 10 lb. and distance between supports of 100 ft. is seen to be roughly 600 times as great as for the 1½-lb. tape, for a pull of 30 lb. and 25 ft. between supports. Other conditions remaining the same, the effect of decreasing the pull from 30 to 10 lb. is to increase the sag correction ninefold; the effect of increasing the distance between supports from 25 to 100 ft. is to increase the sag correction sixteen times. An error of 1 lb. in an assumed tension of 10 lb. introduces an error of 19 per cent in the calculated correction for sag. While in measurements of moderate precision this error is not of much consequence for the lighter tape and shorter distance between supports, it becomes large for the heavy tape and longer distance between supports. Evidently in ordinary chaining, where the pull is applied entirely by estimation, it would rarely be the case that the chainmen could apply the tension without an error greater than 1 lb.

The example further illustrates the serious disadvantage of using a very heavy tape for horizontal measurements over sloping ground.

91. Elimination of Effect of Sag.—By equating the right-hand members of equations (5) and (6) (Arts. 89, 90) the elongation due to increase in tension is made equal to the shortening due to sag. Normal tension P_n, the pull which will produce this condition, is given by the expression

$$P_n = \frac{0.204W\sqrt{AE}}{\sqrt{P_n - P_o}} \tag{7}$$

This equation is conveniently solved with the slide rule, as the following example shows.

Example 3: Given the heavy 100-ft. tape of example 2, Art. 90; distance between supports 100 ft.; $E = 30{,}000{,}000$ lb. per square inch; $A = 0.01$; $P_o = 10$ lb. Determine the tension at which the effect of sag will be eliminated by the elongation of the tape due to increased tension.

The numerator of the right-hand member is

$$0.204 \times 3\sqrt{0.01 \times 30{,}000{,}000} = 334; \quad P_n = \frac{334}{\sqrt{P_n - 10}}$$

Set runner to 334 on D scale; move slide until by inspection $P_n - 10$ on right B scale is at runner, when the C index is at P_n on D scale. The runner reads 41.7 on B scale, when the C index reads 51.7 on the D scale, hence $P_n = 51.7$ lb.

Similarly for a distance of 25 ft. between supports the normal tension is 23.2 lb. For the lighter tape of example 2 the normal tension is 28.0 lb. for a distance of 100 ft. between supports and 14.3 lb. for a distance of 25 ft. between supports. For convenience of comparison these values are tabulated in Table 90.

92. Precision of Measurements with the Tape.—It would be valuable if a definite outline of procedure could be established to produce any desired degree of accuracy in chaining. Unfortunately the conditions are so varied and so much depends upon the skill of the individual that the surveyor must be guided largely by his own experience and by his knowledge of the errors involved.

The usual practice in rough chaining through broken country is to take measurements with the tape horizontal, plumbing from the downhill end, breaking tape when necessary, applying tension by estimation, and making no corrections for sag, temperature, etc. The tape is usually 100 ft. long and weighs about 2 lb. The discussion of the preceding articles makes it evident that the larger errors are likely to arise owing to (1) tape not level, (2) sag in tape, (3) variation in temperature, and (4) poor plumbing. Call these errors per tape length respectively e_h, e_s, e_t and e_v. Of these, e_h and e_s are positive systematic errors, e_t is probably either positive or negative systematic, and e_v is accidental. Let us assume values for these errors, neglecting entirely those arising from other sources, and estimate the limits of the resultant error e per 1,000 ft. From the preceding articles we might perhaps expect the following:

$$e_h = +0.04 \text{ ft.}, \; e_s = +0.03 \text{ ft.}, \; e_t = \pm 0.01 \text{ ft.}, \; e_v = \pm 0.05 \text{ ft.},$$
$$e = 10(+0.04 + 0.03 \pm 0.01) \pm 0.05\sqrt{10} = +0.44 \text{ to } +0.96 \text{ ft.}$$

The corresponding limits of precision are roughly $\frac{1}{2,300}$ to $\frac{1}{1,000}$. In a general way these limits correspond to those usually attained on work of the character specified above where no particular effort is made to secure accuracy. The measured lengths of such lines are nearly always considerably longer than their true lengths, and it may be considered good practice arbitrarily to deduct a reasonable quantity from such measurements. If the chaining is done during extremely cold weather with inexperienced chainmen who fail to keep the tape taut and who do not use reasonable care in keeping the tape horizontal, the precision may be less than $1/500$.

In ordinary chaining over flat, smooth ground the principal errors are those due to variation in temperature and to inclination of tape. "Flat" and "smooth" are relative terms, as here used. Ordinarily the tape, when held to the ground, is neither perfectly straight nor horizontal. Frequently a difference in elevation of 2 or 3 ft. in 100 will go undetected if the slope is smooth. Assuming the error of setting pins as ± 0.007 ft. per tape length, that due to slope and uneven tape as $+0.02$ ft., and that due to temperature as ± 0.01 ft., the limits of precision are roughly $\frac{1}{3,000}$ to $\frac{1}{10,000}$. Usually

$\frac{1}{5,000}$ is considered good chaining for the conditions stated, but to attain this precision a rough correction for variation in temperature must frequently be made. Under extreme weather conditions, change in temperature alone might introduce an error of $\frac{1}{2,000}$ or even greater.

In chaining along a smooth surface as a paved highway, if slope measurements are accurately taken, the principal error is that due to variation in temperature. For measurements of moderate precision when a rough correction for thermal expansion is made without actual observations of temperature, the systematic error per 100 ft. due to temperature variation might be between ± 0.006 and ± 0.01 ft. As compared with this, the accidental errors due to variations in tension and to marking the ends of the tape are of relatively small account. Under these conditions it might reasonably be expected that a precision of $\frac{1}{10,000}$ to $\frac{1}{15,000}$ could be maintained. In practice it is generally assumed that a precision of $\frac{1}{10,000}$ is about the maximum that can be obtained without the aid of special apparatus.

For measurements of higher precision the temperature is determined by a thermometer attached to the tape, and the tension is regulated through the use of a spring balance. Generally the tapes used are light in weight so that when unsupported the uncertainty of the effect of sag will be small. When measurements are corrected for the accumulative effects of sag, slope, etc., the remaining errors are largely accidental in character, though for measurements in sunlight there is likely to be an appreciable difference between the observed temperature and the actual temperature even though the bulb of the thermometer is in contact with the tape. Assuming that the sum of all systematic errors in the same direction might be 0.004 ft. per 100 ft. and that all compensating errors might amount to ± 0.007 ft. per 100 ft., for a line 1 mile long we might expect a precision of $\frac{1}{20,000}$ to $\frac{1}{30,000}$. Experience indicates that lines of considerable length may be measured under the conditions stated above with an accuracy as high as $\frac{1}{30,000}$. Along a smooth course as a railroad or a highway, where measurements are taken with the tape supported its full length, this precision may be maintained by using moderate care in setting the pins or in otherwise marking the position of the ends of the tape on the ground. If the course is rolling

or rough in character it is necessary to employ some device by means of which the tape may be suspended and yet held firmly in position while the tension is applied and the end marks of the tape are projected to the ground points by plumbing; and also it is necessary to plumb with great care.

When steel tapes are used, measurements calling for a very high precision, say $\frac{1}{100{,}000}$ to $\frac{1}{500{,}000}$ or higher, are made at night or on cloudy days so that uncertainties regarding the temperature of the tape are greatly reduced. For work of this character the tapes employed are usually 50 meters or more in length and are very carefully standardized. The successive positions of the forward end of the tape are marked by lines scratched on zinc or copper strips fastened to substantial posts.

Errors due to variations in temperature are greatly reduced by using an invar tape. The coefficient of expansion of invar metal may be as small as 0.0000002 per 1°F.

93. Mistakes.—Some of the mistakes commonly made by inexperienced chainmen are:

1. Adding or dropping a full tape length. This is not likely to occur if both chainmen count the pins, or when numbered stakes are used, if the rear chainman calls out the station number of the rear stake in response to which the head chainman calls out the number of the forward stake as he marks it. The addition of one or more tape lengths may occur through failure of the rear chainman to give the head chainman a pin at breaks marking fractional tape lengths. A tape length may be dropped through failure of the rear chainman to take a pin at the point of beginning.

2. Adding a foot. This usually happens in measuring the fractional part of a tape length at the end of the line. This distance should be checked by the head chainman holding the zero mark on the tape at the terminal point and the rear chainman noting the number of feet and approximate fraction at the last pin set.

3. Wrong points taken as zero or 100-ft. marks on tape. The chainman should note whether these marks are at end of rings or on the tape itself, also whether there is an extra graduated foot at one end of the tape.

4. Reading numbers wrong. It is good practice to observe the number of the foot marks on either side of the one indicating the measurement, especially if the numbers are dirty or are worn. Also the tape should be read with the numbers right side up. Frequently "68" is read "89," or "6" read as "9."

5. Calling numbers incorrectly or so that they are not clearly understood. For example 50.3 might be called "fifty-three" and recorded as 53.0. If called as "fifty, point, three" the mistake would not be likely to occur. When numbers are called to a recorder, he should repeat them as they are recorded. Whenever a decimal point or a zero occurs in a number it should be indicated by the person calling.

Often large mistakes will be discovered or prevented if the chainmen form the habit of pacing distances or of estimating them by eye.

94. Numerical Problems.

1. A line is measured with a 100-ft. steel tape and its measured length is 1,012.3 ft. Afterward the tape is compared with the standard and is found to be 0.03 ft. too long. Calculate the length of the line.

2. A building 80 by 160 ft. is to be laid out with a 50-ft. tape which is 0.016 ft. too long. What ground measurements should be made?

3. The slope measurement of a line is 800 ft. The differences in elevation between successive 100-ft. points, as measured with a hand level, are 1.0, 1.5, 2.5, 3.8, 4.6, 5.0, 7.5, and 6.2 ft. Determine the horizontal distance.

4. A line measured on the slope is 1,246.5 ft. Slope angles measured with a clinometer are as shown below. Calculate the horizontal distance. Calculate by exact and by approximate methods.

Chainage in feet......	0		300		800		1,000		1,246.5
Slope angle in degrees		½		1¼		2½		4	

5. Two points at a slope distance of 100 ft. apart have a difference in elevation of 12 ft. What is the slope distance to be laid off to establish a horizontal distance of 100 ft.? Calculate by exact and by approximate methods.

6. Calculate the effect of sag per tape length for the two tapes of example 2, Art. 90, using tensions of 20 and 40 lb., and distances between supports of 25 and 100 ft.

7. A 100-ft. tape weighing 2 lb. is standard length under a tension of 12 lb. A line is measured with the tape under a tension of 35 lb. and found to be 4,863.5 ft. long. E = 29,000,000 lb. per square inch; 3.56 cu. in. of steel weigh 1 lb. Make the correction for increase in tension.

8. A second line is measured with the tape of problem 7, the tape being supported at intervals of 50 ft. and the pull being 35 lb. The measured length is 1,823.6 ft. Calculate the corrections for sag and variation in tension, and determine the corrected length of the line.

9. Calculate the pull for which the elongation of the tape of problem 7 is equal to the shortening due to sag, the tape being supported at its ends.

10. Chainmen made two independent measurements of a line 10,000 ft. long. The ground was sloping and measurements were taken with the tape horizontal. The tape was 100 ft. long, and weighed 3 lb. One

of the measurements of the line was made by two chainmen supporting the tape at the zero and 100-ft. marks; the second measurement was made by three chainmen supporting the tape at the zero, 100-ft., and 50-ft. points. The discrepancy between the two measurements was 11.7 ft. Several tests with a spring balance indicated that the average pull exerted by the chainmen was 20 lb. How much of the above discrepancy might be attributed to the different modes of supporting the tape?

11. For the purpose of establishing monuments in a city, a line along a paved street is measured on the slope and observations of temperature are made at each application of the tape. The measured length on the slope is 1,320.64 ft., the street grade is 2.5 per cent, the applied tension is 12 lb., and the mean of the observed temperatures is 87.4°F. The 100-ft. steel tape used for the measurements is standardized at 70°F., supported for full length, and is found to be 0.004 ft. too short under a tension of 12 lb. Determine the horizontal length of the line.

12. A line through rough country is chained by horizontal measurements and found to be 2,450 ft. long. On the average, it was necessary to use the plumb line every 50 ft. If the probable error of plumbing from the end of the tape to the ground is ± 0.03 ft. parallel with the line, calculate the probable error due to inaccurate plumbing.

13. If, in problem 12, the average slope of the tape when measurements are taken is 2 in 100, what error is introduced in the length of the line?

14. A line roughly 2 miles long along a railroad track is measured with a steel tape and corrections are made for observed temperatures. What error will be introduced if the actual temperature of the tape is 2°F. higher than the observed temperature? State the error in fractional form with numerator as one.

15. Assume that an invar tape is used under the conditions of problem 14. Calculate the error introduced.

16. A hedge along the line AB makes direct measurement impossible. A point C is established at an offset distance of 20 ft. from the line AB and roughly equidistant from the points A and B. The distances AC and CB are then chained and the following measurements obtained: $AC = 1287.2$ ft., $CB = 1353.0$ ft. By an approximate method calculate the length of the line AB.

17. It is desired to measure a distance of approximately 10 miles with a maximum permissible error of $\frac{1}{10{,}000}$. The country is rolling (average slopes, 5 per cent) and wooded, so that for perhaps half the distance the tape must be held level, unsupported. Make any other necessary assumptions and write specifications for this work.

95. Field Problems.

Problem 1. Pacing

Object.—To determine the length of normal pace, to test the reliability of 2½- and 3-ft. paces, and to determine an unknown distance by pacing.

Procedure.—(1) Walk over an assigned course of known length ten times at an ordinary gait, counting the paces each time. Record the observed number in the field notebook as shown in accompanying form.

PACING.

Standardization of Pace

Natural Pace Trial.	Paces per 300 ft.	2½ ft. Pace. Trial.	Paces per 300 ft.	3 ft. Pace. Trial	Paces per 300 ft.
1	116.5	1	121.4	1	98.5
2	117.3	2	120.8	2	101.4
3	117.8	3	122.3	3	102.4
4	115.9	4	119.7	4	101.6
~~5~~	~~121.4~~	5	118.5	5	97.1
6	118.6	6	121.5	6	100.2
7	117.1	7	120.6	7	101.6
8	117.2	8	118.7	8	103.0
9	119.7	9	119.3	9	100.1
10	118.5	10	120.9	10	102.4
11	115.0	Mean	120.4	Mean	100.8
Mean	117.4	Paces per 100 ft. = 40.1		Paces per 100 ft. = 33.6	
Paces per 100 ft. = 39.1		Length of pace = 2.49′		Length of pace = 2.98′	
Length of pace = 2.56′.					

Estimating Length. Line 251–252.

Trial	1	2	3	Mean	Length (in ft.)
Nat. Pace	371	377	375	374	957
2½ ft. "	378	373	380	377	942
3 " "	318	326	324	324	969
Tape					948

Error in Natural pace 1 in 100.
" " 2½ ft. " 1 in 150.
" " 3 ft. " 1 in 45.

F. F. Smith
Oct. 1, 1927.
Fair & Cold.

Measurements taken with steel tape from Locker No. 46, by F. F. Smith, H.C. and J. C. Brown, R.C.

300 ft course over smooth ground. Strong wind at right angles with course.

Line 251–252 over slightly rolling ground. Crosses highway and two rail fences.

Fig. 95*a*.—Distance by pacing.

Compute the average length of the natural pace and average number of paces for 100 ft. (2) With a tape mark a course 30 ft. long, every 3 ft. Walk over this course several times to obtain the proper stride; then pace

Chaining Over Level Ground. 2.

Sta.	Length in Feet. Forward	B'kward	Mean	Discrepancy	Discrep. to Length.
154					
	1213.61	1213.39	1213.50	0.22	1/5500
155					

Chaining Equipment — F. R. Reed, H.C.
Locker No. 42. — H. P. Bruce, R.C.
K.&E. 100 ft. Steel — Oct. 2, 1928 (2 hrs.)
Tape No. 283. — Cloudy and Cold.

Iron Pipe S.W. Corner of F. Lamb's Property.
Concrete Monument Property Lines S.E. Cor. High and Elm Sts.

Fig. 95*b*.—Notes for chaining over level ground.

the assigned course with this stride, recording the data and making computations as previously explained. (3) Follow a similar procedure for a 2½-ft. pace. (4) Walk over a course of unknown length several times at a natural pace, at a 2½-ft. pace, and at a 3-ft. pace. Estimate

the distance by each method; and then find the true distance with a steel tape. Note the error.

Hints and Precautions.—(1) In attempting to walk at a natural rate, avoid the general tendency to exceed that rate. (2) Count the paces carefully, estimating to the nearest one-tenth pace at the end of the course. (3) Reject observations that vary from the mean by more than 3 per cent. (4) Remember that field notes are a permanent record and should show clearly all the work done in the field. If an observation is rejected, draw a line through it, but do not erase. (5) Never fail to include in the notes of any survey, a complete description or sketch of the work accomplished. Figure 95*a* illustrates a suitable form of notes.

Problem 2. Chaining over Level Ground with 100-ft. Steel Tape

Object.—To chain over an approximately level course about 1,200 ft. long, and to check the distances by chaining in the opposite direction.

Procedure.—(1) Set a flag pole at each end of the line. Follow the procedure indicated in Art. 83. The distance should be read to tenths of feet and estimated to hundredths. Record the data as shown in Fig. 95*b*. Compute the ratio of discrepancy to length. Measurements should check within 1 to 5,000.

Hints and Precautions.—(1) The number of pins held by the rear chainman at any time indicates the number of hundreds of feet chained since the last tally. *The pin in the ground should not be counted.* (2) The rear chainman should refrain from holding onto the tape as he moves from station to station, for, if he moves too slowly, the head chainman is retarded, and if he moves too fast the chain is liable to become kinked. (3) Be careful not to disturb the "*stuck*" pin by allowing the tape to press against it. (4) Avoid injury to the tape by always keeping it straight while in use. (5) Avoid inconsistent errors by checking every measurement. (6) In order not to obstruct the view of the rear chainman in lining, the head chainman should stand beside the tape facing the line. This position also enables the chainmen to better withstand the pull on the tape. (7) If a reel is not provided for winding the tape, do it up in 5-ft. lengths in the form of a figure eight. To do this, stand beside one end of the tape, take the end of the tape in the left hand, and, allowing the tape to slide loosely through the right, extend the arms. As the 5-ft. mark comes along, grasp it with the right hand, and, bringing the hands together, lay it in the fingers of the left hand without permitting the tape to turn over. Then grasp the loop with the left hand and again extend the arm for another 5-ft. length. When the last mark is reached, tie the loop tightly where the ends of the tape come together. By properly twisting the tape it may then be put in circular form.

Problem 3. Standardization of Tape and Chaining over Uneven Ground

Object.—To standardize the tape before going into the field and on returning from work; to find the horizontal length of an assigned course

about 800 ft. long over uneven ground; and to correct for the error in length of tape as determined by the standardization tests.

Procedure.—(1) Standardize the tape by comparing it with the official standard. Maintain the required pull by means of a spring balance attached to one end of the tape. Determine the error with a finely divided scale. (2) Chain the course after the manner described in Art. 84. Maintain a pull estimated to be equal to that used during standardization. (3) For measurements in which the line slopes faster than 5 or 6 ft. in a hundred, "*break chain.*" (4) Correct for the error in length of tape by adding to or subtracting from the observed distance, the observed error

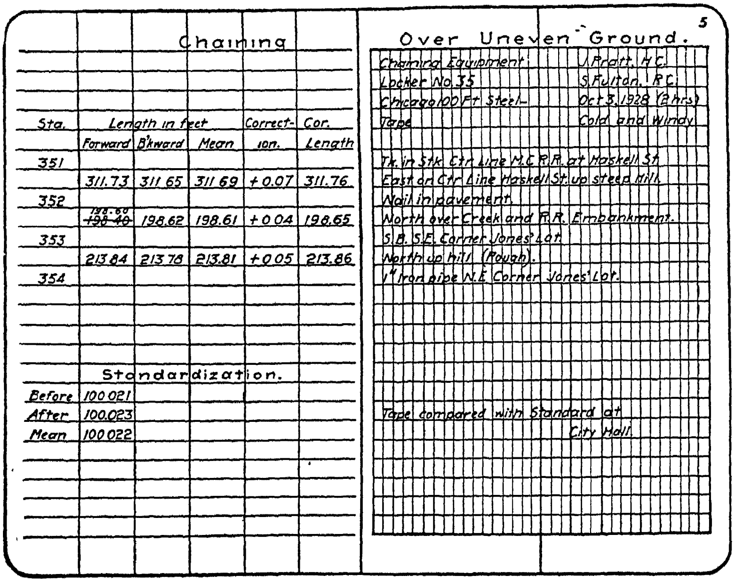


FIG. 95*c*.—Notes for chaining over uneven ground.

multiplied by the number of tape lengths in the course. *The correction should be added when the tape is longer than the standard, and subtracted when it is shorter.* Data should be recorded as shown in Fig. 95*c*.

Hints and Precautions.—(1) The general tendency is to hold the downhill end of the tape too low. Avoid this by comparing the tape with some level line, or if possible, have the two ends in line with the horizon or estimate angle between tape and plumb line. (2) It is usually more accurate to chain downhill than to chain uphill, since in chaining downhill the rear end may be held firmly on the ground.

PROBLEM 4. SURVEY OF FIELD WITH STEEL TAPE

Object.—To collect sufficient data for calculating the area of the field by two sides and included angle, by three sides, and by base and perpendicular methods (see Fig. 95*e*).

Procedure.—(1) Divide the field into triangles, avoiding as far as possible any construction which will result in forming a very acute angle; that is, make the triangles as nearly equilateral as the shape of the field will readily permit.

(2) Measure the sides and altitude of each of the triangles. Erect perpendiculars by one of the following methods:

3:4:5 *Method.*—To erect a perpendicular to AB that shall include point C in the triangle ABC (Fig. 95d) by the 3:4:5 method. Assume point a to be on the perpendicular. Set a pin at a on line AB. With sides 3, 4, 5 ft., or a multiple of these numbers, such as 24, 32, and 40 ft., make a right triangle abc with the tape as follows: Set a pin at b on line AB 24 ft. from a. Loop the tape over the pins at a and b so that the 32-ft. mark on the tape coincides with a, and the 56-ft. mark coincides with b. The head chainman with the zero and 96-ft. marks of the tape grasped in one hand, moves to c setting a pin at the point where the zero and 96-ft.

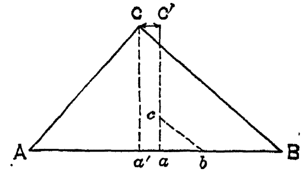
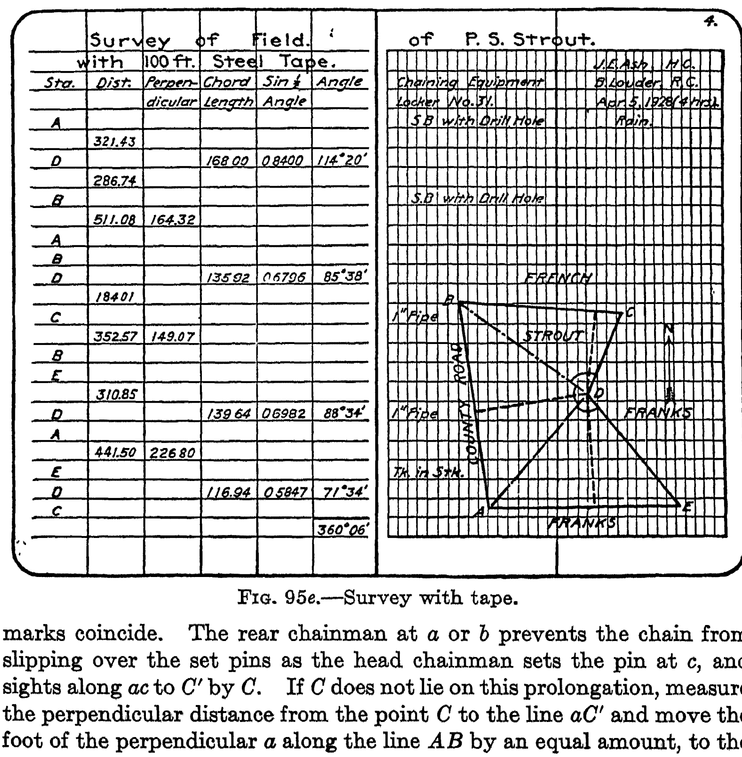


Fig. 95d.—Perpendicular by 3:4:5 method.



Survey of Field. with 100 ft. Steel Tape.					
Sta.	Dist.	Perpendicular	Chord Length	Sin ½ Angle	Angle
A					
	321.43				
D			168.00	0.8400	114°20'
	286.74				
B					
	511.08	164.32			
A					
B					
D			135.92	0.6796	85°38'
	184.01				
C					
	352.57	149.07			
B					
E					
	310.85				
D			139.64	0.6982	88°34'
A					
	441.50	226.80			
E					
D			116.94	0.5847	71°34'
C					
					360°06'

Fig. 95e.—Survey with tape.

marks coincide. The rear chainman at a or b prevents the chain from slipping over the set pins as the head chainman sets the pin at c, and sights along ac to C' by C. If C does not lie on this prolongation, measure the perpendicular distance from the point C to the line aC' and move the foot of the perpendicular a along the line AB by an equal amount, to the

point a'. If the trial perpendicular fails to include the point C by several feet, repeat the process for a', the new point. If, however, the distance $C'C$ is small, the new foot of the perpendicular may be set by making aa' equal to $C'C$ and it may be assumed as correct without further test.

Chord-bisection Method.—Estimate the position of the perpendicular from C upon AB (Fig. 95*f*), and set a pin at d on this estimated perpendicular, less than one tape length from the line AB. With d as center and length of tape as radius, the head chainman describes the arc of a circle. The rear chainman stationed at A or B determines the position of pins at b and c, the points of intersection between the line AB and the arc of the circle EF. The point a bisects the line bc. Prolong ad, place C' by C. and move point a as described in the 3:4:5 method.

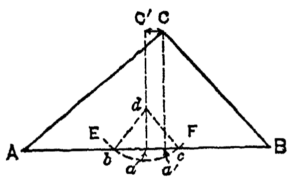


FIG. 95*f*.—Perpendicular by chord-bisection method.

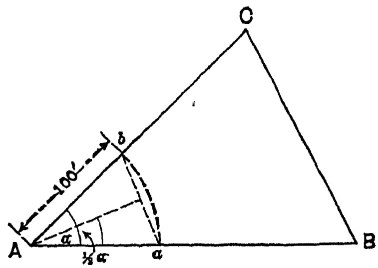


FIG. 95*g*.—Angle with tape.

(3) Determine one angle of each triangle by the chord method as follows: With the vertex of the angle as the center, swing the tape, setting pins a and b (Fig. 95*g*) at points where the arc intersects the lines AB and AC. Measure the chord distance ab.

$$\sin \tfrac{1}{2}\alpha = \frac{ab}{200}$$

Hints and Precautions.—(1) The chord-bisection method of erecting perpendiculars is the more accurate, while the 3:4:5 method requires less time. (2) If the perpendicular and the segments of the base on each side of its foot are measured, sufficient data will have been obtained for determining the angles by the tangent method. However, the chord method of measuring angles is more accurate, and the tangent method should be used only as a check. (3) Considerable care should be taken in lining in points, and intersections should be determined as closely as the eye of the observer will allow. (4) A very good way of finding the approximate position of the perpendicular is to stand on the line AB with arms extended horizontally in the directions of A and B. Now, with eyes closed, bring the arms to the front, palms together, and opening the eyes, see if the vertex of the triangle lies in line with the hands. If it does not, the observer should change his position to some other point along AB nearer the desired perpendicular.

Practical Applications.—It is obvious that surveying by this method would be too slow and inaccurate to be utilized to any great extent except in surveys covering small areas; hence this method is, in general, restricted to rough estimates or checks. In this problem the elementary steps are the more important, since the construction of perpendiculars and the measurement of angles is often necessary when the transit is not at hand. Angles measured by the tape method are usually thought of as being less accurate than those measured with the transit; but for very small angles, the reverse is likely to be true.

Problem 5. Survey of Field with Irregular Boundary

Object.—To collect sufficient data for calculating the area and for plotting the assigned field.

	Survey	of	Land	of	
	With	100 ft.	Steel	Tape.	
Sta.	*Dist.*	*Offset L*	*Sta.*	*Dist.*	*Offset L*
A	0	0			
	25	21.4	D	0	0
	50	37.8		25	10.0
	75	44.6		50	26.2
	100	27.2		75	34.6
	111	0		100	34.5
	125	15.5		125	25.2
	140	16.7		150	14.0
B	159.06 0	0		175	4.1
	30	21.6	E	208.64 0	0
	50	42.0	A	275.98	
	90	51.4	AC	281.14	
	140	27.8	EC	273.61	
	165	13.0			
C	187.88 0	0			
	14	0			
	45	16.3			
	55	31.6			
	75	61.0			
	90	14.2			
	100	7.0			
D	109.89	0			

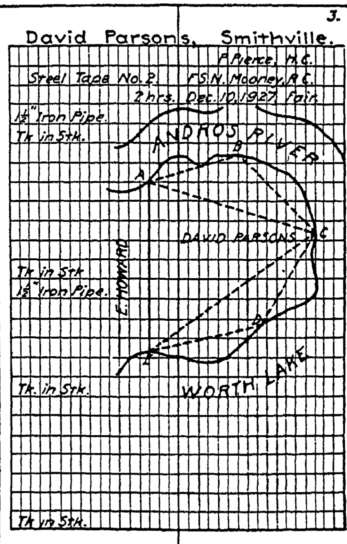


Fig. 95*h*.—Offsets with tape.

Procedure.—(1) Divide as much of the field as its shape will permit into triangles as in problem 4, having in view simplicity of field operations. Lay out the triangles so that no long offsets will be necessary. (2) Collect data for finding the area of the triangles by one of the methods explained in problem 4. (3) From the triangle sides nearest the boundary, take offsets at such intervals as will insure a sufficiently accurate plat. Record the data as shown in sample page of notes (Fig. 95*h*).

Hints and Precautions.—(1) To take the offsets accurately and quickly, set several pins at the desired intervals along the side of the triangle before measuring the offsets. If the distance to the boundary is not

more than 50 ft. and the boundary is fairly regular, erect perpendiculars by estimation with the eye. (2) If the boundary changes abruptly at any point, be sure to take an offset at that point. (3) Do not measure the offsets with unnecessary accuracy. For example, if the sides of the triangles are measured to the nearest 0.01 ft., measure the offsets no closer than the nearest 0.1 ft. (4) Do not record offsets as being on the right of the line when they are on the left.

Practical Applications.—This problem is frequently made use of in land surveying where the boundary line is a stream or lake. Lines are run by transit as near to the stream or lake as is practicable. Where the boundary changes direction frequently, offsets are taken at short intervals; but as long as the boundary continues as one straight line,

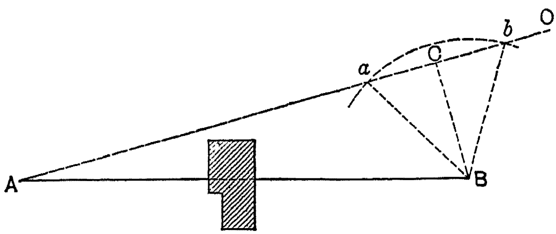


FIG. 95*i*.—Swing offset with tape.

offsets need not be taken. However, because of the greater ease with which the areas can be computed, offsets are frequently taken at regular intervals.

PROBLEM 6. OBSTRUCTED DISTANCE WITH CHAINING EQUIPMENT

Object.—To determine an obstructed distance between two points.

Procedure.—(1) On an assigned course about 800 ft. long, assume that there is some obstacle which makes direct measurement and intervision impossible, and find the distance by the swing-offset method. (2) Find the distance by constructing a line parallel to the given line. (3) Assume that the points are intervisible but direct chaining is impossible, and determine the distance by methods other than those used above. (4) As a check, measure the distance directly. Compare the results.

Swing Offsets.—To find the distance by the swing-offset method, the head chainman attaches the end of the tape to one end of the line as B (Fig. 95*i*), and describes an arc of a circle the radius of which is 100 ft. and the center of which is the point B. The rear chainman stationed at A, lines in the end of the tape with some distant object as O, and sets points a and b where the head chainman crosses line AO. A point C midway between a and b lies on the perpendicular CB. Bisect ab, set a pin at C, and measure the distance AC, to obtain the necessary data for calculating AB.

Parallels.—Parallels may be laid off with a tape by one of the following methods: (1) By equal distances, (2) by alternate angles, (3) by similar triangles.

Method 1.—If points are intervisible, as will be the case when the obstruction is a stream or lake, erect perpendiculars at A and B (Fig. 95*j*)

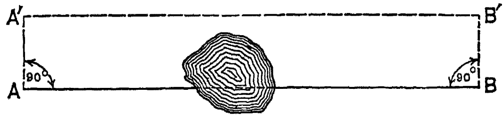


Fig. 95*j*.—Inaccessible distance.

by one of the methods of problem 4, chain distance $AA' = BB'$ to clear the obstacle, and measure the distance $A'B'$. If the perpendiculars have been carefully erected and the distance $AA' = BB'$ is not great, the measured distance should agree closely with the length of the line AB. If the points are not intervisible, this method will be impossible.

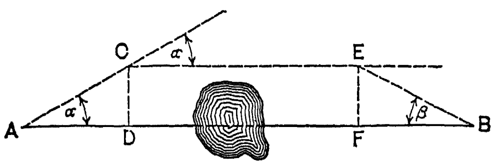


Fig. 95*k*.—Inaccessible distance (long offset).

Method 2.—If the offset necessary to avoid the obstacle is considerable, the preceding method will become inaccurate because of the uncertainty of right angles measured with the tape. If the points are intervisible, establish AC (Fig. 95*k*) so that the estimated value of α will be less than 45°. Determine the chord length of α for radius of 100 ft. (or greater

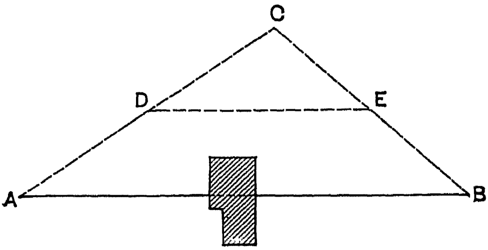


Fig. 95*l*.—Obstructed distance.

if precision makes it necessary). Measure CD, it being roughly perpendicular to AB. At C lay-off an angle equal to the angle at A so that CE will be parallel to AB. Measure CE, E being any convenient point such that β will be less than 45°. Measure BE and EF. $AB = AD + CE + FB$. Solve the right-angle triangles ADC and BFE for AD and BF, and compute the length of the line AB.

The precision of this method over that of the preceding method is due to the fact that the angles laid off are small and that the distances AD and BF are *computed* rather than measured. The reason for computing these distances will readily be seen when it is considered that any variation of the line DC from the true perpendicular will make little difference in its length as compared with the corresponding change of length of AD. To illustrate, suppose α (Fig. 95*k*) equals 45°, that the true length of AD and CD is 200 ft. and that D, supposedly on the perpendicular through C, is 10 ft. in error. As *computed* from AC and CD, AD is about 0.3 ft. in error, but its *measured* length is 10 ft. in error.

Method 3.—If the points are not intervisible, both the preceding methods are impossible. Let C (Fig. 95*l*) be a point from which both ends of the line AB are visible. Measure AC and BC. Lay off CD and CE so that CD will bear the same relation to CA that CE bears to CB. That is, $CD/CA = CE/CB$. (It will generally be convenient to make this ratio a simple one such as ½ or ⅓). The triangles ACB and DCE are similar. Measure DE, and compute AB.

CHAPTER VII

MEASUREMENT OF DIFFERENCE IN ELEVATION

GENERAL METHODS

96. Definitions; Methods.—The difference in elevation between two points on the surface of the earth is the vertical distance between the two level surfaces (imaginary or real) in which the points lie. In Fig. 96 the irregular line represents the profile of the earth's surface in which are the points A and B. The curved lines are level lines representing the profile of imaginary level surfaces in which the points are located.

The elevation of a point is its vertical distance above or below some arbitrarily assumed level surface called the "*datum*" or more often

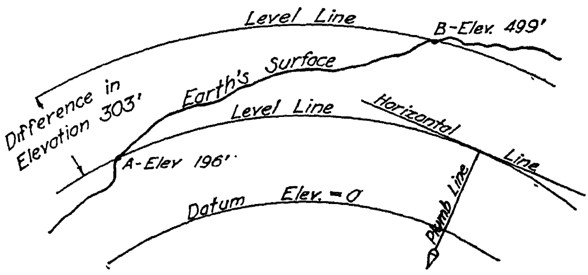


Fig. 96.—Difference in elevation.

the "*datum plane.*" In the figure the datum is represented by the lowest curved line. If the elevation of point A with respect to the datum is 196 ft. and the difference in elevation between A and B is 303 ft. then the elevation of point B is 499 ft. The measurement of differences in elevation therefore deals either directly or indirectly with the determination of vertical distances. The operation is called *leveling.*

The datum most widely used is mean sea level, but most cities have a "city datum" which may or may not agree closely with mean sea level. Thus the Chicago datum might be the mean level of Lake Michigan and the St. Louis datum might be the low-water stage of the Mississippi River. Frequently elevations for a particular survey are referred to some datum which bears no known relation to sea level. For example, the initial point in a survey may be

assumed to have an elevation of 100 ft. and the elevations of all succeeding points may be calculated accordingly. If the relative elevation of points is all that is desired, the relation between the assumed datum and sea level or any other datum in common use is of no consequence.

Difference in elevation may be measured by the following methods:

1. By measuring the variation in the pressure of the earth's atmosphere with a barometer, called *barometric leveling.*

2. By measuring a vertical angle and horizontal distance, called *indirect* or *trigonometric leveling.*

3. By measuring vertical distances directly, called *direct* or *spirit leveling.*

The operation of direct leveling to determine the elevations of points some distance apart, or to establish bench marks, is called *differential leveling.*

Precise leveling is a form of differential leveling for which the instruments and methods are such as to produce a high degree of precision.

The operation of determining elevations of points at short measured intervals along a definitely located line (such as the center line for a highway or a sewer) is called *profile leveling.*

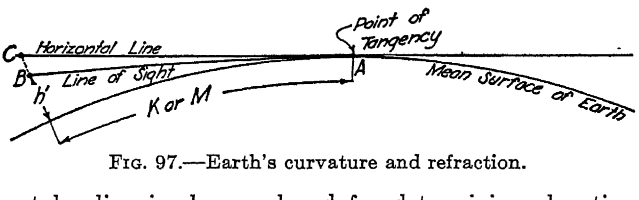


FIG. 97.—Earth's curvature and refraction.

Direct leveling is also employed for determining elevations for cross-sections, grades, and contours. Direct leveling is the most precise method of determining elevations, and is the method most frequently used.

97. Curvature of the Earth.—A *level surface* has been defined as a surface every element of which is perpendicular to the action of gravity. A line tangent to a level surface is called a *horizontal line* and is perpendicular to the action of gravity (to the plumb line) at the point of tangency. In Fig. 97 a horizontal line tangent to a level surface representing the mean surface of the earth is shown. The vertical distance between the horizontal line and mean surface is a measure of the earth's curvature. It varies approximately as the square of the distance from the point of tangency. A ray of light transmitted in the direction of the horizontal line at the point of tangency is refracted or bent downward slightly so that at some distance from the point of tangency it is below the horizontal line,

and in the opposite sense a ray emanating from a given distant point below the horizontal line becomes horizontal at the point of tangency. This phenomenon is called atmospheric refraction. Thus, as viewed from point A (Fig. 97), an object actually at B would apparently be at C. The actual line of sight is along the path of a curve AB.

The combined effect of the earth's curvature and atmospheric refraction is given by the expression:

$$h' = 0.57K^2 = 0.021M^2 \text{ (approximate)}^1 \qquad (1)$$

in which

K is the distance from the point of tangency in miles,

M is the distance from the point of tangency in thousands of feet, and

h' is the combined effect of the earth's curvature and atmospheric refraction in feet.

98. Barometric Leveling.—Since the pressure of the earth's atmosphere varies inversely with the elevation, the barometer may be employed for making rough observations of difference in elevation. If at a given elevation the atmospheric pressure always remained constant, or even approximately so, the barometric method would be one of considerable precision, but the pressure in the course of a day or even in the course of an hour is likely to vary over a considerable range. Usually barometric observations are taken at a fixed station during the same period that observations are made on a second barometer which is carried from point to point in the field. This makes it possible to correct for atmospheric disturbances which could not be readily detected if a single barometer were used.

The difference in elevation between two points A and B is given by Eq. (2), assuming that the mean of the temperatures at A and B is 50°F., and neglecting the effects of humidity and of atmospheric disturbances.

$$z \text{ (uncorrected for temperature)} = 62{,}737 \log \frac{30}{h_a} - 62{,}737 \log \frac{30}{h_b} \qquad (2)$$

in which z is the difference in elevation in feet, and h_a and h_b are respectively the barometer readings (in inches of mercury) at points A and B. Each term of the second member represents the elevation of the corresponding point above a datum plane of barometric pressure 30 inches, or approximately at mean sea level.

For a mean temperature at the two points other than 50°F., and for average conditions of humidity, a proportionate correction is added (algebraically). The amount of this correction is determined

[1] The effect of the earth's curvature alone is about $0.66K^2$. The effect of the refraction is about $0.09K^2$ in the opposite direction.

by multiplying the second member of Eq. (2) by the appropriate factor in the following tabulation.

CORRECTION FACTORS FOR ELEVATION BY BAROMETER

Mean temp., °F.	Factor	Mean temp., °F.	Factor	Mean temp., °F.	Factor
0	−0.1024	35	−0.0273	70	+0.0471
5	−0.0915	40	−0.0166	75	+0.0575
10	−0.0806	45	−0.0058	80	+0.0677
15	−0.0698	50	+0.0049	85	+0.0779
20	−0.0592	55	+0.0156	90	+0.0879
25	−0.0486	60	+0.0262		
30	−0.0380	65	+0.0368		

Example: Given barometer readings at A and B respectively of 26.850 in. and 28.315 in., and corresponding temperatures of 48°F. and 72°F. Determine the difference in elevation.

By Eq. (2)

$$z \text{ (uncorrected)} = 3{,}022.5 - 1{,}575.0 = 1{,}447.5 \text{ ft.}$$

From the table, the correction factor for a mean temperature of 60°F. is +0.0262.

$$1{,}447.5 \times +0.0262 = +37.9 \text{ ft.}$$
$$z \text{ (corrected)} = 1{,}447.5 + 37.9 = 1{,}485.4 \text{ ft.}$$

Some mercurial barometers have an auxiliary scale by means of which the temperature correction is made mechanically. The type of aneroid barometer in most common use has a dial about 3 in. in diameter, graduated both in inches of mercury and in feet of altitude (elevation). It is compensated for temperature. When at a point of known altitude, the pointer may be set at the corresponding reading on the scale, thus placing the instrument in adjustment.

The mercurial barometer is cumbersome and is only suitable for observations at a fixed station. The aneroid barometer is light in weight and is easily transported. By comparing values of pressure indicated by the aneroid barometer with those for a mercurial barometer over a range of temperature, the instrument correction can be determined.

In using the barometer, the instrument should be given time to reach the temperature of the air before an observation is made.

Barometric leveling is employed principally on exploratory or reconnaissance surveys where differences in elevation are large, as in hilly or mountainous country. Under ordinary conditions, eleva-

tions determined by barometric leveling are likely to be several per cent in error. A single aneroid barometer is frequently used by topographers on small-scale surveys where the contour interval is large. Stops are made at frequent intervals during the day and the rate of change in atmospheric conditions is observed; suitable corrections are thus determined and are applied to the observed values. Where distances permit, it is preferable to return to the starting point, and to correct the intermediate readings in proportion to the change in pressure during the interval between observations.

99. Indirect Leveling.—In Fig. 99*a*, A represents a point of known elevation and B a point whose elevation is desired. In employing the method of indirect or trigonometric leveling, α, the vertical angle

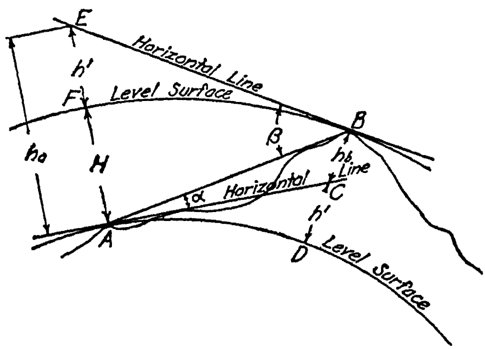


FIG. 99*a*.—Indirect or trigonometric leveling.

at A, is measured and the distance AD is known or is determined by some method of measurement. Within the limits of ordinary practice $AD = AC$ and $\angle BCA = 90°$. Therefore

$$h_b = AC \tan \alpha \tag{3}$$

The correction for curvature and refraction is by Eq. (1),

$$h' = 0.021M^2 = 0.021\left(\frac{AC}{1{,}000}\right)^2, \text{ in which } AC \text{ is in feet.}$$

The difference in elevation is therefore

$$H = h_b + h' = AC \tan \alpha + 0.021\left(\frac{AC}{1{,}000}\right)^2 \tag{4}$$

If the vertical angle β is now taken from B to A, by a similar course of reasoning $h_a = EB \tan \beta$.

For the horizontal distances employed on any ordinary survey EB is the equivalent of AC. Therefore

$$h_a = AC \tan \beta \tag{5}$$

and the difference in elevation is

$$H = h_a - h' = AC \tan \beta - 0.021\left(\frac{AC}{1{,}000}\right)^2 \tag{6}$$

From Eqs. (4) and (6) it will be noted that when the vertical angle is upward or positive the curvature and refraction correction is added; and when downward or negative, the curvature and refraction correction is subtracted.

Adding Eqs. (4) and (6),

$$2H = (h_a - h') + (h_b + h') = h_a + h_b,$$

or

$$H = \frac{h_a + h_b}{2} = \frac{AC}{2}(\tan \alpha + \tan \beta) \tag{7}$$

From Eq. (7) may be deduced the general rule that *when vertical angles are measured from (and to) each of two points whose difference in elevation is desired, the difference in elevation is one half of the horizontal distance between them multiplied by the sum of the tangents of the angles,* and the effect of the earth's curvature and atmospheric refraction is thereby eliminated. In precise trigonometric leveling this is the procedure employed.

In ordinary surveying, indirect leveling furnishes a rapid means of determining the elevations of points in rolling or rough country. On reconnaissance surveys, angles may be measured with the clinometer and distances may be obtained by pacing. On more accurate surveys, angles are measured with the transit and distances by the stadia.

99a. On lines of indirect levels for which angles are measured with the transit the usual procedure is illustrated by Fig. 99*b*. A and D are two points whose difference in elevation is desired. The successive positions of the instrument are indicated by the symbols T_1, T_2, and T_3. With the transit at T_1 the distance and vertical angle to A are determined by a *backsight,* and similar quantities are measured by taking a *foresight* to B. The instrument is then moved ahead to T_2, and observations are taken to B and C. And so the process is repeated until the end of the line is reached. If the transit is *equidistant* from the points on either side of it to which sights are taken, the effect of curvature and refraction will be eliminated. In Fig. 99*c*, the difference in elevation between A and B is

$$H = H_a + H_b = h_a - h' + h_b + h' = h_a + h_b$$

And if D_a and D_b are the respective equal horizontal distances from the transit to points A and B, then

$$H = -D_a \tan \alpha + D_b \tan \beta \tag{8}$$

in which tan α and tan β are negative or positive according to whether the vertical angle is below or above the horizontal.

In practice, with regard to the effect of curvature and refraction in indirect leveling, usually very little attention is paid to keeping the backsight and foresight distances balanced, since the effect of curvature and refraction is negligible (0.02 ft. for a distance of 1,000 ft.) as compared with the accuracy with which elevations can be determined by this method (generally not closer than tenths of feet). The error from such practice is accidental in that while one backsight

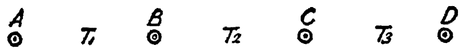


Fig. 99*b*.

distance may be greater than the corresponding foresight distance, normally the next is just as likely to be smaller. Generally the transit is set at some convenient place where good sights can be obtained in both directions, and which is about the same distance from adjacent points on which sights are to be taken.

In small-scale mapping the indirect method of leveling is employed to determine the difference in elevation between the plane table and a point sometimes at a distance of several miles. In such cases the effect of curvature and refraction becomes large and the correction must be applied. For example, if the horizontal distance from the

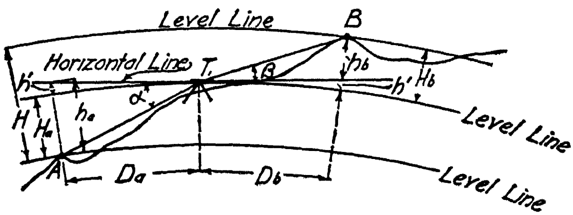


Fig. 99*c*.

plane table to the point sighted were 10 miles, the correction would be 57 ft.

The errors of indirect leveling are chiefly accidental. The precision attainable depends upon the length of sight, the instrument used, and the magnitude of the vertical angles. With the transit, under average conditions, the error may be expected to be not greater than 0.4 ft. times the square root of the distance in miles.

100. Direct Leveling Defined.—In Fig. 100*a*, *A* represents a point of known elevation and *B* represents a point whose elevation is desired. In employing the method of direct or spirit leveling, the

level is set up at some intermediate point as L, and the vertical distances AC and BD are observed by holding a leveling rod first at A and then at B, the line of sight of the instrument being horizontal.[1]

If the difference in elevation between the points A and E is designated as H_a and the difference in elevation between E and point B is designated as H_b,

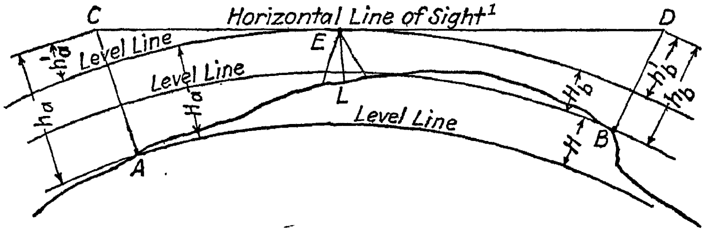


FIG. 100*a*.—Direct leveling.

$H_a = h_a - h'_a$ and $H_b = h_b - h'_b$ in which h_a and h_b are the vertical distances read at A and B respectively, and h'_a and h'_b are the effects of the curvature of the earth and refraction of the earth's atmosphere for the horizontal distances LA and LB respectively.

If H is the difference in elevation between A and B, then

$$H = H_a - H_b = (h_a - h'_a) - (h_b - h'_b)$$

$$H = h_a - h_b - h'_a + h'_b \tag{9}$$

If the backsight distance LA is equal to the foresight distance LB then $h'_a = h'_b$ and

$$H = h_a - h_b \tag{10}$$

Thus, by keeping backsight and foresight distances balanced the difference in elevation between two points is equal to the difference between the rod readings taken to the two points, and no correction for earth's curvature is necessary. In direct leveling, except in rare instances, the work is so conducted that the effect of the earth's curvature is eliminated, or nearly so.

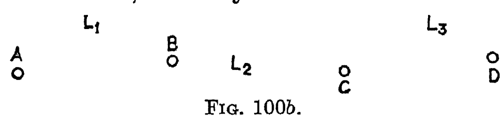


FIG. 100*b*.

On lines of direct levels the usual procedure is as indicated by Fig. 100*b*. A and D are two established points some distance apart, whose difference in elevation is desired. The symbols L_1, L_2, and

[1] Due to refraction, the line of sight is slightly curved as explained in Art. 97.

L_3 indicate successive positions of the level, not necessarily on a line joining A and D. With the level in some convenient position as L_1, a backsight is taken to point A, and a foresight is taken to some convenient point B. The level is then moved ahead to L_2, and a backsight reading is taken to B and then a foresight reading is taken to some accessible point as C. And so the process is repeated until the terminal point is reached.

In accurate leveling the backsight and foresight distances are measured by pacing or by stadia. Usually no special attempt is made to equalize *each* backsight distance with its corresponding foresight distance. The general rule is that their *sums* should balance, or nearly so, between established points that may be used as future references.

In ordinary leveling, *with an instrument in good adjustment*, the backsight and foresight distances are not measured, and no special attempt is made to equalize them. Normally, they will tend to balance in the long run. Even in leveling of moderate precision, however, a succession of very long backsight distances and correspondingly short foresight distances, or *vice versa*, would produce a systematic error of appreciable magnitude unless some correction were made for the earth's curvature and for atmospheric refraction.

For example, in running a line of levels 10 miles long, if the foresight distances were the abnormal length of 1,000 ft. and the backsight distances were 500 ft., the cumulative error due to curvature of the earth and to refraction would amount to about 0.6 ft. On the other hand if the foresight distances were the normal length of 300 ft. and the backsight distances were 150 ft., the error introduced by reason of the earth's curvature would amount only to about 0.16 ft., a comparatively small quantity.

Thus, except in rare instances, direct leveling is carried on without paying any attention to the actual curvature of the earth, its effect being eliminated or at least reduced to a negligible quantity through methods of procedure, and the ordinary variation between backsight and foresight distances producing only a negligible error from this source.

A far more important reason for keeping these distances approximately balanced lies in the fact that certain instrumental errors are eliminated, as will be seen later.

INSTRUMENTS FOR DIRECT LEVELING

101. General.—Any instrument or device used for direct leveling has as its essential features a line of sight and a level tube or some other means of making the line of sight horizontal. The level tube is so mounted that its axis is parallel to the line of sight. The instru-

ment used principally is the engineer's level. The architect's level, a modified form of the engineer's level, but with a telescope which is of lower magnifying power and which has a less sensitive bubble, is used in establishing grades for buildings. The hand level is a simple and useful device for roughly determining differences in elevation. Instruments which are frequently used for direct leveling, but which are not primarily and solely designed for this purpose, are the engineer's transit and the telescopic alidade.

The measurements of difference in elevation are determined by sighting upon graduated wooden rods, called leveling rods. Other accessories are the rod level, which indicates when the rod is plumb, and a metal plate or pin which is useful in establishing temporarily a definite and unyielding point on which the rod may be held.

102. The Engineer's Level.—Figure 102*a* illustrates diagrammatically the principal parts of the engineer's level. The level

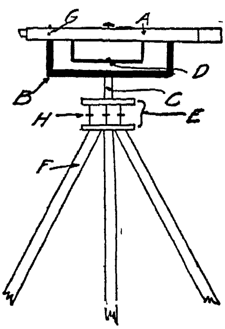


FIG. 102*a*.

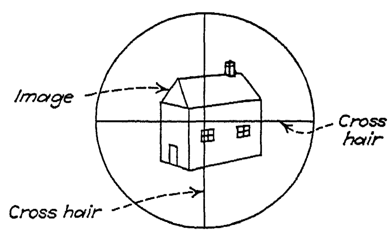


FIG. 102*b*.

consists of the telescope A mounted upon the level bar B, which is rigidly fastened to the spindle C. Attached to the telescope and parallel with it is the level tube D. The spindle fits into a cone-shaped bearing of the leveling head E, so that the level is free to revolve about the spindle C as an axis. The level is screwed to a wooden tripod F. In the tube of the telescope are cross-hairs at G which appear on the image viewed through the telescope as illustrated by Fig. 102*b*. The bubble of the level tube is centered by means of the leveling screws H.

The two distinct types are the *dumpy level* for which the telescope tube is permanently fastened to the level bar, and the *wye level* for which the telescope is removable and rests in Y-shaped supports. Generally the leveling head is equipped with four leveling screws, but the three-screw type is favored by some engineers and surveyors, particularly for instruments of high precision. The level tube is

usually under the telescope, but may be on top or at the side. A reflecting mirror or similar device, by means of which the level tube may be viewed while looking through the telescope, is sometimes employed. Sensitive instruments are often provided with a micrometer altitude-screw, at one end of the level bar, by means of which screw fine settings of the telescope may be made without moving the leveling screws. The details of the telescope are constructed quite differently by the different makers.

103. Level Tube.—A level tube is a glass vial whose inside upper surface (sometimes also its lower surface) is ground to the arc of a circle in a longitudinal direction. Its transverse section is of uniform shape but not of uniform size, since the tube is ground barrel-shaped inside. The tube is nearly filled with sulphuric ether or with alcohol. The remaining space is occupied by a bubble of air which takes up a position at the high point in the tube. The tube is usually graduated in either direction from the middle, so that, by observing its ends, the center of the bubble may be brought to the mid-point of the tube. This operation is called "centering" the bubble.

A line tangent to the level tube at its mid- or zero-point is called the *axis of the level tube.* When the bubble is centered, the axis of the level tube is horizontal.

103a. Sensitiveness of Level Tube.—If the radius of the circle to which the tube is ground is large, a small vertical movement of one end of the tube will cause a large displacement of the bubble; if small, the reverse is true. The radius of the level tube is thus seen to be a measure of its sensitiveness. The tubes of the better grades of engineer's levels are ground to radii varying roughly from 75 to 150 ft., the average being about 85 ft. The tubes of precise levels have a radius of 300 to 1,000 ft. The more sensitive the tube, the longer the time required to center the bubble.

The sensitiveness of the level tube is generally expressed in seconds of the central angle whose arc is one division of the tube. For many levels the length of a division is $\frac{1}{10}$ in., but the practice in this respect is by no means uniform among manufacturers of surveying instruments. For this reason, unless the spacing of graduations is known, the sensitiveness expressed in seconds of arc is not a definite measure. The sensitiveness expressed in this manner is inversely proportional to the number of seconds. Thus a 10-sec. level tube (radius about 170 ft.) is twice as sensitive as a 20-sec. level tube (radius about 85 ft.).

A simple method of determining the radius is explained in field problem 2, Art. 112.

104. Telescope.—Figure 104*a* shows the principal parts of the telescope as it is commonly constructed. Rays of light emanating from an object within the field of view of the telescope are caught by the objective lens *A* and are brought to a focus and form an image in the plane of the cross-hairs *B*. The lenses of the eyepiece *C* form a microscope which is focused on the image at the cross-hairs. The objective lens is screwed in the outer end of the objective slide *D* which fits in the telescope tube *E*. The objective lens is focused by the screw *F* at the inner end of which is a pinion that engages the teeth of a rack fixed to the objective slide. The eyepiece slide is held in position laterally by rings *H*, *J*, through which it may be

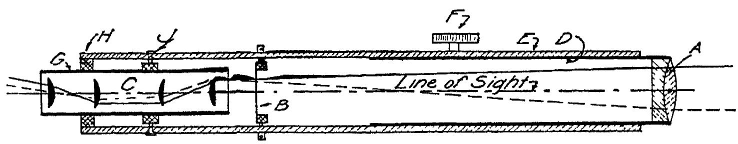


FIG. 104*a*.—Longitudinal section of telescope.

moved in a longitudinal direction for focusing. By means of screws the ring *J* may be moved laterally so that the intersection of the cross-hairs will appear in the center of the field of view.

The line of sight is defined by the intersection of the cross-hairs and the optical center of the objective lens. The instrument is so constructed that the optical axis of the objective lens coincides (or practically coincides) with the axis of the objective slide; in other words, a given ray of light passing through the optical center of the objective always occupies the same position in the telescope tube regardless of the longitudinal position of the lens. The cross-hairs may be so adjusted that the line of sight and the optical axis coincide.

104a. Focusing.—In using the telescope, the eyepiece is first moved in or out until the cross-hairs appear sharp and distinct. This adjustment of the eyepiece should be tested frequently, as the observer's eye becomes tired. When an object is sighted, the objective slide is moved in or out until the image appears clear, when it *should* be in the plane of the cross-hairs.

If a slight movement of the eye from side to side produces an apparent movement of the cross-hairs over the image, the plane of the image and the plane of the cross-hairs do not coincide and *parallax* is said to exist. Since parallax is a source of error in all observations it should be eliminated by refocusing the objective until further trial shows no apparent movement. The objective lens must be focused for each distance sighted. The nearer the object sighted, the greater must be the distance between the objective and the cross-hairs.

While for short sights there must be a considerable movement of the objective for a comparatively small change in distance, for the longer sights only a small movement of the objective is necessary regardless of the distance.

Frequently the telescope will be so badly out of focus that the outline of the object cannot at first be detected. It often facilitates the work of focusing if the telescope is directed approximately in the proper direction by sighting along the outside of the tube. Some instruments are equipped with peep sights for this purpose.

Figure 104*b* illustrates the manner in which rays from an object are deviated by the objective and brought to a focus to form the image. It will be noted that the image is inverted.

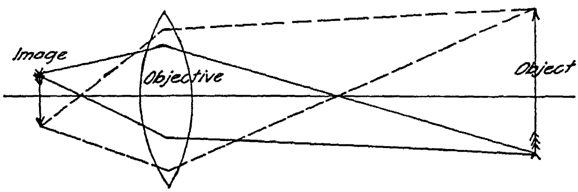


FIG. 104*b*.

104b. Objective.—The principal function of the objective is to form an image for sighting purposes.

For accuracy of measurements the objective should produce an image which is well lighted, accurate in form, distinct in its outlines, and free from discolorations. A single bi-convex lens answers the first two of these requirements, but is faulty in regard to the other two for the following reasons:

1. Rays entering the lens near its edge come to a focus nearer the objective than do those entering near its center. The image does not lie in a plane, but in the surface of a sphere. Hence as viewed through the telescope, portions of the object are blurred. This defect is called *spherical aberration.*

2. Rays of the various colors of the spectrum are deviated by different amounts as they pass through the lens, hence the field of view appears discolored by lights of various hues. This is called *chromatic aberration.*

These two objectionable features of the single lens are nearly eliminated in most surveying instruments by providing an outer double-convex lens of crown glass and an inner concavo-convex lens of flint glass. The two lenses are usually cemented together with balsam but are sometimes separated by a thin spacer ring.

The *optical center* of the objective may be defined as that point in the lens through which a ray of light will pass without permanent

deviation, regardless of the direction of the object from which the light emanates. In other words, the direction of the ray is the same after leaving the lens as before entering it. In a bi-convex lens with faces of equal curvature the optical center and geometrical center coincide.

The *optical axis* is defined as the line taken by a light ray that experiences no deviation either on entering or on leaving the objective. It passes through the optical center and the centers of curvature of the lens.

The *principal focus* is a point on the optical axis back of the objective where rays entering the telescope parallel with the optical axis

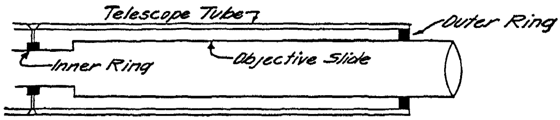


Fig. 104*c*.

are brought to a focus; or it is a point in front of the objective from which diverging light rays entering the lens emerge from it parallel with the optical axis. Stated in another form, the image of a point on the optical axis and an infinite distance away is at the principal focus back of the objective. If a point is at the principal focus in front of the objective, however, it will have no image.

The *focal length* of the objective is the distance from its optical center to the principal focus. When the telescope is focused on a distant point, the focal length is very nearly the distance from the plane of the cross-hairs to the optical center of the objective, for reasons which the preceding paragraph makes clear.

104c. Objective Slides.—Any lateral movement of the objective causes a deviation in the position of the optical axis and also in the line of sight, thereby introducing errors in measurements. The objective slide should therefore fit as neatly as possible and still admit readily of longitudinal movement for focusing. The workmanship on any good instrument is sufficiently precise to insure practical elimination of errors of this sort when the telescope is new, but in the course of long use, wear develops between the sliding parts, and the slide becomes loose. This produces uncertainties in observation which no amount of adjustment can overcome.

For most instruments, the objective slide fits neatly into the telescope tube so that there is nearly perfect contact between these two parts for a considerable length near each end of the slide. Any wear that develops through use is therefore distributed over most of

the length of the slide. Other instruments are provided with objective slides which are held in position by two metal rings as in Fig. 104*c*. One ring is screwed in the end of the telescope tube. The other ring is placed in the rear of the rack and pinion and is held in position by four screws passing through the telescope tube. The inner ring is of somewhat smaller diameter than the telescope tube, so that by means of the screws just mentioned the objective slide may be adjusted laterally.

Particular care should be taken to protect the objective slide from dust, water, and other foreign matter. Most instruments are equipped with a guard which affords at least partial protection. If the slide is lubricated at all, only a drop of the finest watch oil should be used and all excess oil should be removed with a soft cloth. If the objective lens is removed it should be replaced as nearly as possible in its original position, and after replacing it the adjustment should be checked.

104d. Cross-hairs.—Threads from the cocoon of the brown spider are usually used as cross-hairs. These threads are nearly opaque, are brown in color, and are both strong and small. Often the cross-hairs are made of very fine platinum wire. Two threads, one vertical, the other horizontal, are fastened to a metal ring called the cross-hair ring or reticule, as shown in Fig. 104*d*. The ring is held in position by four capstan-headed screws which pass through the telescope tube and tap into the ring. The holes in the telescope tube are slotted so that when the screws are loosened, the ring may be rotated through a small angle about its own axis. The ring is smaller than the inside of the tube, and it may be moved either horizontally or vertically by means of the screws. Thus, to move it to the left, the right-hand screw is loosened, and the left-hand screw is tightened.

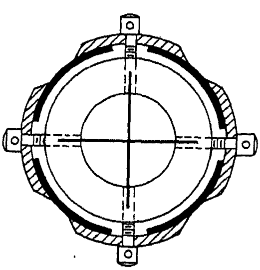


FIG. 104*d*.

Spider threads, unless properly stretched, are likely to sag during wet weather, and being fragile they are easily broken. The cross-hair ring can be removed from the tube as follows: Two opposite capstan-headed screws are removed and the ring is rotated 90° about the remaining two screws by means of a pointed stick inserted through the end of the telescope. The stick is then inserted in a screw hole, the remaining screws are taken out, and the ring is withdrawn without damage. Broken cross-hairs are readily replaced if a magnifying glass is available. They are usually cemented to the ring with shellac. They should be carefully stretched nearly to the breaking point before being applied, and should be

held securely in position until the shellac dries. The operations of replacing the cross-hair ring are in the reverse order of those employed in removing it.

104e. Stadia Hairs.—Two additional horizontal threads which are placed one above and one below the horizontal cross-hair are called stadia hairs. Usually they are mounted in the same plane with the cross-hairs and hence when the eyepiece is in focus all four hairs appear in the field of view. Sometimes they are mounted in another plane so that when the cross-hairs are in focus the stadia hairs are invisible, or *vice versa.* They are then called *disappearing* stadia hairs.

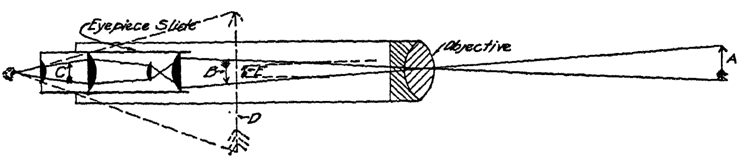


FIG. 104*e.*—Erecting eyepiece.

104f. Eyepiece.—Attention has previously been drawn to the fact that the image formed by the objective is inverted. Eyepieces are of two general types.

The *erecting* or *terrestrial* eyepiece, the more common of the two types, reinverts the image so that the object appears to the eye in its normal position. In the common form it consists of four plano-convex lenses placed in a metal tube, called the eyepiece slide (Fig. 104*e*). In the figure *A* represents the object, *B* the image in the plane of the cross-hairs, *C* the image which is magnified by the lens nearest the eye, and *D* the magnified image as it appears to the eye.

The *inverting* or *astronomical* eyepiece simply magnifies the image without reinverting it. It is composed of two plano-convex lenses generally arranged as shown in Fig. 104*f.* The arrangement is seen to be identical with that of the two lenses farthest apart in the erecting eyepiece. The magnified image *D* is seen to be inverted, and the object as viewed through the telescope is upside down.

For either eyepiece the ratio of the angle at the eye subtended by the magnified image, to that subtended by the object itself, is the magnifying power of the telescope. If, in either Fig. 104*e* or Fig. 104*f*, *D* is the apparent length of the magnified image at a given distance from the eye and *E* is the apparent length of the object as seen by the naked eye, the ratio of *D* to *E* is the magnifying power.

Each lens which is interposed between the object and the eye absorbs some of the light which strikes it. Hence, other things being equal, the object is more brilliantly illuminated when viewed through

the inverting eyepiece, and this is a great advantage, particularly when observations are made during cloudy days or near nightfall. Another important advantage of the inverting eyepiece is that the telescope is shorter and the instrument is lighter in weight. The beginner experiences some inconvenience on viewing things apparently upside down, but this difficulty is overcome with a little practice.

The single advantage of the erecting eyepiece is that objects appear in their natural position, and this is the reason why its use is so common. Most American engineers and surveyors prefer the erecting eyepiece.

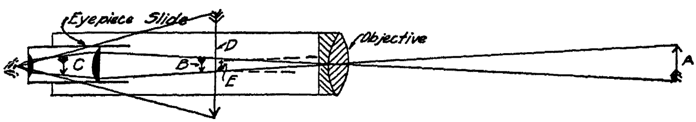


FIG. 104*f*.—Inverting eyepiece.

The slide of the erecting eyepiece is usually held in position by rings similar to those described for the objective slide (Art. 104*c*). In most instruments the slide is held tightly by spring friction and it is focused by a screw-like motion.

The slide of the inverting eyepiece is usually held by a single wide ring which is fixed in the end of the telescope tube and which admits of no lateral adjustment.

104g. Properties of the Telescope.—The illumination of the image depends upon the effective size of the objective, the quality and number of lenses, and the magnifying power. Other conditions remaining the same, the larger the objective, or the smaller the magnifying power, the better the illumination, *i.e.*, the better lighted appears the object.

Distortion of the field of view so that it does not appear flat is mainly caused by what is termed the *spherical aberration* of the eyepiece. While this introduces no appreciable error in ordinary measurements, it is not desirable when two points in the field are to be observed at the same time, as in stadia measurements.

The *definition* of a telescope is its power to produce a sharp image. It depends upon the quality of the glass, the accuracy with which the lenses are ground and polished, and the precision with which they are spaced and centered. Light rays passing through the lenses near their edges are particularly troublesome, and to improve the definition these rays are intercepted by diaphragms or screens placed between the lenses of the eyepiece and also in the rear of the

objective. The effect of these screens is to decrease the illumination somewhat.

The angular width of the field of view is the angle subtended by the arc whose center is nearly at the eye and whose length is the distance between opposite points of the field viewed through the telescope. For a particular instrument this angle may be readily determined by observation. It is independent of the size of the objective. In general the larger the telescope and the greater the magnifying power, the less the angle of the field of view. For most surveying of moderate precision, the work is greatly retarded if the instrument does not have a fairly large field of view, and this is one of the reasons why the telescopes are not usually made of high magnifying power. Usually the angular width of the field ranges from about 1°30′ for a magnifying power of 20, to 45′ for a magnifying power of 40.

The magnifying power of the telescope may be determined by observations as outlined in field problem 1, Art. 112. For the better grade of engineer's levels it is about 30 diameters. Some precise levels have a magnifying power as high as 50 diameters. The U. S. Coast Survey type of precise level has a magnification of 43 diameters. The magnification of transit telescopes ranges from 18 to 25 diameters.

105. Relation between Magnifying Power and Sensitiveness of Level Tube.—It is desirable that the sensitiveness of the level tube be such that for the smallest noticeable movement of the bubble there is an apparent movement of the cross-hairs on a level rod held at an average distance from the instrument; and likewise for the smallest noticeable movement of the cross-hairs there should be an observable movement of the bubble. The least noticeable movement of the cross-hairs depends to some extent upon the definition and illumination of the image, but principally upon the magnification.

If the level tube is more sensitive than is necessary, time is wasted in centering the bubble. If the magnifying power is higher than it need be, unnecessary labor is expended by reason of the more limited field of view and by reason of the increased difficulties of focusing the objective properly. A satisfactory test may be conducted by one person sighting at a rod while a second person bears down slightly on one end of the telescope and at the same time observes the level tube. If the first noticeable movement of the bubble is accompanied by an apparent movement of the cross-hairs, there is a satisfactory balance between sensitiveness and magnification. If the cross-hairs move first, a level tube of greater radius might properly be employed.

106. Dumpy Level.—Figure 106 shows the details of a dumpy level with inverting eyepiece. The telescope is rigidly attached to the crossbar and the instrument is so constructed that the optical axis of the telescope is perpendicular to the axis of the spindle. The

level tube is permanently placed so that its axis lies in the same vertical plane as the optical axis, but it is adjustable in altitude by means of a capstan-headed screw at one end. The spindle revolves

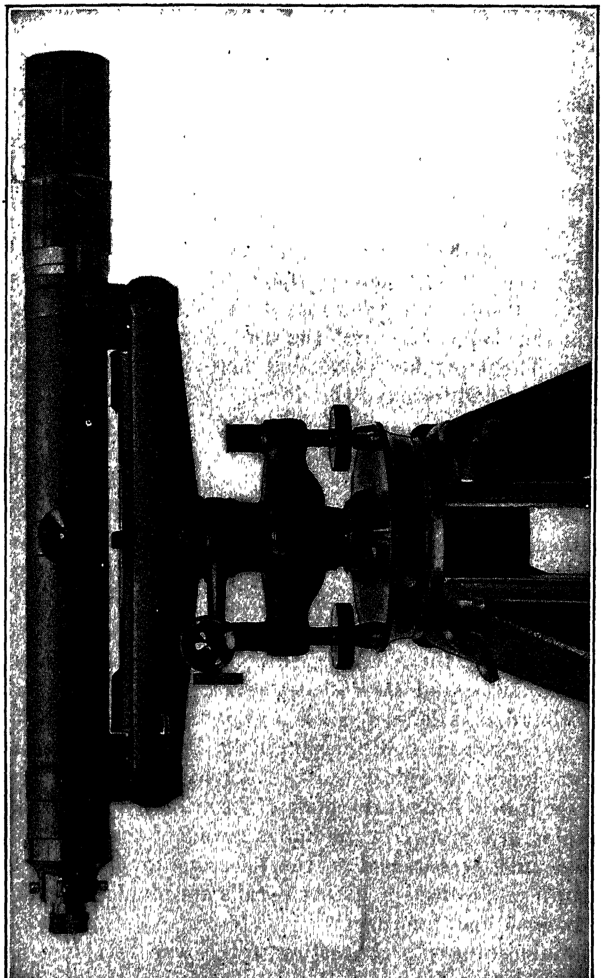


FIG. 106.—Engineer's dumpy level (inverting eyepiece).

in the socket of the leveling head which is controlled by the leveling screws. At the lower end of the spindle is a ball-and-socket joint which makes a flexible connection between the instrument proper and the foot plate. When the leveling screws are turned, the level

is moved about this joint as a center. Above the leveling head is a collar and a clamp-screw by means of which the spindle may be clamped to the leveling head. The tangent-screw controls small movements of the level about its vertical axis after the clamp-screw is tightened. The sunshade protects the objective from the direct rays of the sun. The adjusting screws for the cross-hair ring are near the eyepiece end of the telescope.

The dumpy level was formerly more often equipped with an inverting eyepiece and therefore for the same magnifying power it was considerably shorter than the wye level which is usually supplied with an erecting eyepiece; this accounts for its name. Its advantages over the wye level are that it is simpler in construction, has fewer parts that are subjected to wear, requires a lesser number of adjustments, and stays in adjustment better. It is the type commonly employed in Europe. Until recently it has not met with much favor among American engineers, but the use of the dumpy level is now increasing. The most highly refined instruments used in precise leveling are modified forms of the dumpy level.

107. Wye Level.—Figures 107*a* and 107*b* show the details of a wye level with erecting eyepiece. The telescope rests in Y-shaped bearings called the *wyes.* The leg of each wye is threaded. It passes through the crossbar and is secured in position by capstan-headed nuts. By means of these nuts, one of the wyes may be raised or lowered. The telescope is secured in position by the wye clips. When the clips are raised the telescope may be revolved in the wyes or it may be lifted from the wyes and turned end for end. The enlarged portions of the telescope barrel which rest in the wyes are cylindrical in shape and are called the *rings.* The line joining the centers of the rings is defined as the *axis of the rings* or the *axis of the wyes.* The telescope is held longitudinally by a flange on each ring which bears against the side of the wye. When the clips are fastened, the telescope is held from turning about its axis by a lug on one of the clips engaging in a groove in the flange of one of the rings.

The level tube is attached to the telescope and is adjustable in both azimuth and altitude. Other details are much the same as for the dumpy level just described.

The distinguishing characteristics of the wye level are that the telescope may be revolved in its bearings and may be turned end for end. While in the work of leveling, these features are of no particular advantage, as will be seen later, they facilitate the making of adjustments provided the bearings are not worn. Each ring is in contact with the wye at two points shown as No. 64 in Fig. 107*b*. In process of use these points and the rings become worn and flattened so that

the rings are no longer cylindrical nor are they likely to be of the same size or shape; the axis of the rings does not maintain a fixed position as the telescope is revolved in the wyes and does not lie on

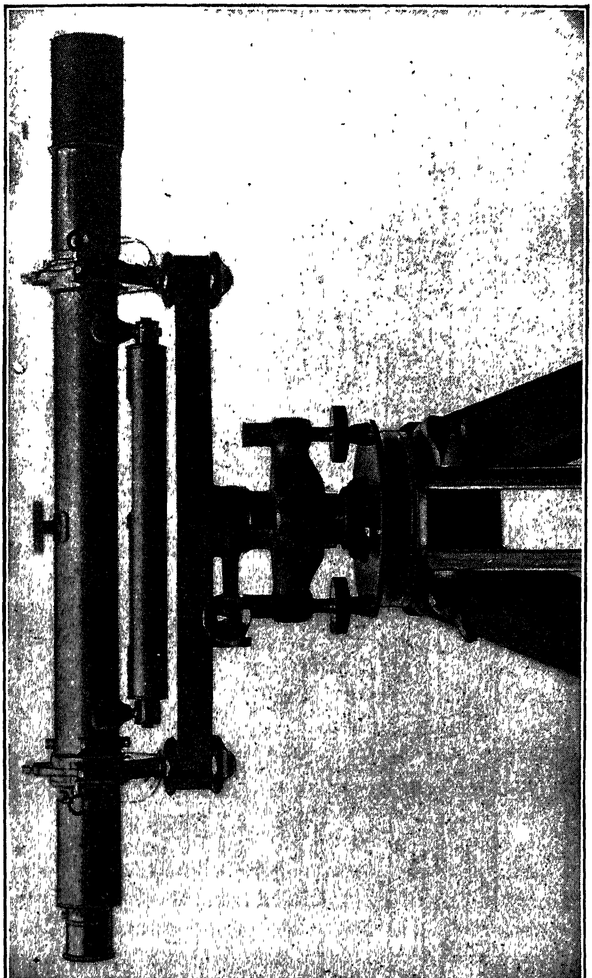


FIG. 107*a*.—Engineer's wye level (erecting eyepiece).

the same line when the ends of the telescope are reversed in the wyes. Under these conditions it is impossible to adjust the instrument by the usual methods, and adjustments, if correctly made, are the same as for the dumpy level.

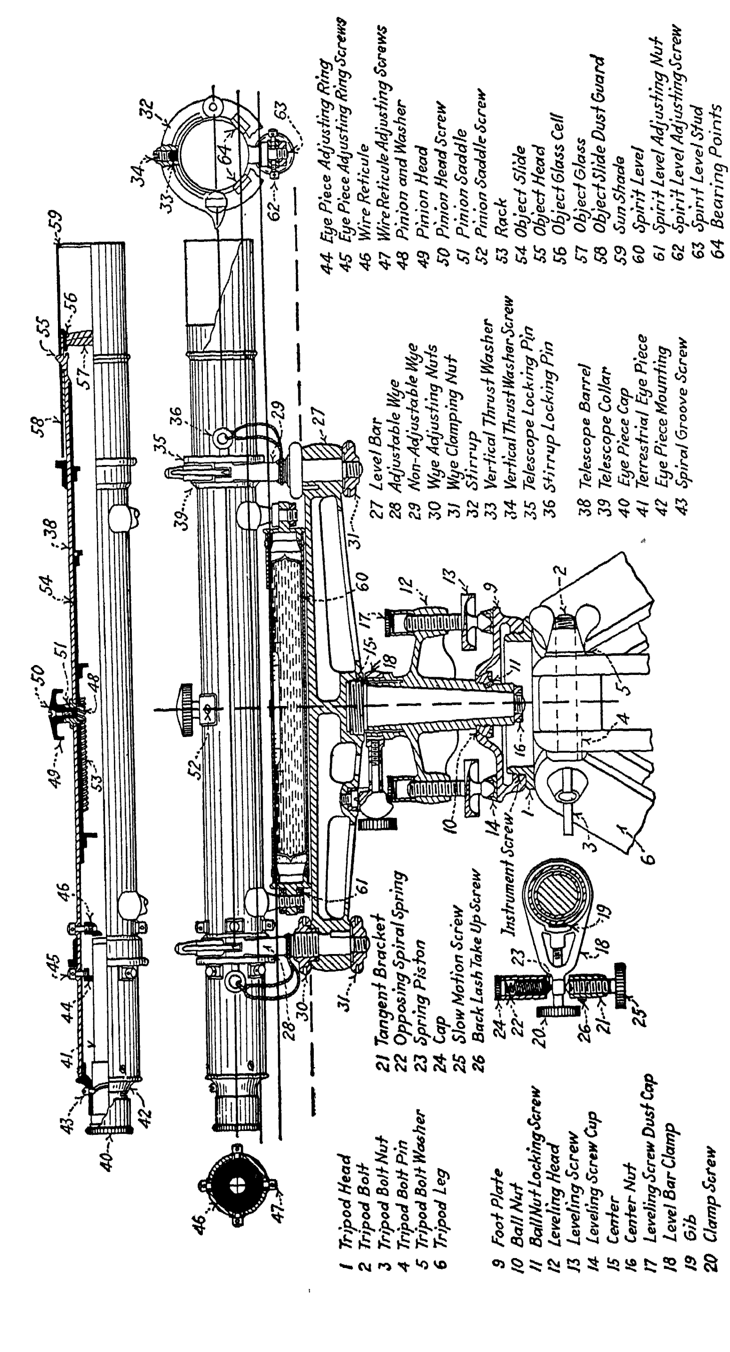


FIG. 107b.—Cross-section of wye level (erecting eyepiece).

108. Hand Levels; Locke Level.—The Locke hand level is a serviceable device which is widely used for rough leveling. It consists of a brass tube about 6 in. long on which is mounted a level bubble as shown in Fig. 108*a*. In the tube beneath the bubble is a prism. An opening is left in the tube, and the image of the bubble is reflected by the prism to the eye end of the level. Just beneath the bubble vial is a cross-wire which is adjustable by means of a screw, the head of which protrudes through the end of the case enclosing the vial. The eyepiece consists of a peep-hole mounted

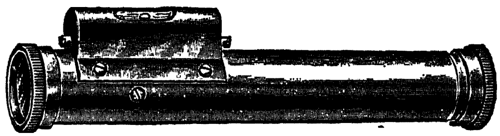


Fig. 108*a*.—Locke hand level.

in the end of a slide which fits inside the tube and is held in a given position by friction. Mounted on the right half of the inner end of the slide is a semicircular convex lens which magnifies the image of the bubble and cross-wire as reflected by the prism. Both the object and the eye ends of the tube are closed by disks of plain glass so that dust will not collect on the prism and lens. The magnifying lens is focused by moving the eyepiece slide in or out. In using the level the object is viewed directly through the left half

Fig. 108*b*.—Abney level and clinometer.

of the tube, without magnification, while with the same eye and at the same time the position of the bubble with respect to the cross-wire is observed apparently in the right half of the field of view. The level is held with the bubble vial uppermost and is tipped up or down until the cross-wire bisects the bubble, when the line of sight is horizontal. After a little practice one may make observations with greater facility by keeping both eyes open.

108a. Abney Level and Clinometer (Fig. 108*b*).—As its name indicates, this level is suitable both for direct leveling and for measuring the angles of slopes. When it is used as a level, the index of

the vernier is set at zero, and it is then used in the same way as the Locke hand level. When it is used as a clinometer, the object is sighted and the level tube is moved by means of a screw until the cross-wire bisects the bubble as viewed through the eyepiece.

109. Leveling Rods.—These are graduated wooden rods of rectangular cross-section by means of which difference in elevation is measured. In the United States, the rods are ordinarily graduated into feet and decimals. On some Government surveys the

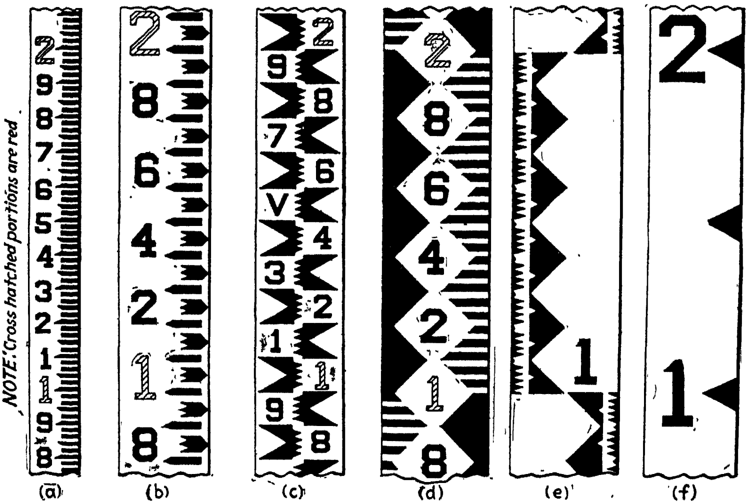


FIG. 109a–f.—Graduations for self-reading rods.

rods are graduated into decimals of the meter or the yard. The lower or ground end of the rod is shod with metal to protect it from wear and usually, though not always, is the point of zero measurement from which the graduations are numbered.

The rod is held vertically, and hence the reading of the rod as indicated by the horizontal cross-hair of the level is a measure of the vertical distance between the point on which the rod is held and the line of sight.

Rods are obtainable in a variety of patterns and graduations, and are either in single pieces or in sections which are jointed together or slide past each other and are clamped together. A common length is 12 ft. The two general classes of leveling rods are: (1) self-reading rods, those that may be read directly by the observer as he looks through the telescope of the level; and (2) target rods, those for which a target, sliding on the rod, is set by the rodman as directed by the leveler.

109a. Self-reading Rods.—With the self-reading rods, the rodman simply holds the rod vertically; the leveler observes the graduation at which the line of sight intersects the rod, and records the reading. In most of the operations of leveling, the self-reading rod is much superior to the target rod in point of the speed with which observations may be made, and is nearly as accurate. It is the type most widely used at the present time, even for leveling of the highest precision. Observations closer than the smallest division on the rod are made by estimation.

The self-reading rod should be so marked that the graduations appear sharp and distinct for any normal distance between level and rod. Most commonly the background is white with graduations 0.01 ft. wide painted in black as shown in Fig. 109*a*. The numbers indicating feet are in red and those indicating tenths of feet are in black. This makes a very satisfactory style of graduation when the length of sight is less than 400 or 500 ft. For sights of greater length the graduations of Fig. 109*a* become hazy and rods with larger blocks of contrasting color are desirable. The authors have found the graduations of Fig. 109*b* simply and easily read up to distances of 800 ft. Those of Figs. 109*c*, *d*, and *e* are more complicated but are satisfactory when once the leveler has become used to them. There are many types of rods which are designed for accurate reading at short distances and at the same time for clear reading at long distances and are adapted for use either as leveling rods or as stadia rods.

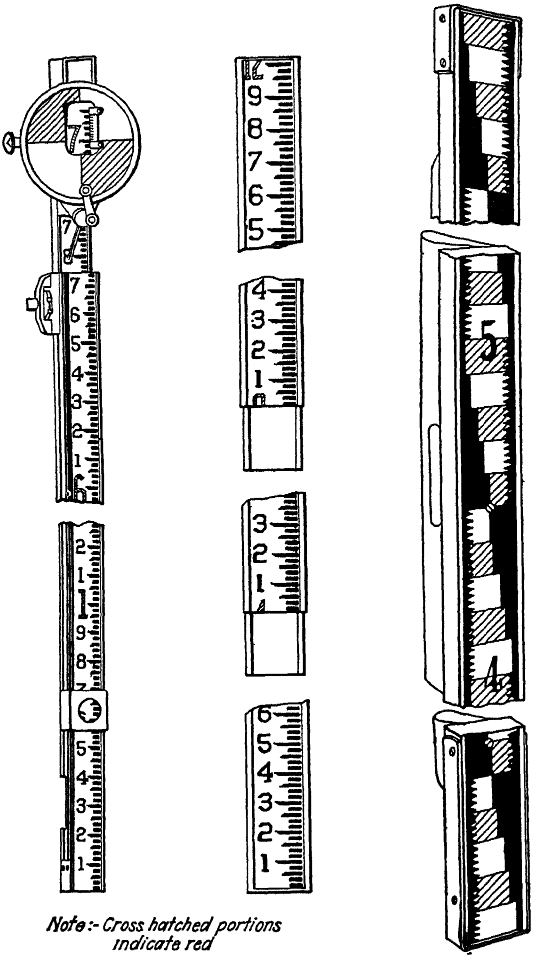
Figure 109*f* shows a graduation suitable for leveling with the hand level. The zero point is at the height of the observer's eye, and the graduations are 0.5 ft. apart. The numbers increase in both directions from the zero mark.

The Philadelphia rod (Fig. 109*g*) is the most widely used of any rod. It is equipped with a target which makes it possible to use it also as a target rod. It is usually in two sections, the parts being held in contact by two brass sleeves. By means of a screw attached to the upper sleeve the two parts of the rod may be clamped together in any relative position desired. For readings of less than 7 ft. the back strip is clamped in its normal position; for greater readings, the rod is extended its full length, when the graduations on the front face of the back strip are a continuation of those on the front strip.

The Chicago rod (Fig. 109*h*) is in three sections with slip joints.

The Florida rod (Fig. 109*i*) is a one-piece rod 10 ft. long with a tapering rib fastened to its back. The cross-hatched portions of the face are in red. It is equally well adapted to short and to long sights.

Flexible ribbons of enameled and waterproofed fabric which are graduated in some of the patterns illustrated above (as well as others



(g) (h) (i)

Fig. 109g–i.—Self-reading rods.

not shown) may be obtained from dealers. Such a ribbon fastened to a plain wooden strip makes a serviceable and accurate leveling rod.

109b. Target Rods.—With the target rod, the leveler signals the rodman to slide the target up or down until it is bisected by the line of sight. With the target clamped in this position, the rodman,

leveler, or both observe the indicated reading. Usually the target is equipped with a vernier or other device by means of which fractional measurements of the rod graduations may be read without estimation. The principal advantage of the target rod, in most leveling operations, is that mistakes are less likely to occur, particularly if both rodman and leveler read the rod. Under certain conditions its use materially facilitates the work; for example, when very long sights are taken, when the rod is partially obscured from view, or when it is necessary to establish a number of points all at the same elevation. When it is desired simply to secure readings on points of unknown elevation under the normal conditions of leveling, however, the use of the target rod greatly retards progress without adding much, if anything, to the precision obtained.

The Philadelphia rod (Fig. 109*g*) described in the preceding article, although designed as a self-reading rod, may also be used as a target rod. Lugs on the target engage in a groove on either side of the front strip. For readings on the lower half of the rod the target is moved in these grooves to the desired position. The reading is made by means of a vernier attached to the target. Graduations on the back of the rear strip are a continuation of those on the front strip and read downwards. On the back of the top sleeve is a vernier employed for observations with the rod extended. For readings greater than can be taken with the "short" rod, the target is set at the uppermost graduation on the face, and the rod is extended until the target is bisected by the line of sight. The vertical distance from foot of rod to target is then indicated by the back vernier reading. The verniers read to thousandths of feet.

The New York rod is similar to the Philadelphia rod except in the manner in which it is graduated. It is unpainted, and distances from the foot of the rod are indicated by fine lines at intervals of 0.01 ft. and by longer lines and numbers at intervals of 0.1 ft. It cannot be used as a self-reading rod and is chiefly employed in building construction for setting grades or under poor lighting conditions. The graduations for reading the rod in the extended position are on one side of the back strip, and the accompanying vernier is cut in the wood of the front strip.

The architect's rod is similar to the New York rod but is graduated in $\frac{1}{8}$ inches and is equipped with verniers reading to $\frac{1}{64}$ in. Its use is confined to building construction.

109c. Targets.—The usual target (Fig. 109*j*) is a circular disk about 5 in. in diameter with horizontal and vertical lines formed by the junction of alternate quadrants of red and white. A rectangular space in the front of the target exposes a portion of the rod to view

so that readings may be taken. The attached vernier fits closely to the rod, its zero point or index being at the horizontal line of the target.

109d. Verniers.—The vernier is a small scale placed parallel and in contact with the main scale by means of which fractional parts of the least division on the rod may be accurately measured. Using the Philadelphia rod as an example, the main scale is divided into hundredths of feet. If a plain target were used and it were set so that its index lay between, let us say, 6.15 ft. and 6.16 ft., a reading to thousandths of a foot would necessarily be estimated and would be uncertain in the last decimal place. Thus, one person might read 6.152 ft. and another might read 6.153 ft. If a vernier target were employed the reading could be made to the nearest thousandth of a foot without estimation and would be correct in the last decimal place. The accuracy of the vernier depends upon the fact that the eye can more closely determine when two lines coincide than it can determine by estimation the distance between two parallel lines.

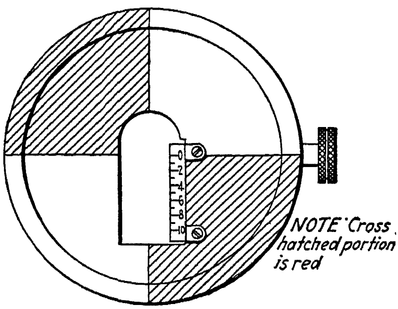


FIG. 109*j*.—Rod target.

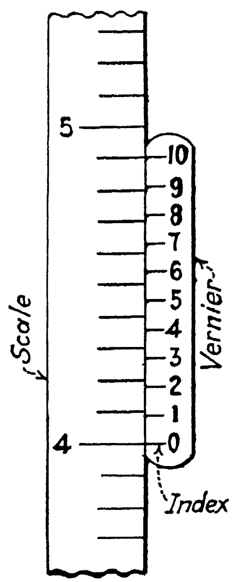


FIG. 109*k*.—Direct vernier.

Verniers used on leveling rods are of two types: (1) the *direct vernier*, which is so graduated that there is one more space within its length than exists in the corresponding distance on the main scale (Fig. 109*k*); and (2) the *retrograde vernier*, which has *one less* space within its length than exists in the corresponding distance on the main scale (Fig. 109*m*).

The direct vernier is so graduated that 10 spaces are equal to 9 spaces or 0.09 ft. on the scale. Thus each space on the vernier is equal to 0.009 ft. In Fig. 109*k* the index of the vernier is set at 0.400 ft. If the vernier were moved upward 0.001 ft., its gradua-

tion numbered 1 would coincide with 0.41 ft. on the scale, and its index would be at 0.401 ft.; if it were moved upward 0.002 ft., its graduation numbered 2 would coincide with 0.42 ft. and the index would be at 0.402 ft. It is thus seen that the position of the index to the vernier is determined to thousandths of feet without estimation simply by noting where a graduation on the vernier coincides with one on the scale.

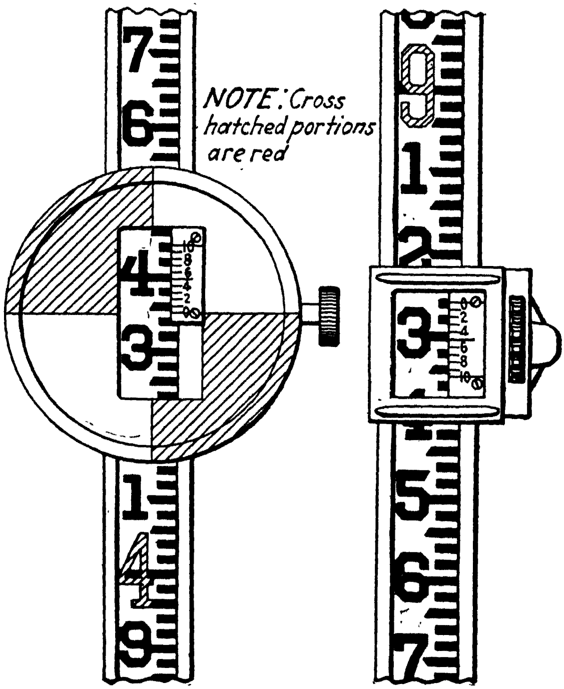


FIG. 109*l*.—Direct vernier settings.

In general, if n is the number of spaces on the vernier whose total length is equal to that of $(n - 1)$ spaces on the scale, and if v is the length of a space on the scale, then the fineness of reading or *least count* of the vernier is $\frac{v}{n}$.

Figure 109*l* illustrates a setting of the vernier on the target and also on the back of a Philadelphia rod. The rod reading indicated by the target (4.347 ft. in figure) is determined by first observing the position of the vernier index on the scale to hundredths of feet (4.34 in figure), next by observing the number of spaces *on the vernier* from the index to the coinciding graduations (7 spaces in figure), and finally by adding the

vernier reading (0.007 ft. in figure) to the scale reading (4.34 ft.). It should be noted that the coinciding graduation on the rod scale *does not* indicate the rod reading. Observations of the vernier on the back of the rod are made in the same manner except that readings of both scale and vernier are made by counting down the rod. Thus in Fig. 109*l* the scale reading is seen to be 9.26 ft. and the vernier reading is 0.004 ft.; hence the rod reading is 9.264 ft. It will be observed that the readings of the direct vernier increase in the same direction as the scale readings, that is, up the rod when the target is set on the "short rod" and down the rod when the "long rod" is used.

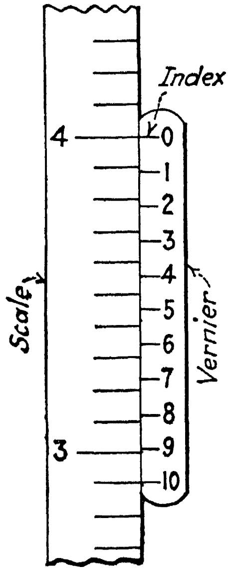


Fig. 109*m*.—Retrograde vernier.

Figure 109*m* illustrates the manner in which the retrograde vernier is graduated. Ten spaces on the vernier are equal to 11 spaces or 0.11 ft. on the scale; hence each space on the vernier is equal to 0.011 ft. and the difference in length between one space on the vernier and one on the scale is 0.001 ft. The index is set at 0.400 ft. If the index is moved upward 0.001 ft. the vernier graduation numbered 1 will coincide with 0.39 ft.; if moved up 0.002 ft., the vernier graduation numbered 2 will coincide with 0.38 ft.; and so on. Thus the retrograde vernier is read in the same manner as the direct vernier.

In general, for the retrograde vernier, if n is the number of spaces on the vernier in a length of $(n + 1)$ spaces on the scale, and v is the length of one space on the scale, then the least count of the vernier is $\frac{v}{n}$. It will be noted that the vernier readings are counted in the *opposite* direction to rod readings; that is, down on the target vernier, and up on the back vernier.

The only advantage of the retrograde vernier is that its spaces are longer and hence it can be read somewhat more easily than the direct vernier. For leveling rods, both types of vernier are in common use.

109e. Topographer's Rod (Fig. 109*n*).—This rod is especially adapted for use on topographic surveys where the contours are located directly with the hand level. The distinguishing features of the rod are graduations numbered in either direction from a point near the middle of the rod, and an adjustable base by means of which the zero point on the rod may be fixed at the height of the observer's eye

above the ground. The rod is graduated in half-feet and numbered each foot. The numbers are large and the graduating marks are heavy so that they can be readily distinguished at considerable distances. Readings to tenths of feet are made by estimation. Usually the rod is a home-made device, and frequently the feature of the adjustable base is not incorporated, the rod being so graduated that for one particular person the zero point is at the proper distance from the base.

The hand leveler, by standing with the base of the rod held at his toe makes the proper adjustment to bring the zero point to his eye. The rod reading for any given position of the rod therefore indicates the distance above or below the point on which the hand leveler stands when the observation is made.

The hand level is sometimes fixed at the top of a stick or a Jacob's staff, about five feet long.

109f. Tape Rod.—In its most improved form (Fig. 109*o*), the tape rod consists of a one-piece wooden rod with a metal roller near each end. Passing over these rollers is a continuous metal ribbon on which is a painted scale graduated to half-tenths of feet. Readings to hundredths of feet may be made by estimation. The bearings of one of the rollers are held in position by spiral springs which maintain a constant tension in the ribbon. In direct leveling, the rod is held so that the numbers increase downward. In cross-sectioning, it is sometimes held one end up and sometimes the other, for reasons that will be explained later. The tape may be moved on the rollers so that a particular graduation occupies a desired position with

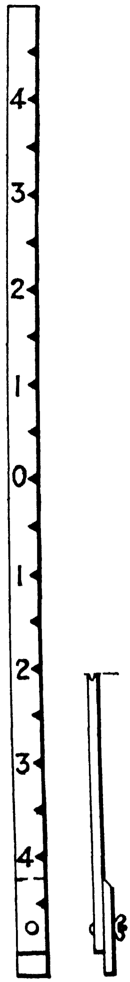


Fig. 109*n*.—Topographer's rod.

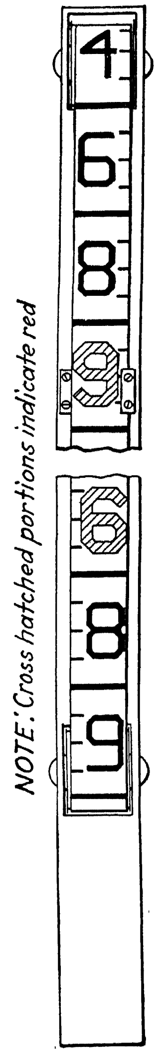


Fig. 109*o*.—Tape rod.

respect to the foot of the rod. A clamp is provided for clamping the ribbon to the rod.

The tape rod should be more generally employed than it now is. It is not suitable for accurate leveling, but for work of ordinary precision where many elevations are to be determined, its use results in a large saving of time and in fewer mistakes. It is very useful in connection with levels for earthwork.

The procedure in ordinary leveling is as follows: With the rod on a point of known elevation (say 751.2) the leveler directs the rodman to move the ribbon until a corresponding rod reading (say 11.2) is cut by the line of sight, when the ribbon is clamped. Since the numbers increase down the rod, the rod reading (say 9.5) at any point of unknown elevation indicates the elevation directly by simple addition $(740.0 + 9.5 = 749.5)$. If mental calculations are necessary they merely consist in adding the rod reading to some multiple of 10 ft.

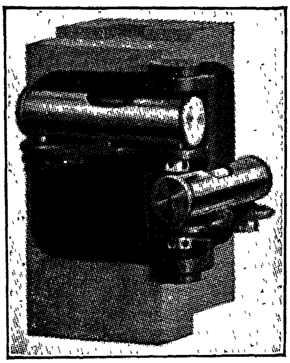


FIG. 109*p*.—Rod level.

109g. Rod Levels.—The rod level is an attachment for indicating the verticality of the leveling rod. One type (Fig. 109*p*) consists of a hinged casting on each wing of which is mounted a level tube. The level is held so that the plane faces of its two wings are in full contact with adjacent sides of the rod. When both of the bubbles are centered, the rod is plumb. It is suitable for use with any type of rod. The hinge makes it possible to fold the level compactly when it is not in use.

Another type consists of a circular or "bull's-eye" level vial mounted on a metal angle or bracket which is attached by screws to the side of the rod.

110. Turning Points.—A metal plate or pin which will serve temporarily as a solid or stable object on which the leveling rod may be held at turning points (the purpose of which will later be described) is a useful, if not necessary, part of the leveling equipment for careful lines of differential levels. The iron pin shown in Fig. 110*a* is adapted for use in firm ground. Often a railroad spike is used.

In soft ground the steel plate of Fig. 110*b* makes a satisfactory turning point. The plate is also adapted for use where the ground is so solid as to make driving the pin impossible or at least impracticable, as along highways. Under these conditions the plate with the dogs at its corners acts as a tripod, no special attempt being

made to secure bearing between the lower surface of the plate and the ground.

111. Numerical Problems.

1. What is the combined effect of the earth's curvature and mean atmospheric refraction in a distance of 300 ft.? In a distance of 3,000 ft.? In a distance of 6 miles? In a distance of 60 miles?

2. Two points, A and B, are each distant 2,000 ft. from a third point from which vertical angles to A and B are taken. The vertical angle to A is $+3°21'$ and that to B is $+0°32'$. What is the difference in elevation between A and B?

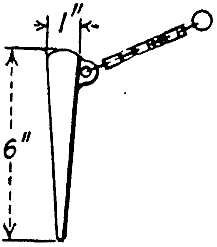


FIG. 110*a*.—Turning point.

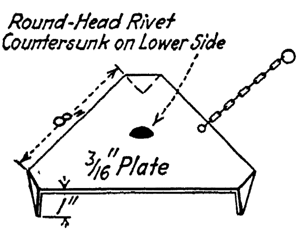


FIG. 110*b*.—Turning plate.

3. Let A be a point of elevation of 100.00 ft. and B and C be points of unknown elevation. By means of an instrument set 4.00 ft. above B, vertical angles are observed, that to A being $-1°55'$ and that to C being $+3°36'$. If the horizontal distance AB is 1,500 ft. and the horizontal distance BC is 5,000 ft. what are the elevations of B and C, making due allowance for earth's curvature and atmospheric refraction?

4. Two points, A and B, are 1,000 ft. apart. The elevation of A is 615.03 ft. A level is set up on the line between A and B and at a distance of 250 ft. from A. The rod reading on A is 9.15 and that on B is 2.07. Making due allowance for curvature and refraction what is the elevation of B? What would be the magnitude and sign of the error introduced if the correction for curvature and refraction were omitted?

5. Two points, A and B, are 400 ft. apart and the elevation of A is 615.037 ft. A level is set up on line and distant 100 ft. from A in the direction of B. The rod reading on A is 5.812 and on B is 7.358. What is the elevation of B, neglecting the correction due to curvature and refraction? What is the magnitude and sign of the error introduced by not considering the effect of curvature and refraction?

6. A sight is taken with an engineer's level at a rod held 300 ft. away and an initial reading of 6.323 ft. is observed. The bubble is then moved through five spaces on the level tube, when the rod reading is 6.589 ft. What is the sensitiveness of the level tube in seconds of arc? What is the radius of curvature of the level tube if one space is $\frac{1}{10}$ in.?

7. In viewing the rod through the telescope of a level the magnified image of the portion between 5.00 and 5.20 apparently covers the unmagnified image between 2.1 and 8.4. What is the magnifying power of the telescope?

8. Design a direct vernier reading to thousandths of feet, each space on the rod being equal to 0.025 ft.

9. Design a retrograde vernier, other conditions remaining as in problem 8.

10. Design a direct vernier for an architect's rod that shall have a least count (*a*) of $\frac{1}{32}$ in.; (*b*) of $\frac{1}{64}$ in.

112. Field Problems.

PROBLEM 1. MAGNIFYING POWER OF TELESCOPE

Object.—To determine the number of diameters an object viewed through the telescope is magnified.

Procedure.—(1) Sight at the rod held erect about 15 ft. in front of the instrument. (2) With both eyes open turn the instrument until the images, as seen by the naked eye and as seen through the telescope, appear to fall one upon the other. (3) Compare 0.1 ft. on the rod as seen through the telescope with a space as seen with the naked eye. The number of tenths apparently covered on the unmagnified image is the magnifying power of the telescope.

Hints and Precautions.—(1) Some practice will probably be necessary before the student will be able to see both images at the same time. First sight the image through the telescope; then still keeping the image in sight, look at the rod with the other eye. After a little practice both images will appear distinct. Turning the level slightly, if necessary, will cause the unmagnified image to fall upon the magnified image. For observation select the tenth on the magnified image which is located wholly in the field of vision. Observe the reading of the upper line of this tenth on the unmagnified image, and then observe the lower. The difference of these readings in tenths of feet will be the magnifying power.

PROBLEM 2. RADIUS OF CURVATURE OF LEVEL TUBE

Object.—To determine, in the field, without the use of special apparatus, the radius of the curvature of the level tube of transit or level.

Procedure.—(1) Hold the rod on a solid point 300 ft. from the instrument. With one end of the bubble at a division near the end of the tube, take a careful rod reading to the nearest 0.001 ft. Note the exact position of each end of the bubble. (2) Adjust the foot-screws until the other end of the bubble falls near the other end of the tube. Take another rod reading, and measure the exact distance traversed by each end of the bubble. (3) Determine the bubble movement (this should be expressed to the nearest 0.001 ft.) and the difference between the two target readings or target movement. (4) In this manner obtain a series of five bubble

movements and their corresponding target movements. (5) Compute the radius of curvature by the following formula: $R = \frac{b}{t}D$, in which R is the radius of curvature, D is the distance from the instrument to the rod, b is the mean of the five bubble movements, and t is the mean of the five target movements. (6) Compute the value of one division of the level tube in seconds of arc.

CHAPTER VIII

DIRECT LEVELING

USE AND ADJUSTMENT OF THE LEVEL

113. Setting Up the Level.—When observations are to be made the level is placed in a desired position with tripod legs well spread and firmly pressed into the ground and with the tripod head somewhere near level. The telescope is brought over one pair of leveling screws and the bubble is approximately centered; then the process is repeated with the telescope over the other pair. By repetition of this procedure the leveling screws are manipulated until the bubble remains centered, or nearly so, for any direction in which the telescope is pointed. If the instrument is in adjustment the line of sight is then horizontal.

Following are items deserving attention:

1. The legs should be spread at such an angle that the tripod is stable and that the object may be viewed through the telescope from a convenient posture.

2. The friction bearings between tripod head and legs should be tightened by means of the wing nuts until each leg when held horizontally will barely fall from its own weight.

3. The leveling screws should be tightened no more than is necessary to secure firm bearing. If the bubble is considerably off center it is best to leave all four screws rather loose until it is brought approximately level. Usually the final centering of the bubble will be facilitated by turning one screw rather than by attempting to manipulate two opposite screws at the same time.

114. Reading the Rod.—For accurate observations the rod is held on some well-defined point of a stable object. The rodman holds the rod vertical either by observing the rod level or by estimation. The leveler revolves the telescope about the vertical axis until the rod is about in the middle of the field of view, focuses the objective for distinct vision, and carefully centers the bubble. If the self-reading rod is used, the leveler notes and records the reading indicated by the line of sight (apparent position of the horizontal cross-hair on rod). As a check he again observes the bubble and the rod. If the target rod is used, the procedure is identical except that the target is set by the rodman as directed by the leveler.

For less accurate leveling, as for example when rod readings for points on the ground are determined to the nearest 0.1 ft., the observations usually are not checked and proportionally less care is exercised in keeping the rod vertical and the bubble centered, always bearing in mind the errors involved and the precision with which measurements are desired.

The following are pertinent suggestions:

1. No part of the body or clothing should come in contact with the level when an observation is being made, and care should be taken not to step near the feet of the tripod on soft or yielding ground. Between the times when the bubble is centered and the rod is read, the leveler

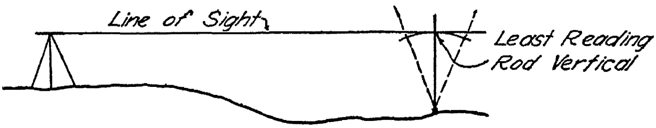


FIG. 114.—Waving the rod.

should not move his feet except when the stability of the ground is assured.

2. The eyepiece should be correctly focused on the cross-hairs before observations are begun, and the leveler should move his head up and down slightly while viewing the rod to assure himself that no parallax is present (Art. 104*a*).

3. If the air is calm the rodman can plumb the rod quite accurately by balancing it upon the point on which it is held. By means of the vertical cross-hair the leveler can determine when the rod is held in a vertical plane passing through the instrument but has no way of telling whether it is not tipped forward or backward in this plane. If it is in either of these positions, the rod reading will be greater than the true vertical distance, as illustrated by Fig. 114. To eliminate this error, the rodman slowly swings the rod forward and backward as shown in the figure, and the leveler takes the least reading, which occurs when the rod is vertical. This movement is called *waving the rod.* The larger the rod reading, the larger the error due to the rod's being held at a given inclination, and hence it is more important to wave the rod for large readings than for small readings.

115. Adjustments.—Regardless of the precision of manufacture, all levels, as well as other surveying instruments, in process of use require certain field adjustments from time to time. In the interest of accuracy it becomes an important duty of the surveyor to test his instrument at short intervals and to make with facility such adjustments as are found necessary.

With a few exceptions the nuts and heads of screws are capstan shaped and hence are tightened with a pin. The instrumentman

should possess an adjusting pin which fits the holes accurately, and should carry it at all times. To prevent damage to the instrument, it is imperative that the screws should not be unduly tightened. This is particularly true of those threading into the cross-hair ring; it requires no considerable force to strip their threads or to twist off the screw. When a screw has been set to a firm bearing, nothing is gained by further tightening.

The inverting eyepiece is incapable of lateral adjustment. The erecting eyepiece, when adjustable, is set so that the intersection of the cross-hairs appears in the center of the field of view. The ring through which the inner end of the eyepiece moves is controlled by screws which are manipulated in the same manner as those for adjusting the cross-hairs. On some levels these screws are slot-headed and are covered by a cylindrical sleeve. When the sleeve is unscrewed, the heads of the screws are exposed. This adjustment has no effect on the precision of the measurements, and ordinarily requires no attention except when the telescope is being assembled.

The cross-hair ring should be rotated about its axis by loosening two adjacent screws slightly and then tapping one of the screws lightly. The ring should be tightened by turning the same two screws in order that the position of the intersection of the cross-hairs will be disturbed a minimum amount.

The cross-hair ring is adjusted vertically or horizontally by first loosening one screw and then tightening the opposite one. The ring is moved towards the screw which is being tightened.

In some instances one adjustment is likely to be altered by, or depends upon, some other adjustment made subsequently. For example, the lateral movement of the cross-hair ring may likewise produce a small rotation. And the lateral adjustment of the level tube depends upon the vertical adjustment. Hence if an instrument is badly out of adjustment, related adjustments must be repeated until they are gradually perfected.

116. Adjustments of the Dumpy Level.—For a dumpy level in perfect adjustment the following relations should exist:

1. The axis of the level tube should be parallel with the optical axis and line of sight.

2. The horizontal cross-hair should lie in a horizontal plane when the instrument is leveled.

3. The vertical axis should be perpendicular to the axis of the level tube and the line of sight.[1]

[1] Also the optical axis, the axis of the objective slide, and the line of sight should coincide, but for the type of level commonly used in the United States, the axis of the objective slide and the optical axis are

The parts which are capable of, and require, adjustment are the cross-hairs and the level tube. The basis for adjustments is the vertical axis. The adjustments are as follows:

1. *To Make the Axis of the Level Tube Perpendicular to the Vertical Axis.*—Set up the level and approximately center the bubble over each pair of foot screws; then bring the bubble carefully to center over one pair. Revolve the level 180° about its vertical axis. If the level tube is in adjustment, the bubble will retain its position. If the tube is not in adjustment the displacement of the bubble indicates double the actual error, as is shown by Fig. 116*a*. If $(90° - \alpha)$ represents the angle between the vertical axis and axis of level tube,

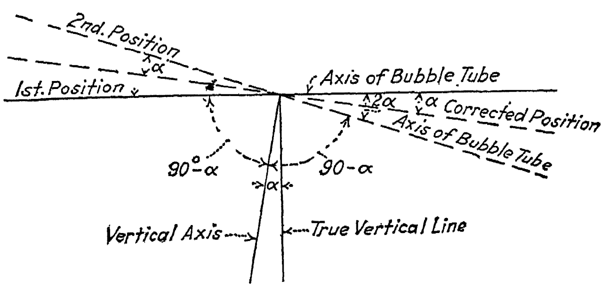


FIG. 116*a*.

then when the bubble is centered the vertical axis makes an angle of α with the true vertical. When the level is reversed, the bubble is displaced through the arc whose angle is 2α. Hence the correction is the arc whose angle is α. The correction is made by bringing the bubble halfway back to the center by means of the capstan-headed nuts at one end of the tube. The instrument is then releveled with the foot screws, and the process is repeated until the adjustment is perfected. Usually three or four trials are necessary. As a final check the bubble should remain centered over each pair of foot screws.

2. *To Make the Horizontal Cross-hair Lie in a Plane Perpendicular to the Vertical Axis (and Hence Horizontal When the Instrument Is Leveled).*—Sight the horizontal cross-hair on some clearly defined point (as *A*, Fig. 116*b*) and rotate the instrument slowly about its vertical axis. If the point appears to travel along the cross-hair no adjustment is needed.

If the point departs from the cross-hair and takes some position as *A′* on the opposite side of the field of view, rotate the cross-hair

permanently fixed perpendicular to the vertical axis by the manufacturer, and no provision for further adjustment is made.

ring until by further trial the above condition is satisfied. The instrument need not be level when the test is made.

3. *To Make the Line of Sight Parallel with the Axis of the Level Tube (Two-peg Method).*—Set two stakes 200 to 300 ft. apart on approximately level ground. Set and level the instrument in a position such that the eyepiece is ½ in. or less in front of the rod

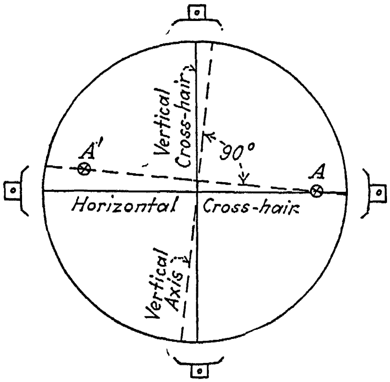


Fig. 116*b*.

held on one of the stakes as at A, Fig. 116*c*. With the rod held at A, take a rod reading a by sighting through the objective end of the telescope (the eyepiece next to the rod). The cross-hairs will not be visible, but the field of view will be so small (one or two hundredths of a foot) that its center may be determined within one or two

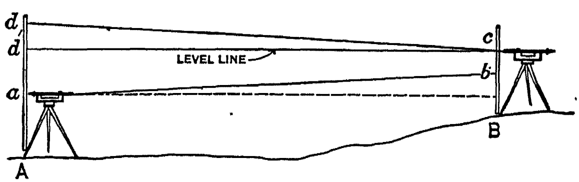


Fig. 116*c*.—The two-peg test.

thousandths of a foot by holding the point of a pencil on the rod; and this may with sufficient accuracy be called the true rod reading.

Move the rod to the other stake B, and take a rod reading b in the usual manner.

Move the instrument to B, set up as before, and take rod readings c and d.

If $e = dd'$ represents the error in the line of sight for the distance A to B, then considering first the rod readings taken with the instrument at A, the true difference in elevation is

$$\text{Diff. El.} = a - (b - e) \tag{1}$$

and considering the rod readings with the level at B is

$$\text{Diff. El.} = (d - e) - c. \tag{2}$$

Adding (1) and (2) there results

$$\text{Diff. El.} = \frac{(a - b) + (d - c)}{2} \tag{3}$$

Equation (3) shows that the true difference in elevation is the mean of the difference between rod readings taken with the instrument at A and the difference between those taken with the instrument at B.

If the two differences in elevation thus determined are equal, that is, $(a - b) = (d - c)$, the line of sight is in adjustment. If not, then the correct rod reading at A for instrument with position unchanged at B is

$$d' = c + \frac{(a - b) + (d - c)}{2}\text{[1]} \tag{4}$$

The adjustment is made by moving the cross-hair ring vertically until the line of sight cuts the rod at d'. The preceding steps are then repeated as a check on the accuracy of the adjustment.

It should be carefully noted that Eqs. (3) and (4) must be solved algebraically, *i.e.*, with due regard to signs; otherwise, if the error of adjustment should happen to be greater than the difference in elevation of the two points, the mean difference will not be the true difference in elevation.

116a. A modified type of the dumpy level not previously described but which is used to some extent in the United States has the telescope hinged to one end of the level bar and resting on the point of a micrometer screw in the other end of the level bar. Levels of at least one American maker and of several foreign manufacturers are of this pattern, and the precise level of the U. S. Geological Survey and the U. S. Coast and Geodetic Survey is a refined form of this type. The level tube is attached to the telescope in the usual manner or it may be placed either at the side or above. The screws controlling the cross-hair ring are usually protected by a metal sleeve so that they cannot be disturbed, and the line of sight is made to coincide with the optical axis by the manufacturer, and is supposed to require no further attention.

[1] Strictly speaking, the effect of the earth's curvature and atmospheric refraction should be added (see Art. 97). For a length of sight of 200 ft. this correction would amount to 0.0008 ft. and for a distance of 300 ft. would be about 0.0018 ft. These quantities are so small as to be of no consequence in ordinary leveling and are negligible for that reason.

The only field adjustment consists in making the axis of the bubble tube parallel with the line of sight. This is performed by the two-peg method as just described, except that the line of sight is set on the true rod reading by means of the micrometer screw, and then the bubble is centered by means of the capstan nuts at one end of the level tube.

Some surveyors prefer to determine the true difference in elevation between the two pegs by setting up midway between them. Regardless of whether or not the line of sight is inclined, the difference between rod readings will give the correct difference in elevation, as may readily be shown by a sketch. Having found the difference in elevation, the leveler then moves to one end of the line and proceeds to make the adjustment of the line of sight as described in Art. 116.

Instead of viewing the near rod through the objective end of the telescope, unless the instrument is badly out of adjustment, the level may be set up 6 or 8 ft. from the near rod and the near rod reading may be observed in the customary manner.

117. Adjustments of the Wye Level.—For a wye level in perfect adjustment the following relations should exist:

a. The axis of the objective slide, the optical axis, the line of sight, and the axis of the wyes should coincide.

b. The axis of the level tube should lie in the same plane with and should be parallel with the axis of the wyes.

c. The horizontal cross-hair should be truly horizontal when the instrument is level.

d. For convenience in leveling, the vertical axis should be perpendicular to the axis of the level tube and the line of sight.

For the wye level with objective slide permanently fixed so far as lateral movement is concerned, the axis of the wyes, the optical axis, and the axis of the objective slide are fixed in the proper relation by the manufacturer, which relation is presumed to be maintained without further attention. The remaining relations are established by the following adjustments:

1. *To Make the Axis of the Level Tube Lie in the Same Plane with the Axis of the Wyes.*—Set up and level the instrument. Raise the wye clips and rotate the telescope a few degrees in the wyes. If the relation exists, the bubble will remain centered.

If the bubble moves, bring it back to the center by means of the lateral adjusting screws at one end of the level tube.

2. *To Make the Axis of the Level Tube Parallel with the Axis of the Wyes.*—Level the instrument carefully, raise the wye clips, lift the telescope from the wyes, and turn it end for end. If the desired relation exists, the bubbles will remain centered.

If the bubble moves, the displacement is double the error (Fig. 116*a*). Hence bring it back halfway to the center by means of the vertical adjusting nuts at one end of the level tube. Level the instrument by means of the foot screws, and repeat the test until the adjustment is perfected.

3. *To Make the Horizontal Cross-hair Lie in a Plane Perpendicular to the Vertical Axis (and Hence Horizontal When the Instrument is Level).*—This adjustment is exactly the same as for the dumpy level except that for some instruments the adjustment may be made by rotating the telescope in the wyes instead of the cross-hair ring being rotated in the barrel of the telescope. The telescope is fixed in the desired position by means of an adjustable stop attached to one of the wyes.

4. *To Make the Line of Sight Coincide with the Axis of the Wyes (and Thus Parallel with the Axis of the Level Tube).*—Raise the wye clips, sight the intersection of the cross-hairs on some well-defined point, and clamp the vertical axis. Revolve the telescope 180° in the wyes. If the line of sight still remains on the point, the desired relation exists.

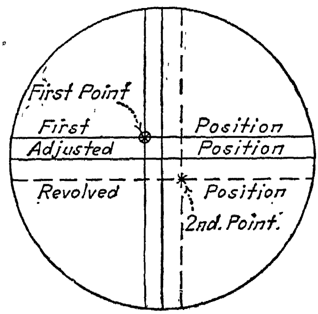


Fig. 117*a*.

If not, the cross-hair ring is adjusted until the line of sight takes a position midway between the point first sighted, and the one defined by the line of sight with the telescope in the revolved position (see Fig. 117*a*). The above process is repeated until the proper relation is obtained. The adjustment is made by manipulating opposite screws, first bringing one cross-hair and then the other to what is estimated to be its correct position. If the required movement is large, however, it is best to loosen two adjacent screws slightly before attempting to aline the cross-hairs.

5. *To Make the Axis of the Wyes Perpendicular to the Vertical Axis.* Inasmuch as the preceding adjustments have established parallelism or coincidence between the axis of the level tube, the line of sight, and the axis of the wyes, this adjustment in no wise adds to the precision of observations. The adjustment makes it possible, however, to level the instrument so that the bubble will remain centered for any direction in which the telescope may be pointed.

The test is made by leveling the instrument and revolving the telescope 180° about the vertical axis. If the bubble moves, one-half its displacement indicates the error (see Fig. 116*a*). The correction

is made by means of the capstan nuts controlling the vertical position of one of the wyes. The adjustment is seen to be identical with that of the level tube of the dumpy level, with the exception that for the wye level, both the telescope and the level tube are moved in a vertical plane.

117a. The adjustments of the level with adjustable objective slide are identical with those just described, but in addition, the objective slide may occasionally require attention. The screws controlling the inner ring, through which the slide passes, are usually slot-headed and are protected by a metal sleeve, which when unscrewed exposes the screw heads to view. The ring may be moved laterally in the same

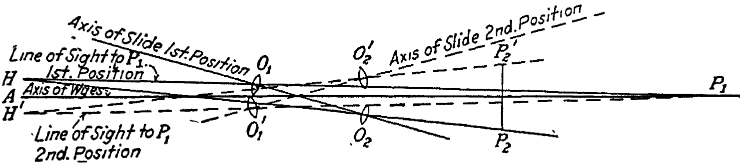


FIG. 117*b*.

manner as the cross-hair ring. Assuming that the telescope has been properly constructed and that the objective has not been disturbed, the optical axis and the axis of the slide will coincide.

In Fig. 117*b* let P_1 be some distant point on the axis of the wyes Let O_1O_2 represent one position of the axis of the objective slide, and $O_1'O_2'$ represent a second position after the telescope has been rotated 180° in the wyes. Suppose O_1 and O_1' represent the corresponding positions of the optical center of the objective when it is focused on P_1. From the figure it is evident that there will be one position of the intersection of the cross-hairs, namely at H with the telescope in its normal position and H' when it is rotated 180°, for which the line of sight will continuously strike the point P_1 as the telescope is rotated. Yet neither the intersection of the cross-hairs nor the optical center of the objective need necessarily be on the axis of the wyes. In other words, the test of adjustment 4 of Art. 117 may be satisfied at a given distance and yet neither parallelism nor coincidence between the line of sight and the axis of the wyes is assured.

Suppose the cross-hairs have been so adjusted that the line of sight will continually remain on a distant point P_1, that their intersection is at H and that the line of sight passes through the center of the objective at O_1. If now the objective is focused for a short distance (say 15 or 20 ft.) the objective will be drawn out to the position O_2 and the line of sight will be defined by the line HO_2P_2. If the axis of the objective slide is out of adjustment as indicated in the figure, the line of sight will fall at P_2' when the telescope has been rotated 180° in the wyes. Hence, after the cross-hairs have been adjusted as previously described for a distant

point, sight on an object a short distance away and rotate the telescope 180° in the wyes. If the line of sight remains on a point, the objective slide is in adjustment. If not, a correction of one-half the apparent error is to be applied by moving both the cross-hair ring and the object slide ring. The separate amount that each ring should be moved depends on a number of factors and is not readily calculated. Hence the corrections are applied by estimation until the condition that the intersection of the cross-hairs should remain on a point is satisfied both for a long and for a very short distance. Figure 117*b* indicates the directions in which the intersection of the cross-hairs and the center of the objective must be moved to place them on the axis of the wyes. When the objective is to be moved in one direction, the object slide ring must be moved in the other.

117b. Regardless of the type of level, when the rings or the wyes become worn the adjustments described in Art. 117 are inadequate, and the two-peg test must be made as for the dumpy level. The

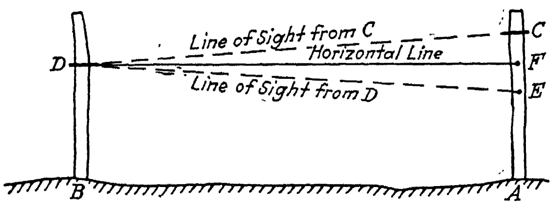


FIG. 118.—Adjustment of the hand level.

common procedure is first to perform the adjustments for the cross-hairs in the usual manner, making the line of sight coincide as nearly as may be with the axis of the wyes. Then the true difference in elevation between two points having been determined by the "peg method," the line of sight is set to the proper rod reading for a horizontal line with the foot screws, and the bubble is centered by means of the capstan nuts at one end of the level tube.

118. Adjustment of the Hand Level.—The simplest procedure is to hold the hand level alongside an engineer's level which has been leveled and sighted at some well-defined point. The line of sight of the hand level should strike the same point when the bubble is centered.

The Locke level is adjusted by means of the screw at one end of the level tube, which screw moves the cross-wire defining the line of sight. The adjustment of the Abney level is made by raising or lowering one end of the level tube until the bubble is centered, the index having first been set at zero on the graduated arc.

The hand level may be used to establish a horizontal line by employing the principle of the two-peg test (see Art. 116). Let A and

B (Fig. 118) be two posts, trees, or other convenient objects on nearly level ground. The level is held at *C* and with bubble centered is sighted to the point *D*. The level is then held at *D*, and the point *E* is established in a similar manner. The distance *EC* represents double the error. The point *F* established halfway between *E* and *C* is therefore on the horizontal line through *D*.

DIFFERENTIAL LEVELING

119. General.—Every construction enterprise of magnitude requires the establishment of more or less permanent monuments of known elevation, called *bench marks*, to which levels in a given locality may be referred. The nature, location, and interval between monuments of this character depend upon the uses for which they are intended. Throughout the United States—in nearly all cities, and at scattered points in the less populated areas—are the permanent bench marks established by the U. S. Geological Survey. They are of bronze set securely in stone or concrete and marked with the elevation above mean sea level. Though intended primarily as control points for the topographic mapping carried on by the Survey, they are very useful as points from which other bench marks may be established either for private or public enterprises. Similar monuments have been established by various other government, state, and municipal agencies as well as by such private interests as railroads and water companies; so that the surveyor has not far to go before he can find some point of known elevation.

For a new project, levels are run from a bench mark of known or assumed elevation to scattered points in desirable locations for future reference. Such points may be either natural or artificial objects of a nature to be readily identified and of sufficient permanence to serve the purpose for which they are intended. When the elevation of such points has been determined and their location has become a part of the record of the survey, the points become bench marks. The operation of leveling to establish bench marks is called *differential leveling*. Besides the permanent monuments already mentioned, objects frequently used as bench marks are stones, stakes and pipes driven in the ground, nails in roots of trees, painted marks on street curbs, and spikes in telegraph poles. Differential leveling between two bench marks requires a series of set-ups of the instrument along the general route and, for each position of the level, a rod reading back to a point of known elevation and forward to a point of unknown elevation.

120. Definitions.—From the preceding article a *bench mark* (B.M.) is seen to be a definite point of more or less permanent character whose elevation is known.

A *turning point* (T.P.) is an intervening point between two bench marks, upon which foresight and backsight rod readings are taken. It may be a pin or plate (see Art. 110) which is carried forward by the rodman after observations have been made, or it may be any stable object, as a street curb, railroad rail, or stone. The nature of the turning point is generally indicated in the notes, but no record is made of its position.

A *backsight* (B.S.) is a rod reading taken on a point of known elevation, as a bench mark or a turning point. Generally, though not always, it will be taken with the level sighting back along the line, hence the name.

A *foresight* (F.S.) is a rod reading taken on a point whose elevation is to be determined, as on a turning point or on a bench mark that is to be established.

The *height of instrument* (H.I.) is the elevation of the line of sight of the telescope when the instrument is leveled.

121. Procedure in Differential Leveling.—In Fig. 121, B.M.$_1$ represents a point of known elevation, as a bench mark, and B.M.$_2$ represents a bench mark to be established some distance away, the

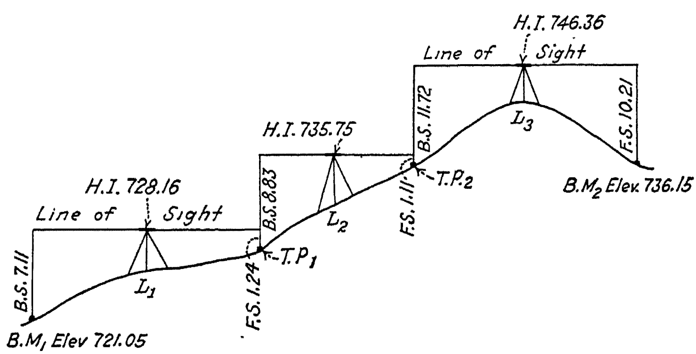


FIG. 121.—Differential leveling.

elevation of which it is desired to determine. The rod is held at B.M.$_1$ and the level is set up in some convenient location, as L_1, along the general route, B.M.$_1$ to B.M.$_2$. The level is placed in such a location that a clear rod reading is obtainable, but no attempt is made to keep on the direct line joining B.M.$_1$ and B.M.$_2$. A backsight is taken on B.M.$_1$. The rodman then goes forward and, at the direction of the leveler, chooses a turning point T.P.$_1$ at some convenient spot

within the range of the telescope along the general route B.M.$_1$ to B.M.$_2$. It is desirable, but not necessary, that each foresight distance be approximately equal to its corresponding backsight distance. When convenient, the distance L_1-T.P.$_1$ is by estimation made roughly equal to the distance B.M.$_1$-L_1. The chief requirement is that the turning point shall be a stable object at an elevation and in a location favorable to a rod reading of the required precision. The rod is held on the turning point, and a foresight is taken. The leveler then sets up the instrument at some favorable point as L_2, takes a backsight to the rod held on the turning point, the rodman goes forward to establish a second turning point T.P.$_2$; and so the process is repeated until finally a foresight is taken on the terminal point B.M.$_2$.

Referring again to Fig. 121 it will be seen that the *backsight added to the elevation* of a point on which the backsight is taken gives the height of instrument, and the *foresight subtracted from the height of instrument* determines the elevation of the point on which a foresight is taken. Thus if the elevation of B.M.$_1$ is 721.05 ft. and the B.S. is 7.11 ft., then the H.I. with the instrument set up at L_1 is $721.05 + 7.11 = 728.16$. And if the following F.S. is 1.24 ft., the elevation of T.P.$_1$ is $728.16 - 1.24 = 726.92$ ft. Also the difference between the backsight taken on a given point and the foresight taken on the following point is equal to the difference in elevation between the two points. It follows that the difference between the sum of all backsights and the sum of all foresights taken between two bench marks gives the difference in elevation between the bench marks.

When several bench marks are to be established along a given route, each bench mark is made a turning point in the line of levels. Elevations of bench marks are checked sometimes by rerunning levels over the same route but more often by "tying on" to a previously established bench mark near the end of the line or by returning to the initial bench mark. A line of levels which ends at the point of beginning is called a *level circuit*. The final observation in a level circuit is therefore a foresight on the initial bench mark. If each bench mark in a level circuit is also a turning point, and the circuit checks within the prescribed limits of error, it is regarded as conclusive evidence that the elevations of all points in the circuit are correct within prescribed limits. In a level circuit any bench mark might be established by a foresight and the line of levels might be continued without taking a backsight on the point, provided some other object were used as a turning point. The fact that a level circuit checks is no indication that the elevation of a bench mark is correct unless it has been employed as a turning point.

121a. Balancing Backsight and Foresight Distances.—As mentioned in Art. 121, in ordinary leveling no special attempt is made

to balance each foresight distance against the preceding backsight distance. As to whether or not such distances are roughly determined and are approximately balanced between bench marks will depend upon the desired precision. It has already been pointed out (Art. 100) that the effect of the earth's curvature and atmospheric refraction is very slight unless there is an abnormal difference between the backsight and foresight distances. The effect of instrumental errors is likely to be of considerably greater consequence with regard to the balancing of these distances. No matter how carefully adjustments have been performed, the chances are that there is not absolute parallelism between the line of sight and the axis of the level tube, so that if the instrument were perfectly leveled, the line of sight would be inclined always slightly upward or always slightly downward. Evidently the error in a rod reading due to this imperfection of adjustment would be proportional to the distance from the instrument to the rod and for a given distance would be of the same magnitude and sign for a backsight as for a foresight. Since backsights are added and foresights are subtracted, it is clear that instrumental errors of this nature are completely eliminated if, *between bench marks*, the sum of the backsight distances is made equal to the sum of the foresight distances.

In ordinary leveling no special attempt is made to equalize these distances if there is assurance that the instrument is in fair adjustment. Normally, for levels run over flat or gently rolling ground, the line of sight will fall within the length of the rod regardless of the position of the instrument, and the distance between instrument and rod is governed by the optical qualities of the telescope. When the leveler moves forward to set up beyond the point where the rod is held he generally paces or estimates by eye a distance he assumes to be about the proper maximum length of sight; and when the rodman moves forward, he similarly estimates the proper distance from the instrument to the next turning point which he establishes. In leveling uphill or downhill the length of sight is usually governed by the slope of the ground. In order that maximum distances between turning points may be obtained and hence progress be most rapid, the leveler sets up the instrument in a position such that the line of sight will intersect the rod near its top if the route is uphill, or near its bottom if the route is downhill; and he directs the rodman to a similarly favorable location for the turning point.

While between bench marks at the same (or nearly the same) elevation the backsight and foresight distances will tend to balance in the long run, regardless of the character of the terrain, it is instructive to note that in levels which are run between two points having a large

difference in elevation, a very small inclination of the line of sight may normally be expected to produce a marked error unless some attempt is made to balance backsight and foresight distances.

Consider two points 40 miles apart, one being, say, at sea level and the other at an elevation of 5,000 ft.; the intervening ground is of a fairly uniform slope averaging, say, 2½ ft. in 100 ft., so that, starting from the lower point the backsights would be near the top of the rod (say at 11.5 ft.) and the foresights would be near the bottom (say at 1 ft.). If the instrument were placed directly between turning points and the telescope were 4.5 ft. above the ground at each set-up, the backsight distance would on the average be about twice as great as the foresight distance, hence the effect of any error in the line of sight would be twice as great for backsights as for foresights. Suppose that, when the bubble was exactly centered, the line of sight was inclined upward 0.002 ft. in 100 ft. (fair adjustment). Then, due to this error alone, the sum of backsights would be about 2⅔ ft. too large and the sum of foresights would be about 1⅓ ft. too large; hence the calculated elevation of the terminal bench mark would be approximately 1.3 ft. too great. This is an error considerably larger than would ordinarily be permissible except in rough leveling. The example is not an extreme case for some sections of the country. It serves to illustrate how a relatively small error in adjustment may produce a large systematic error in elevation when backsight distances are consistently larger than foresight distances, or *vice versa;* and further, that under certain conditions it is not good practice to neglect the balancing of these distances, at least roughly, even for leveling of relatively low precision and with a level in fair adjustment.

It is also to be observed that the effect of the earth's curvature and atmospheric refraction may be considerable, under the conditions of the example just cited, even though the lengths of sight are within the range of those ordinarily used in leveling.

Thus, if the backsight distances were 300 ft. in length and the foresight distances were 150 ft. in length, the constant error per set-up of the instrument would be 0.0014 ft. or for the distance of 40 miles approximately 0.7 ft. This is about the maximum error that would be permissible in ordinary leveling (see Art. 127). For the example, both the error due to imperfect adjustment and that due to the earth's curvature happen to be of the same sign, hence the resultant error from these two sources is 2.0 ft.

Considering the fact that no instrument is likely to be in perfect adjustment and further that the effect of the earth's curvature is not a negligible quantity, it is clear that for accurate leveling the equalization of backsight and foresight distances between bench marks is a necessity and hence these distances, as well as the rod readings, become a part of the record of accurate leveling operations. In less refined leveling, distances are usually determined by pacing; in

precise leveling, they are usually measured with the stadia. In leveling uphill or downhill, a balance between foresight and backsight distances may be obtained with a minimum number of set-ups by following a zigzag course.

122. Differential Level Notes.—For ordinary differential leveling when no special effort is made to equalize backsight and foresight distances between bench marks, the record of field work is usually kept in the form indicated by Fig. 122*a*. The left-hand page is divided into columns for numerical data, and the right-hand page is reserved for descriptive notes concerning bench marks and turning points. In the same horizontal line with each turning point or bench mark shown in the first column, are all data concerning that point. The levels from B.M.$_1$ to B.M.$_2$ are the same as shown by Fig. 121. The H.I.'s and elevations are calculated as the work progresses. Thus, when the backsight (7.11) has been taken on B.M.$_1$ it is added to the elevation (721.05) to determine the H.I. (728.16). The height of instrument is recorded on the same line with the backsight by means of which it is determined. When the first foresight (1.24) is observed, it is recorded on the line below and is subtracted from the preceding H.I. (728.16) to determine the elevation of T.P.$_1$ (726.92). And so the notes are continued. Usually at the foot of each page of level notes, the calculations are checked by taking the difference between the sum of the backsights and the sum of the foresights, and the difference between the initial and final elevation, as illustrated by the numerical work at the bottom of Fig. 122*a*. When these two differences agree it signifies that the additions and subtractions are correct but does not check against mistakes either in observing or recording.

Bench marks should be briefly but accurately described and should be so marked in the field that they can be readily identified. They are usually marked with paint or with crayon that will withstand the effects of the weather. When the bench mark is on stone or concrete the position is often indicated by a cross cut with a cold chisel. A bench mark may or may not be marked with its elevation. Whenever there might arise any question as to the exact position of the point on which the rod was held, its nature should be clearly indicated in the notes. A description of turning points is of no particular importance unless the points are on objects that can be identified and might therefore become of some value in future leveling operations. Such points are usually marked with crayon, and very briefly described in the notes.

122a. When backsight and foresight distances are to be balanced the form of notes is the same, except that these distances are usually

(Left-Hand Page)
LEVELS FOR BENCH MARKS ALONG
RIDGE ROAD

(Right-hand Page) 38
Dec 31, 1924 J G Sutter ⊼
Fair W R Knowles Rod

Sta.	B.S.	H.I.	F.S.	Elev.		Remarks
B.M.₁	7.11	728.16		721.05		Top of hydrant Cor. Oak St.
T.P.₁	8.83	735.75	1.24	726.92		Curb
T.P.₂	11.72	746.36	1.11	734.64		
B.M.₂	4.32	740.47	10.21	736.15		Spike in Pole North of Williams house Marked B.M.736.15
T.P.₃	3.06	733.57	9.96	730.51		
T.P.₄	2.74	727.40	8.91	724.66		Stone
T.P.₅	0.81	716.59	11.62	715.78		
B.M.₃			12.42	704.17		Concrete Monument No. of road County Line
Σ B.S. =	38.59	Σ F.S. =	55.47	721.05		
			38.59			
		Diff. =	16.88 =	16.88	ck.	

FIG. 122a.—Differential level notes.

Differential Levels. 13.
Self-Reading Rod.
Eng. Hall to Ag. Bldg & Return.

Berger Dumpy No. 114. Locker No 30.
J.K. Pratt, ⊼
C.P. Kidd, Rod
2 hrs. Oct 21, 1912.
Cold & Cloudy.

Sta.	B.S.	H.I.	F.S.	Elev.	Dist. BS.	Dist. FS.	
B.M. 1	6.102	419.212		413.11	97		U.S.G.S. B.M. Cor. Eng. Hall.
T.P. 1	9.842	425.858	3.196	416.016	96	$\overset{+11}{108}$	Top Hydrant Front E.N. House.
T.P. 2	11.276	432.950	4.184	421.674	85	$\overset{+15}{100}$	Plug.
T.P. 3	3.616	435.654	0.912	432.038	104	$\overset{-6}{64}$	N.E. Cor. Lower Step Fellows Hall.
B.M. 2	2.256	431.796	6.114	429.540	84	$\overset{-2}{108}$	Concrete Monument S.W. Cor. Big Barn.
T.P. 4	3.387	423.535	11.648	420.148	44	$\overset{+10}{96}$	Plug.
T.P. 5	1.110	414.958	9.687	413.848	115	$\overset{+42}{76}$	Plug.
B.M. 3	5.302	415.887	4.373	410.585	96	$\overset{0}{73}$	Copper Bolt in Sidewalk Front of Ag. Bldg.
T.P. 6	10.415	423.655	2.647	413.240	80	$\overset{-5}{91}$	S.E. Cor. Brick Pier at Observatory.
T.P. 7	6.7[illegible]	428.330	2.113	421.542	93	$\overset{-5}{80}$	Plug.
T.P. 8	0.6[illegible]	423.837	5.162	423.168	75	$\overset{-16}{82}$	Plug.
B.M. 1			10.768	413.069		$\overset{+2}{93}$	U.S.G.S. B.M.
	60.763		60.804	413.110	969	971	ck.
			60.763	413.069			
			0.041 =	0.041	ck.		
	Error = 0.04 $\sqrt{\text{dist. in miles}}$						

Distances Expressed in Paces.
37 Paces = 100 ft.

$$\text{Total length} = \frac{1940 \times 100}{37 \times 5280} = 1.0 \text{ mi.}$$

FIG. 122b.—Differential level notes.

recorded in the last column of the left-hand page, as illustrated by Fig. 122*b*.

The numbers preceded by plus or minus signs indicate the cumulative excess or deficiency of foresight over backsight distances.

123. Precise Differential Leveling.—While the subject of precise leveling as practiced on government surveys is not to be considered here, it is appropriate to call attention to certain refinements by means of which a relatively high degree of precision may be obtained with the ordinary wye or dumpy level and the self-reading rod.

For work of this nature the rods should be treated in some manner to prevent expansion or contraction through change in moisture content and at intervals should be compared with a standard length. They should have attached rod levels for plumbing. It is particularly important that turning points be on solid objects with rounded tops so that the base of the rod may be held in the same position for both backsight and foresight. For example, the turning pin or turning plate described in a preceding article would be superior to a street curb.

Sta.	Back Sights			H.I.	Fore Sights			Elev.
	Hairs	Mean	Dist.		Hairs	Mean	Dist.	
	9.316							
	7.942							
B.M.1	6.565	7.941	2.751	329.561				321.620
	11.742				4.112			
	10.635				2.911			
	9.528	10.635	2.214		1.716	2.913	2.396	326.648

FIG. 123.—Precise level notes.

The level should be equipped with stadia hairs in addition to the regular cross-hairs. Preferably it should be of the dumpy type with inverting eyepiece and reflecting mirror by means of which the bubble may be viewed at the instant the rod is read. To prevent unequal thermal expansion the level should be protected from the sun's rays by an umbrella. It should be set up very firmly so that no settlement will occur. To eliminate as far as possible the effects of any change in atmospheric refraction, settlement of the tripod, or warping of the level it is desirable that the shortest possible time elapse between a backsight and the succeeding foresight. The backsight and foresight distances are determined preferably by stadia but sometimes by pacing and are balanced very closely between bench marks.

Excellent results have been obtained by employing two rods and two rodmen, each occupying alternate turning points (of the same set). If the instrument is not equipped with a reflecting mirror, an assistant should keep the bubble centered while the leveler is making the observa-

tions. The assistant also acts as a recorder. All three horizontal hairs should be read by estimation to thousandths of feet and the readings should be recorded. The mean of the readings for the three hairs is taken as the correct rod reading for each sight. The interval between the reading of the upper hair and that of the lower hair is a measure of the distance from instrument to rod. In order further to eliminate possible systematic errors the order of readings may be interchanged at alternate set-ups of the level; that is, at one set-up, the backsight may be observed before the foresight, and at the next set-up the foresight may be determined before the backsight. This would be practicable only when two rodmen are employed.

Figure 123 shows a suitable form for numerical data. A portion of the right-hand page can be reserved for explanatory notes as in the notes of Fig. 122*a*.

124. Leveling with Two Sets of Turning Points.—This method was formerly used quite extensively on some of the government surveys, where two rods and two rodmen were generally employed. For this reason levels run in this manner are often designated as "double-rodded" lines. Though the speed will be considerably lessened, a single rod may be used with nearly as good results. The advantage of this method does not lie so much in the increased precision over using one set of turning points as in checking the levels as the work progresses. Hence it is particularly useful in running levels that do not close on points of known elevation. The target rod has generally been used in the past, but the self-reading rod may be employed equally well.

Two sets of turning points are established so that at each set-up of the level two independent backsights and two independent foresights are taken. The turning points on one line are usually a foot or more higher than corresponding points on the other line, so as to eliminate the possibility of making the same mistake in reading the foot marks on both rods. When two rodmen are employed, one gives readings for points along the "high" line and the other for points along the "low" line.

An appropriate form of notes is illustrated by Fig. 124. The observations are seen to give two independent determinations for the height of instrument at each set-up. Were it not for errors of observation these H.I.'s should exactly agree. If at any set-up the discrepancy between the two H.I.'s shows a material variation from the discrepancy between H.I.'s for the preceding set-up, observations are repeated. In careful leveling the maximum allowable variation between the discrepancies at two successive set-ups is usually two or three thousandths of a foot. Normally the difference between H.I.'s may be expected to increase as the length of the line increases, and hence the two independent determinations of the elevation of a bench mark along the route may be expected to show a difference which in general will increase with the distance from the point of beginning. Thus in the notes, the discrepancies between the two H.I.'s are seen to be successively 0.002, 0.005, 0.007, 0.004, and 0.006. The variations between the discrepancies for succeeding

set-ups are therefore 0.003, 0.002, 0.003, and 0.002. On the right-hand page of the notes are calculations for checking the additions and subtractions. The difference between the total of all backsights and the total of all foresights is double the difference in elevation between the initial and final bench marks.

It is desirable that both rods be read on the terminal bench marks. Intermediate bench marks are employed as turning points on either of the two lines. When the discrepancy between H.I.'s becomes sufficiently large to be of consequence the elevation of each bench mark is adjusted to conform to the mean of the two heights of instrument for the set-up from which a foresight to the bench mark was taken.

Levels, Double Rodded Line Along P. & R F. Ry.

Sta.	B S	H I.	F S.	Elev.
B. M.₁	5.241	532.871		527 630
B. M.₁	5.239	532.869		
TP₁H	6.943	535 898	3.916	528 955
TP₁L	7 897	535 893	4.873	527.996
TP₂H	8 337	541 804	2.431	533.467
TP₂L	9 746	541.797	3.842	532.051
TP₃H	5.173	541.508	5 469	536 335
TP₃L	7.549	541 504	7.842	533.955
TP₄H	3 411	536.731	8.188	533 320
BM₂L	4.963	536.725	9.742	531.762
TP₅H	2 344	531.837	7 238	529.493
TP₅L	5.729	531.830	10 624	526 101
BM₃H	7.004	531 043	7.798	524 039
TP₆L	8 021	531.039	8 812	523 018
	87 597		80.775	

Dixfield to Peru. 26.

Gurley Wye Level. K.K. Burton, ⊼
2- Phila. Rods. J.J. Hamel, Recorder.
Lowe & Smith, Rods.
July 9, 1927.
U.S.G.S. in Culvert Fair & Warm.
800 ft. S. of Mile Post 13.

	531.043	531.039
	527.63	527.630
87.597	3.413	3.409
80.775	3.409	
6.822 =	6.822 ck.	

Spk in Tel Pole at Road to Abbotts Mills.

d H. in Culvert at Alder Brook.

FIG. 124.—Notes for double-rodded line.

125. Reciprocal Leveling.—On occasion it becomes necessary to determine accurately the relative elevations of two intervisible points a considerable distance apart, between which points levels can not be run in the ordinary manner. For example, it may be desired to transfer levels from one side to the other of a deep canyon, or from bank to bank of a wide stream. With certain modifications the method employed in getting the true difference in elevation between two points prior to the adjustment of the line of sight of the dumpy level (*i.e.*, the two-peg test) may be utilized in situations of this kind.

If A and B are two such points, then the level is set up near A and one or more rod readings are taken on both A and B. Then the

level is set up in a similar location near B and rod readings to near and distant points are taken as before. The mean of the two differences in elevation thus determined is taken to be the true difference between the two points. Usually the distance between points is large (often a half-mile or more) so that it is necessary to use a target on the distant rod. If precise results are desired, a series of foresights is taken on the distant rod and sometimes also a series of backsights on the near rod, the bubble being recentered and the target reset after each observation. The difference in elevation is then computed by using the mean of the backsights and the mean of the foresights.

This method assumes that the conditions under which observations are taken remain unaltered for the two positions of the level. Two factors which may appreciably alter the results, if the sights are long, are unequal expansion of the parts of the instrument and variations in atmospheric refraction. On this account it is best to make observations on cloudy days when atmospheric and temperature conditions do not vary greatly; or if this is impossible, to protect the instrument from the sun's rays and to allow the minimum possible time to elapse between observations taken with the level in one position and those taken with it in the other.

When one point cannot be quickly reached from the other, the effect of variation in refraction may be eliminated by taking simultaneous observations with two instruments, one being set up near one point and the other near the other point. The instruments are then interchanged and simultaneous readings on near and far points are taken as before. All things being equal, the mean of the difference in elevation obtained with one level and that obtained with the other is assumed to be the true difference; if there is clear indication that one set of observations is inferior to the other, each set may be weighted (see Art. 70). Preferably the two instruments should have about the same magnifying power and sensitiveness of bubble tube.

126. Errors.—In leveling, errors are due to some or all of the following causes:

1. *Imperfect Adjustment of the Instrument.*—In so far as results are concerned, the only essential adjustment is that the line of sight shall be parallel to the axis of the level tube. Any inclination between these lines causes a systematic error, for then if the bubble were perfectly centered, the line of sight would be inclined always slightly upwards or downwards. Evidently the error in a rod reading due to this imperfection would be proportional to the distance from the instrument to the rod, and for a given distance would be of the same magnitude and sign for a backsight as for a foresight.

Since backsights are added and foresights are subtracted, it is clear that the error in elevations will be completely eliminated to the extent that, between bench marks, the sum of the backsight distances is made equal to the sum of the foresight distances. And conversely, a systematic error will result to the extent that these distances are not equalized between any two bench marks. And while often, in the long run, these distances will be sufficiently balanced, regardless of the terrain, to yield a satisfactory final result, that fact does not insure a corresponding accuracy for the bench marks established along the line.

The amount of the error arising from this source for a particular case is calculated in Art. 121*a*.

The effect of imperfect adjustment of the instrument is minimized (1) by adjusting the instrument and (2) by balancing backsight and foresight distances. In precise leveling this error is also further reduced by computations.

2. *Parallax.*—This condition produces an accidental error. It may be nearly eliminated by careful focusing.

3. *Earth's Curvature.*—This produces an error only when backsight and foresight distances are not balanced. Under ordinary conditions these distances do not tend to vary greatly and whatever error arises from this source is accidental in nature and in ordinary leveling is so small as to be of no consequence. When backsight distances are consistently made greater than foresight distances, or *vice versa,* it produces a systematic error of considerable magnitude, particularly when the sights are long. Its effect is the same as that due to the line of sight being inclined upward. It varies as the square of the distance from instrument to rod and hence will be eliminated not merely by equalizing the sum total of backsight and foresight distances between bench marks, but rather by balancing each length of foresight by a corresponding length of backsight.

4. *Atmospheric Refraction.*—This varies as the square of the distance, but is, under normal conditions, only about 1/7 that due to the earth's curvature and its effect is opposite in sign. It is usually considered together with the earth's curvature, but though the effect of the latter will be entirely eliminated if each backsight distance is made equal to the following foresight distance, the atmospheric refraction often changes rapidly and varies greatly in a short distance. It is particularly uncertain when the line of sight passes close to the ground. Hence it is impossible to eliminate entirely the effect of refraction even though the backsight and foresight distances are balanced. In ordinary leveling its effect is negligible. In accurate leveling the change in refraction may be minimized by keeping

the line of sight well above the ground (say at least 2 ft.) and by taking the backsight and foresight readings in quick succession. In the long run the error is accidental, but over a short period, as a day, it may be systematic. So-called heat waves are evidences of rapidly fluctuating refraction. Errors from this source may be reduced by shortening the length of sight until the rod appears steady.

5. *Variations in Temperature.*—The sun's rays falling on top of the telescope, or on one end and not on the other, will produce a warping or twisting of its parts and hence may influence rod readings through temporarily disturbing the adjustments. While this is not of much consequence in leveling of ordinary precision, it may produce an appreciable error in more refined work. The error is usually accidental, but under certain conditions it may become systematic. It is practically eliminated by shielding the instrument from the rays of the sun.

6. *Rod Not Standard Length.*—This produces a systematic error that varies directly as the difference in elevation and bears no relation to the length of the line over which levels are run. The error may be eliminated by comparing the rod with a standard length and applying the necessary corrections. The case is analogous to measurements of distance with a tape that is too long or too short. If the rod is too long the correction is added to a measured difference in elevation; if the rod is too short, the correction is subtracted.

Most manufactured rods are nearly of standard length, but where large differences of elevation are to be determined, few rods are near enough to the standard that corrections can be ignored in accurate leveling.

7. *Expansion or Contraction of the Rod.*—Due to change in moisture content or to change in temperature the leveling rod may expand or contract. The resultant error is systematic. Wood when well seasoned and painted will shrink or swell very little in the direction of the grain. Likewise its coefficient of thermal expansion is small. The error is of no particular consequence in ordinary leveling. For precise leveling, gage points may be established by inserting metal plugs in the rod, and corrections for shrinkage may be determined by observing any change in distance between the gage points. Corrections for thermal expansion may be based upon observed temperatures of the rod, as indicated by an attached thermometer, the temperature being recorded in the notes.

8. *Rod Not Held Plumb.*—This condition produces rod readings that are too large. In running a line of levels uphill or downhill it becomes a systematic error, inasmuch as the backsights are larger than the foresights, or *vice versa.* Over rolling or level ground the

resultant error is accidental since the backsights are, on the average, about equal to the foresights. The error varies directly with the magnitude of rod reading and directly as the square of the inclination. Thus, if a 10-ft. rod is 0.2 ft. out of plumb, the error amounts to 0.002 ft. for a 10-ft. reading and 0.0002 ft. for a 1-ft. reading; but if the rod were 0.4 ft. out of plumb, the corresponding errors would be 0.008 and 0.0008 ft. It is therefore evident that appreciable inclinations of the rod must be avoided. The error may be eliminated by using a rod level, or by waving the rod.

9. *Faulty Turning Points.*—This refers to turning points that are not well defined. A flat, rough stone, for example, does not make a good turning point for accurate leveling for the reason that no definite point exists on which to hold the rod, which is not likely to be held in the same position for both backsight and foresight. Errors from this source are accidental.

10. *Settlement of Tripod or Turning Points.*—If the tripod settles in the interval that elapses between taking a backsight and the following foresight, the foresight will be too small and the observed elevation of the forward turning point will be too large. Similarly, if a turning point settles in the interval between foresight and backsight readings, the height of instrument as calculated from the backsight reading will be too great. It is thus seen that by the normal leveling procedure, if either the level or the turning point settles, as may be the case to some extent when leveling over soft ground, the error will be systematic and the resulting elevations will always be too high. Few occasions arise when turning points may not be so selected or established as to eliminate the possibility of settlement, but care should be taken not to strike the bottom of the rod against the turning point between sights.

On the other hand, some settlement of the instrument is nearly certain to occur when leveling over muddy, swampy, or thawing ground or over melting snow. The errors due to such settlement may be greatly reduced by employing two rods and two rodmen, one rodman setting the turning point ahead while the other remains at the turning point in the rear. Backsight and foresight readings may then be made in quick succession. Small errors remaining from this source may be made accidental by reversing the order of sights at alternate set-ups, as described in Art. 123.

11. *Bubble Not Exactly Centered at Instant of Sighting.*—This produces an accidental error which tends to vary as the distance from the instrument to the rod. Consequently the longer the sight the greater the care that should be observed in leveling the instrument.

12. *Inability of Observer to Read the Rod or to Set the Target Exactly on the Line of Sight.*—This causes an accidental error of a magnitude depending upon the instrument, weather conditions, the length of sight, and the observer. It may be confined within reasonable limits through proper choice of length of sight.

126a. Looking over the errors just listed it will appear that under normal conditions the important ones are accidental, provided the proper leveling procedure is observed. Hence the resultant error may be expected to vary as the square root of the number of set-ups of the instrument or as the square root of the distance. Experience in general bears out this conclusion, and for this reason it is customary to express limiting errors of leveling in terms of the square root of the distance in miles, kilometers, or other unit of measure. It has been demonstrated, however, that on very long lines of precise levels, the errors are proportional to some power of the distance between one-half and one, indicating that in spite of every precaution there are certain small systematic errors which cannot be eliminated by any known method of procedure.

127. Precision of Differential Leveling.—This depends perhaps upon more factors than does any other operation of surveying. While it is influenced by the instrument employed, it depends chiefly upon the care and skill of the leveler and upon the degree of refinement with which the work is executed. Other conditions remaining the same, the error for a given length of line will tend to vary as the number of set-ups above a certain minimum, hence the precision may be expected to be lower in hilly country where the sights are limited to short distances than in flat country where normal backsight and foresight distances are employed. Above a certain length of sight, however, the error of reading the rod increases very rapidly with the distance, therefore the precision will be lower for long sights than for those of normal length. Likewise, due to erroneous length of rod, unequal refraction, and other causes, the precision of leveling between two points of large difference in elevation is likely to be less than between two points, the same distance apart, at or near the same elevation. Atmospheric disturbances also bear an important relation to the accuracy attainable.

While conditions are so variable that no hard and fast rules can be laid down by means of which a desired precision can be maintained, practice indicates that under average conditions, with a level in good adjustment, the maximum error may be kept within the limits shown below.

1. *Rough leveling,* such as is practised on rapid reconnaissance or preliminary surveys. Sights up to 1,000 ft. in length. Rod read-

ings to tenths of feet. No particular attention paid to balancing backsight and foresight distances. Maximum error in feet, $\pm 0.4\sqrt{\text{distance in miles}}$.

2. *Ordinary leveling*, such as is necessary in connection with the location and construction of railroads, highways, and most other engineering works. Sights up to 500 ft. in length. Rod readings to hundredths of feet. Backsight and foresight distances roughly balanced when running for long distances uphill or downhill, but no attention paid to these distances when sights of normal length can be secured. Turning points on solid objects. Maximum error in feet, $\pm 0.1\sqrt{\text{distance in miles}}$.

3. *Accurate leveling* for important city bench marks, or for the principal bench marks on extensive surveys. Sights up to 300 ft. in length. Rod readings to thousandths of feet either with the target or self-reading rod. Backsight and foresight distances measured by pacing and approximately balanced between bench marks. Rod waved for large rod readings. Bubble carefully centered before each sight. Turning points on metal pin or plate, or on well-defined points of solid objects. Tripod set on firm ground. Maximum error in feet, $\pm 0.05\sqrt{\text{distance in miles}}$.

4. *Precise leveling* for establishing bench marks with great accuracy at widely distributed points. High-grade level equipped with stadia hairs and with sensitive bubble. Adjustments carefully tested daily. Rod standardized frequently. Sights up to 300 ft. in length. Rod readings of three horizontal hairs to thousandths of feet. Level protected from the sun. Turning points on metal pin or plate. Two rodmen. Backsights and following foresights taken in quick succession. Bubble very carefully centered and under observation at instant of taking sight. Rod plumbed with rod level. Backsight and foresight distances balanced between bench marks by stadia readings. Level set up securely on firm ground. Levels not run when the air is boiling badly nor during high winds. Maximum error in feet, $\pm 0.02\sqrt{\text{distance in miles}}$.

It should be borne in mind that the above limits of error for the several classes of leveling represent the *maximum* errors for average conditions. The *average* errors will usually be materially less.

128. Adjustment of Elevations.—When a line of levels makes a complete circuit the final elevation of the initial bench mark as computed from the level notes will not agree with the initial elevation of this point. The difference is the true error of running the circuit and is called the *error of closure*. It is evident that elevations of bench marks established while running the circuit will also be in error, and

there arises the problem of determining the probable errors of these intermediate points and of adjusting their elevations accordingly. It has been shown that the principal errors of leveling are accidental in character, hence the probable error tends to vary as the square root of the number of opportunities for error or in other words as the square root of the number of set-ups. In the adjustment of elevations it will usually be sufficiently exact to assume that the number of set-ups per mile is the same for one portion of the circuit as for any other, and that therefore the probable error varies as the square root of the distance. Since weights are inversely proportional to the square of the probable errors (Art. 70*a*) the reliability of the observed elevation of a given bench mark in the circuit is inversely proportional to the first power of the distance from the bench mark to the point of beginning. Thus if E_c is the error of closure of a level circuit of length L and C_a, C_b . . . C_n are the respective corrections to be applied to observed elevations of bench marks A, B, . . . N, whose respective distances from the point of beginning are a, b, . . . n, then

$$C_a = -\frac{a}{L}E_c; \quad C_b = -\frac{b}{L}E_c; \quad \ldots \text{ and } C_n = -\frac{n}{L}E_c \qquad (5)$$

Example 1: The accepted elevation of the initial bench mark B.M.$_i$ of a level circuit is 470.46 ft. The length of the circuit is 10 miles. The final elevation of the initial bench mark as calculated from the level notes is 470.76. The observed elevations of bench marks established along the route, and the distances to the bench marks from B.M.$_i$ are as tabulated in the third and second columns below. The most probable values of the elevations of these intermediate points are required.

Point	Distance from B.M.$_i$, miles	Observed el., ft.	Correction, ft.	Adjusted el., ft.
B.M.$_i$	0	470.46	0.0	
B.M.$_a$	2	780.09	−0.06	780.03
B.M.$_b$	5	667.41	−0.15	667.26
B.M.$_c$	7	544.32	−0.21	544.11
B.M.$_i$	10	470.76	−0.30	470.46

$$E_c = 470.76 - 470.46 = +0.30 \text{ ft.}$$

By Eq. (5),

$$C_a = -2/10 \times 0.30 = -0.06 \text{ ft.}$$
$$C_b = -5/10 \times 0.30 = -0.15 \text{ ft.}$$
$$C_c = -7/10 \times 0.30 = -0.21 \text{ ft.}$$

These corrections subtracted from the corresponding observed elevations give the adjusted elevations as tabulated above.

It is to be noted that, if the error of closure is positive, all corrections are to be subtracted, and *vice versa.*

The same principles apply to the adjustment of elevations of bench marks on a line of levels run between two points whose difference in elevation has previously been determined by more accurate methods and is assumed to be correct.

128a. A somewhat similar problem occurs in the adjustment of the elevation of a bench mark which is established by lines of levels run over several routes. For a point established in this manner there will be as many observed elevations as there are lines terminating at the point. Assuming that the probable error of each of the individual observed values varies as the square root of the length of the line of levels by means of which the determination is secured, then the weight to be applied to a given observed elevation will vary inversely as the length of the corresponding line. The most probable value of the elevation will then be the weighted mean of the observed values. The following example illustrates the procedure in securing the most probable value by weighting differences, rather than by weighting the observations themselves (see Art. 70*a*, example 2, solution *b*).

Example 2: Lines of levels between B.M.$_1$ and B.M.$_2$ are run over four different routes. The length of the lines and the observed values of the elevation of B.M.$_2$ are tabulated below. It is required to determine the most probable value of the elevation of B.M.$_2$.

Route	Length, miles	Observed el., B.M.$_2$	Diff. el., less 640.00	Weight	Weighted difference
a..........	2	640.72	0.72	$\frac{1}{2}$	0.36
b..........	4	640.56	0.56	$\frac{1}{4}$	0.14
c..........	10	641.08	1.08	$\frac{1}{10}$	0.11
d..........	20	640.26	0.26	$\frac{1}{20}$	0.01
				$\Sigma = \frac{18}{20}$	$\Sigma = 0.62$

The weights may be represented by the reciprocals of the corresponding distances as shown. The products of the weights and the differences are shown in the last column of the table. The ratio of the sum of the weighted differences to the sum of the weights gives the difference between 640.00 and the most probable value. Thus, if P is the most probable value

$$P - 640.00 = \frac{0.62}{\frac{18}{20}} = 0.69,$$

or

$$P = 640.00 + 0.69 = 640.69 \text{ ft.},$$

which is the adjusted value of the elevation of $B.M._2$.

Referring to the above example, if bench marks were established along any of the lines joining $B.M._1$ and $B.M._2$, the elevations of these bench marks in turn would require adjustment after the most probable value of the elevation of $B.M._2$ had been determined. This would be done by assuming that the adjusted value of $B.M._2$ represented the true elevation and then proceeding as for a line closing on the point of beginning (see example 1, Art. 128).

Thus, suppose the observed elevation of $B.M._d$ (example 2) on route d at a distance of 15 miles from $B.M._1$ is 520.33 ft. The adjusted value of $B.M._2$ is 640.69 ft. and the observed value for route d is 640.26 ft. The discrepancy is $640.26 - 640.69 = -0.43$ ft. Hence,

$$C_d = -{}^{15}\!/_{20} \times (-0.43) = +0.32 \text{ ft.},$$

the correction to be applied to $B.M._d$. The adjusted elevation of $B.M._d$ is therefore $520.33 + 0.32 = 520.65$ ft.

128b. Where elevations of bench marks in an interconnecting network of level circuits are to be adjusted, a rigid *least squares* adjustment considers solving the group of circuits as a whole, which for a particular bench mark makes necessary the solution of as many equations of condition as there are separate figures in the net. A simpler method sometimes employed, quite satisfactory for usual conditions, consists in distributing the error of each of the individual circuits in proportion to the distances between connecting bench marks, considering first the circuit having the largest error of closure, then the one having next to the largest error of closure, and so on until all circuits have been adjusted.

Figure 128 represents a level net made up of the circuits *ABCA*, *ACDFA*, *FDEF*, *FEAF*, and *EBAE*. Within each circuit are shown its error of closure and length. The connecting bench marks are *A*, *B*, *C*, *D*, *E*, and *F*. On lines of the circuits are given the lengths between adjacent connecting bench marks and also the calculated errors existing in the observed difference in elevation between these adjacent points. Looking at the figure it will be seen that the maximum error of closure (-0.40 ft.) is in the circuit *ACDFA*. The sum of the lengths of the sides of this figure is 70, hence the calculated error in side *AC* is ${}^{12}\!/_{70} \times (-0.40) = -0.07$. Similarly the calculated error in side *CD* is ${}^{28}\!/_{70} \times (-0.40) = -0.16$ ft. and so on around the figure. The circuit containing the next largest error ($+0.36$) is *ABCA*. In this figure the error in the side *AC* has already been calculated, and this value will not be changed. Hence the error to be distributed between the sides *AB* and *BC* will be the *algebraic difference* between the error of closure of the figure and the calculated error in side *AC*, or $+0.36 - (-0.07) = +0.43$ ft. This error will be distributed in proportion to the lengths as before. The sum of the lengths of the two sides *AB* and *BC* is 46. The calculated error in *AB* is then ${}^{16}\!/_{46} \times (+0.43) = +0.15$ ft. and that in *BC* is

$^{30}\!/_{46} \times (+0.43) = +0.28$ ft. Thus the process is continued for figure *DEFD*; and so on.

The adjusted differences in elevation between adjacent connecting bench marks are obtained by applying corrections with signs opposite to those of the calculated errors. When the elevations of connecting bench marks have been determined in this manner, they may be assumed to be correct, and elevations of intermediate bench marks along any of the lines may be adjusted in the manner described in Art. 128.

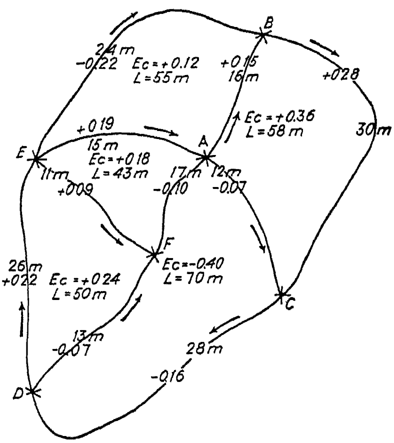


FIG. 128.—Adjustment of level net.

129. Mistakes in Leveling. Some of the mistakes that are commonly made in leveling are:

1. Confusion of numbers in reading the rod, as for example, reading and recording 4.92 when it should be 3.92. The mistake is not likely to occur if the numbers on either side of the observed reading are noted.
2. Recording backsights in foresight column, and *vice versa.*
3. Reduction of notes. Faulty additions and subtractions; adding foresights and subtracting backsights. Such mistakes will be detected if the difference between the sum of the backsights and the sum of the foresights is computed.
4. Rod held on wrong point. Unless the turning points are marked or otherwise clearly defined the rodman may not hold the rod on the same point for both foresight and backsight.
5. With the Philadelphia rod, not having the rod fully extended when reading the long rod. Before a reading on a turning point is taken the clamp should be inspected to see that it has not slipped.
6. Wrong reading of the vernier when the target rod is used.
7. Not having target set properly when the long rod is used. For the long rod, the vernier on the target should be set to read exactly the same as the vernier on the back of the rod when the rod is short.

130. Numerical Problems.

1. A line of differential levels was run between two bench marks 20 miles apart and the measured difference in elevation was found to be 2,163.4 ft. Later the rod whose nominal length is 13 ft. was found to be

0.003 ft. too short, the error being distributed over its full length. Correct the measured difference in elevation for erroneous length of rod.

2. Suppose the levels of problem 1 had been run by using a rod which was 0.003 ft. too short due to wear on the lower end. What would have been the error?

3. Suppose the line of levels of problem 1 were continued to form a circuit closing on the initial bench mark. Due to erroneous length of rod what error of closure would be expected?

4. Differential levels are run from B.M.$_1$ (el. 470.07 ft.) to B.M.$_2$, a distance of 100 miles. The backsight distances are 400 ft. in length and the foresight distances are 200 ft. in length. The elevation of B.M.$_2$ as deduced from the level notes is 3,652.74 ft. Calculate the error due to earth's curvature and atmospheric refraction, and correct the elevation of B.M.$_2$.

5. Later the levels of problem 4 were rerun using an average backsight distance of 200 ft. and an average foresight distance of 100 ft. The elevation of B.M.$_2$ as deduced from the level notes was 3,651.38. Compute the error due to curvature and refraction and correct the elevation of B.M.$_2$.

6. Suppose the instrument used in running the levels of problems 4 and 5 was out of adjustment, so that when the bubble was centered the line of sight was inclined 0.001 ft. upward in a distance of 100 ft. Correct the results of problems 4 and 5 for inclination of line of sight.

7. A line of levels 10 miles long is run over thawing ground. Backsight and foresight distances average 300 ft. in length. What error would be introduced and what would be the sign of the correction to be applied to the elevation of the terminal bench mark if sights were taken in their normal order, and the average settlement of the instrument was 0.004 ft. between the instant of backsight reading and the instant of the following foresight reading? Suppose that at alternate set-ups the order of reading were reversed, what would be the error of the line of levels?

8. In looking through the telescope of a level a magnified tenth on the rod apparently covers 3.2 ft. of the unmagnified image. What is the magnifying power of the telescope?

9. The diameter of the field of view of a level is 5.25 ft. when the rod is 300 ft. from the objective. What is the angular width of the field of view?

10. A rod is held 300 ft. from the instrument and with the end of the bubble at a graduation a reading of 3.234 ft. is observed. The bubble is then displaced five spaces and a reading of 3.342 ft. is observed. If one space on the level tube is equal to 0.1 in., what is the radius of curvature of the level tube? What is the sensitiveness of the level tube expressed in seconds of arc per division?

11. If in running levels between two points, the rod were inclined 0.3 ft. in a height of 13 ft. what error would be introduced per set-up when backsight readings averaged 12 ft. and foresight readings averaged 1 ft.?

12. If levels are run from B.M.$_1$ (el., 2,000.00 ft.) to B.M.$_2$ (observed el., 3,000.00 ft.) and the rod is, on the average, 0.2 ft. out of plumb in a height of 12 ft., what error is introduced due to the rod's not being plumb? What is the correct elevation of B.M.$_2$?

13. Suppose that in problem 12 both bench marks were at the same elevation. What would be the error?

14. The error of closure of a level circuit 100 miles long is 0.53 ft. The average length of sight is 250 ft. If all systematic errors have been eliminated, what is the probable error per set-up of the level? What is the probable error of a single observation of the rod?

15. If sights average 200 ft. in length and the probable error of a single observation is 0.004 ft. what is the probable error of running a line of levels 25 miles long? What is the probable error of running a line of levels 100 miles long?

16. In the two-peg test of a dumpy level the following observations are taken:

	Instrument at *A*	Instrument at *B*
Rod reading on *A*	4.937	3.077
Rod reading on *B*	6.736	4.752

What is the true difference in elevation between the two points? With the instrument in position at *B*, to what rod reading on *A* should the line of sight be adjusted? What is the error in the line of sight for the distance *A* to *B*?

17. Complete the differential level notes shown below. Perform the customary check:

Station	B.S.	H.I.	F.S.	El.
B.M.$_1$	6.11	....	....	416.23
T.P.$_1$	9.25	....	7.36	
T.P.$_2$	11.48	....	3.12	
T.P.$_3$	8.30	....	2.98	
B.M.$_2$	12.29	....	4.37	
T.P.$_4$	7.73	....	5.16	
T.P.$_5$	8.24	....	3.38	
T.P.$_6$	10.66	....	0.47	
B.M.$_3$	.....	....	4.33	

18. Complete the differential level notes shown on page 178. Determine the error of closure of the level circuit and adjust the elevations of B.M.$_2$ and B.M.$_3$, assuming that the error is a constant per set-up.

Station	B.S.	H.I.	F.S.	El.
B.M.1	4.127	....	......	100.000
T.P.1	3.831	....	9.346	
T.P.2	4.104	....	10.725	
T.P.3	2.654	....	12.008	
B.M.2	4.368	....	7.208	
T.P.4	6.089	....	6.534	
T.P.5	8.863	....	4.736	
B.M.3	12.356	....	2.100	
T.P.6	10.781	....	3.662	
T.P.7	12.365	....	4.111	
B.M.1	......	....	9.059	

19. Lines of differential levels are run from B.M.$_1$ to B.M.$_2$ over three different routes. Below are the lengths of the routes and the observed elevations of B.M.$_2$. Determine the most probable value of the elevation of B.M.$_2$.

Route	Length, miles	El. of B.M.$_2$
a.....................	10.5	742.81
b.....................	16.8	742.58
c.....................	36.3	743.27

131. Field Problems.

PROBLEM 1. DIFFERENTIAL LEVELING WITH ENGINEER'S LEVEL AND SELF-READING ROD

Object.—To determine the elevations of points in an assigned level circuit (for sample notes, see Fig. 122*b*).

Procedure.—Follow the procedure outlined in Art. 121. Keep notes as explained in Art. 122. Estimate rod readings to thousandths of feet. Check each rod reading by recentering the bubble and taking a second observation.

Hints and Precautions.—(1) Select solid ground for instrument stations. Plant the tripod feet firmly in the soil. Spread the tripod well, especially if the wind is strong. (2) In leveling up, the bubble travels in the same direction as the left thumb. (3) Do not try to get the bubble in the center over one pair of leveling screws before bringing it approximately to the center over the other pair. It is a waste of time to center the bubble exactly before the rod has been sighted. (4) If one pair of leveling screws turns hard, loosen the other pair slightly. Finish leveling up with all foot screws bearing solidly but not so tightly as to spring the plate. (5) Test for parallax (see Art. 104*a*). (6) Look at

the bubble just before taking each reading, and glance at it again just after, to be sure that it has not moved. The observer should stand in such a position that this will be possible without moving his feet. (7) Be sure the rod is held vertical while a sight is being taken; keep the foot of the rod free from dirt. (8) When the Philadelphia rod or similar rod is extended (long rod), it should be firmly clamped in the proper position; and between rod readings the rodman should examine it to see that no slip has occurred. Do not let the long rod down "on the run." (9) Slowly *wave the rod* toward and from the instrument when the long rod is used, and take the least reading on the rod. (10) Remember that the reading is recorded opposite the station number of the station on which the rod is held and has nothing to do with the instrument station. (11) Backsights are always taken upon points of *known* elevation; foresights, on points of *unknown* elevation. (12) Check all calculations by showing that the difference between the sum of the backsights and the sum of the foresights equals the difference in elevation between the initial and terminal stations. (13) Give a clear and concise description of each bench mark and its location, on the right-hand page in line with numerical notes concerning that bench mark. (14) Make the sum of backsight distances between bench marks equal the sum of foresight distances as nearly as conditions will conveniently permit, to eliminate the effect of imperfect adjustment of the instrument and of curvature of the earth. (15) There should be a definite, well-understood system of signals between the instrumentman and rodman (see Art. 23). (16) The rodman should choose the position of the turning point with an eye to simplicity of field operations.

Problem 2. Differential Leveling with Engineer's Level and Target Rod

Object.—To determine the elevation or difference in elevation of points assigned.

Procedure.—The procedure differs from problem 1 only in that the rodman sets the target as directed by the instrumentman. Both men read the rod from the attached vernier to the nearest 0.001 ft. The rodman should record the distances and rod readings in a book for that purpose. The field notes will be kept by the leveler as explained in the preceding problem (see Fig. 122*b*). Compare the results of this method with those of the preceding problem. Note the relative errors of closure of the circuits and the time required for each method per set-up of the instrument.

Hints and Precautions.—In addition to those for problem 1: (1) The leveler should make his signals easily distinguishable, holding his hand well up to raise the target and well down to lower the target. It is not necessary to wave the hand. (2) The rodman should move the target rapidly at first until the opposite signal is given by the instrumentman; he should then move it slowly until the instrumentman indicates that it is in the proper position by the "all right" signal (extending arms hori-

zontally). (3) After the target is clamped, the rod should be waved slowly toward and from the instrumentman, particularly when the long rod is used. If the horizontal line on the target appears above the horizontal hair, the target should be lowered. (4) When using the long rod the target must be clamped to read exactly the reading of the vernier on the back of the rod when the rod is short.

Problem 3. Reciprocal Leveling

Object.—To determine accurately the difference in elevation between two points (B.M.$_a$ and B.M.$_b$) on opposite sides of a wide stream or ravine.

Procedure.—(1) Set up the level in such a position that rod readings may be taken on each bench mark. (This usually necessitates the instrument's being much closer to one point than to the other.) Carefully take a series of five consecutive readings on B.M.$_a$. The mean of these is to be used as a backsight. (2) Take ten careful readings on B.M.$_b$, the distant point. The mean of these readings is to be used as a foresight. (3) Now set up the level on the opposite side of the stream in such position that the distances from the instrument to *a* and *b* are respectively the same as the distances to *b* and *a* from the former position of the instrument. Take a series of readings on the near and distant points as before. (4) The difference between the mean of the backsight readings on *b* and the mean of the foresight readings on *a* from this series will also give a difference in elevation between the two points. (5) The mean of the two differences in elevation secured from the two settings of the instrument should be the true difference. The precaution given in regard to the two-peg method (Art. 116), namely, that Eqs. (3) and (4) must be solved algebraically, should be given special attention in this case. (6) If the stream or ravine is imaginary, run a line of differential levels between the two points and note the discrepancy.

Hints and Precautions.—(1) Be sure that the bubble is accurately centered at the time of each reading. The effect of bubble displacement will be particularly great on long-distance readings. For distant sights the bubble and also the target should be moved and reset after each observation. (2) If the instrument can be set up near the bench mark used as a backsight, only one observation need be taken to that point. (3) If the rod has no target, a white card may be used as such a target for long sights. The target should be moved and reset for each observation.

Problem 4. Test of Accuracy of Setting Level Target

Object.—To determine the probable error of setting the level target at distances of 100, 300, and 600 ft. from the instrument, and to determine what length of sight will give best results in running a line of levels.

Procedure.—(1) Set the level in position to permit a 600-ft. sight. Drive stakes solidly at 100, 300, and 600 ft. from the instrument (distances by pacing). (2) Take a series of ten consecutive rod readings upon each stake, reading the target vernier to the nearest 0.001 ft. Center the bubble and reset the target at each observation. (3) Compute the mean

rod reading for each distance. (4) Record in the column for residuals the difference between each rod reading and the mean of all the rod readings for each distance. (5) Compute the probable errors of each set of observations (see Chap. V). (6) From the probable error of a single observation at 100, 300, and 600 ft., compute the probable error in running a line of levels of any given length when the sights are in one case all 100 ft. long, in a second case all 300 ft. long, and in a third case all 600 ft. long.

Hints and Precautions.—(1) The bubble should be moved and then recentered after each rod reading. (2) The rodman should move the target several inches between observations and reset it without prejudice, as directed by the instrumentman. (3) Note carefully the effect of distance or length of sight upon the precision of rod readings. In considering the effect of distance upon the accuracy of a line of levels, it must be remembered that three times as many 100-ft. sights are necessary as 300-ft. sights, and that the probable error of a line of levels varies as the square root of the number of set-ups.

CHAPTER IX

PROFILE LEVELING; GRADES; CROSS-SECTIONS

132. Profile Leveling.—The process of determining the elevations of points at short measured intervals along a fixed line is called profile leveling. In connection with the location and construction of railroads, highways, canals, sewers, drains, etc., stakes or other marks are placed at regular intervals along an established line, usually the center line. Ordinarily the interval between stakes is

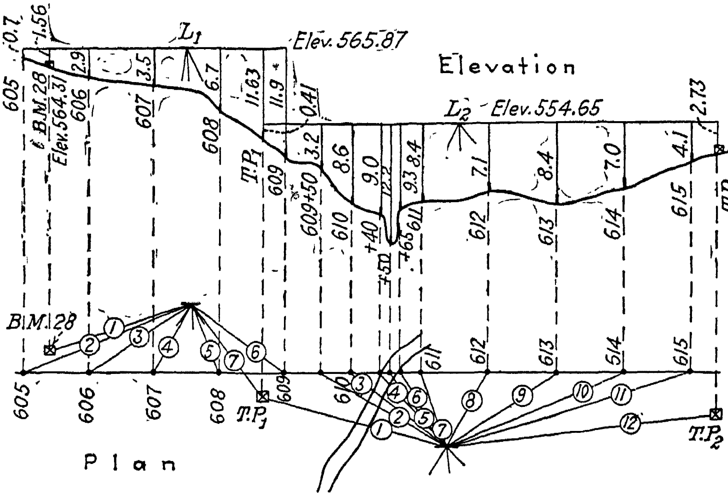


Fig. 132.—Profile leveling.

100 ft. or some simple subdivision thereof, such as 50 ft. or 25 ft. The 100-ft. points, reckoned from the beginning of the line, are called *full stations* and all other points are designated as *plus stations*. Each stake is marked with its station and plus. Thus a stake set at 1,600 ft. from the point of beginning is numbered "16" or "16 + 00," and one set at 1,625 ft. from the point of beginning is numbered "16 + 25." Elevations by means of which the profile may be constructed are obtained by taking level-rod readings on the ground at each stake and at intermediate points where marked changes in the slope occur.

Figure 132 illustrates in plan and elevation the steps in leveling for profile. In this case stakes are set every 100 ft. according to the common practice in railway and highway location. The instrument is set up in some convenient location not necessarily on the line (as at L_1), the rod is held on a bench mark (B.M. 28, El. 564.31), a backsight (1.56) is taken, and the height of instrument (565.87) is obtained as in differential leveling. Readings are then taken with the rod held on the ground at successive stations along the line. These rod readings, being for points of unknown elevation, are foresights regardless of whether they are back or ahead of the level, in the forward direction of the line. They are frequently designated as intermediate foresights to distinguish them from foresights taken on turning points or bench marks. The intermediate foresights (0.7, 2.9, . . . 11.9) subtracted from the H.I. (565.87) give elevations of stations. When the rod has been advanced to a point beyond which further readings to ground points cannot be observed, a turning point (T.P.$_1$) is selected, and a foresight (11.63) is taken to establish its elevation. The level is set up in an advanced position (L_2), and a backsight (0.41) is taken on the turning point just established (T.P.$_1$). Rod readings on ground points are then continued as before. The rodman observes where changes of slope occur (as 609 + 50, 610 + 40 . . . 610 + 65), and readings are taken to these intermediate stations. The "plus," or the distance from the preceding full station to the intermediate point, is measured by pacing or with a tape or the rod according to the precision required.

It is seen that elevations are carried forward in the same manner as in leveling to establish bench marks, that is, by a succession of turning points. The care exercised in taking observations on turning points depends upon the distance between bench marks, the elevations of which have deen determined previously, and also upon the required precision of the profile. For a ground profile usually the backsights and foresights are read to hundredths, and no particular attention is paid to balancing backsight and foresight distances; the intermediate foresights to ground points are read to tenths of feet only. Occasions arise when it is desirable or necessary to determine intermediate foresights to hundredths of feet, for example, in securing the profile of railroad track or of the water grade in a canal; rod readings on turning points are then generally taken to thousandths of feet, and backsight and foresight distances are often balanced.

132a. As the work of leveling for profile progresses, bench marks are generally established to facilitate later work. These are made turning points wherever possible. To check the elevation of turning points, which is a desirable measure, it is sometimes necessary to run

short lines of differential levels connecting the main line of profile levels with bench marks previously established by some other survey (for example, the bench marks of the U. S. Geological Survey); often the only means of checking is to run a line of differential levels back to the point of beginning. It is evident that the checking of turning points makes cumulative mistakes in the profile impossible, but does not detect mistakes in the individual intermediate foresight readings and hence in the elevations of individual ground points. The only manner in which a profile can be absolutely checked is by rerunning profile levels over the line. Except on work of more than

Profile Levels — I.N.R. Location. 20.

Cox Brook to Big Forks. — J. C. Brown,

Buff Dumpy Level. — F. Graham, Rod.

Sept 16, 1927.

Sta.	B.S.	H.I.	I.F.S.	F.S.	Elev.	Fair.
B.M. 28	1.56	565.87			564.31	On Spruce Root 50 ft Lt. Sta. 605.
605			0.7		565.2	
606			2.9		563.0	
607			3.5		562.4	
608			6.7		559.2	
609			11.9		564.0	
T.P.	0.41	554.65		11.63	554.24	On Stone
609+50			3.2		551.5	
610			8.6		546.1	
+40			9.0		545.7	Bank Cox Brook.
+50			12.2		542.5	Ctr. " " Water 1.5 ft deep.
+65			9.3		545.4	Bank " "
611			8.4		546.3	
612			7.1		547.6	
613			8.4		546.3	
614			7.0		547.7	
615			4.1		550.6	
T.P.	8.02	559.94		2.73	551.92	On Plug.
616			9.7		550.2	
+40			6.3		553.6	Ctr. Highway to St. Leonards.
	9.99		564.31	14.36		
			559.94	9.99		
			4.37	= 4.37	ck.	

FIG. 133.—Profile level notes.

ordinary importance, the effect of an occasional error in the elevations of points on the profile is not of sufficient moment to justify the additional work which this course would make necessary, and if turning points are checked it is regarded as sufficient.

133. Profile Level Notes.—The notes for profile leveling may be recorded as shown in Fig. 133, where foresights to turning points and bench marks are in a separate column from intermediate foresights to ground points. The values shown in the notes are the same as those illustrated in Fig. 132.

It will be observed that the notes for turning points are kept in the same manner as for differential leveling. H.I.'s and elevations of turning points are ordinarily computed as the work progresses. The difference

between the sum of the backsights and the sum of the foresights taken between any two bench marks or turning points along the line should equal the difference in elevation between these points, which check is applied to each page of notes, as in differential leveling.

The calculations shown at the foot of the notes of Fig. 133 check all computations for H.I.'s and elevations of T.P.'s on the page and thus for the notes shown the difference between the sum of all backsights and the sum of all foresights is equal to the difference in elevation between B.M.28 and the last H.I.

The intermediate foresights taken from any given H.I. are recorded on lines below that on which the H.I. is shown and above that on which the succeeding H.I. is shown, so that elevations of ground points are always obtained by subtracting the corresponding intermediate foresights from the preceding H.I. For reasons that are obvious these elevations are recorded to the *number of decimal places contained in the intermediate foresights* regardless of the number of places in the H.I.

The right-hand page is reserved for concise descriptions of bench marks and for other items of moment. For example, in the notes of Fig. 133 the stream and highway crossings are noted, either of which might materially influence the elevation of the roadbed at these points. Occasionally simple sketches are employed in conjunction with the explanatory notes.

134. Cross-section Levels.—In connection with problems in drainage, irrigation, grading of earthwork, location and construction of buildings, and similar enterprises, the shape of the surface of a piece of land is frequently desired. This may be obtained by dividing the given area into a system of squares and then determining the elevations of the corners and other points where changes in slope occur. The lengths of the sides of the squares are usually 100 ft. or some simple subdivision thereof, such as 50 ft. or 25 ft. Directions of the lines may be obtained with the tape or transit, distances may be laid off with the tape or by stadia, and elevations may be determined either with the engineer's level or the hand level, depending upon the required precision. Stakes are set at the corners of the squares. The data secured by a survey of this character may be employed in the construction of a contour map (see Art. 434, p. 636).

Figure 134 illustrates a suitable form of notes. The elevations are carried forward and the rod readings on ground points are determined as in profile leveling. The sketch on the right-hand page of the notes shows the area divided into 100-ft. squares, the lines running in one direction being numbered and those running in the other direction being lettered. The coordinates of a given corner may then be stated as the

letter and number of the two lines intersecting at the corner. Thus A-8 is a point at the intersection of the A line and the 8 line. The coordinates of a point not at the corner of a square are designated by its letter and plus in one direction and its number and plus in the other direction. Thus $(B + 50)$-7 is a point 50 ft. from line B toward line C, and on line 7; and E-$(5 + 40)$ is a point on line E, 40 ft. from line 5 towards line 6.

When the engineer's level is used it is set in any convenient position and a backsight is taken on a point of known elevation; turning points are established as each new set-up is required. Rod readings to ground points are taken to tenths of feet, as in profile leveling.

Survey of Proposed Rice Farm, C.K. Perry. 34.

For Contour Map.
Scale 1" = 50', Contour Int. = 1 ft.

Sta.	B.S.	H.I.	F.S.	El.	
B.M.1	5.31	105.31		100.00	(Assumed)
A-8			7.3	98.0	
B-8			6.0	99.3	
C-8			4.1	101.2	
D-8			2.2	103.1	
E-8			1.4	103.9	
E-7			2.9	102.4	
D-7			4.2	101.1	
C-7			5.0	100.3	
(B+50)-7			5.6	99.7	
B-7			4.8	100.5	
A-7			3.2	102.1	
A-(7+60)			6.2	99.1	
A-6			2.5	102.8	
(A+50)-6			0.4	104.9	
B-6			1.9	103.4	
C-6			3.6	101.7	
D-6			4.2	101.1	
E-6			3.5	101.8	
E-(5+40)			6.7	98.6	
E-5			6.1	99.2	

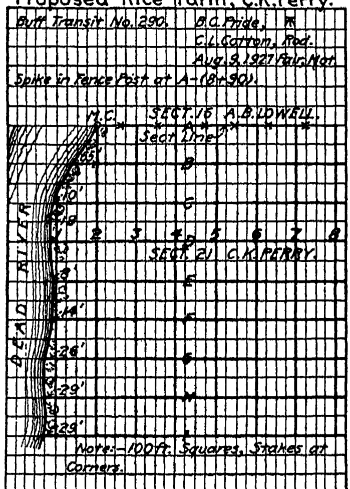


FIG. 134.—Cross-section notes.

In rough, wooded country where long sights cannot be obtained and where approximate elevations (say to the nearest half-foot or foot) will answer the purpose, more rapid progress may be secured by using the hand level and topographer's rod (see Art. 466, p. 683). Usually the elevations of one or more stations on each cross-section line are accurately determined with the engineer's level. These stations then serve as vertical control points to which cross-section levels run with the hand level may be tied.

135. Route Cross-sections.—Preliminary surveys for railroads, highways, and canals are often made by running a chained traverse line along the proposed route, stations being established by stakes set every 100 ft. as illustrated by the full line of Fig. 135. The eleva-

tions of the stations are then determined by profile leveling, as already described. To furnish data for location studies and for estimating volumes of earthwork, the shape of the ground on either side of the traverse line may be determined by running levels over lines at right angles to the traverse. Usually these crosslines intersect the traverse at each station, as indicated by the dash lines of Fig. 135. The elevations may be determined with either the engineer's or the hand level, depending upon the desired precision and the length of the crosslines. Generally the hand level is used in rough country and the engineer's level is employed when the ground is comparatively

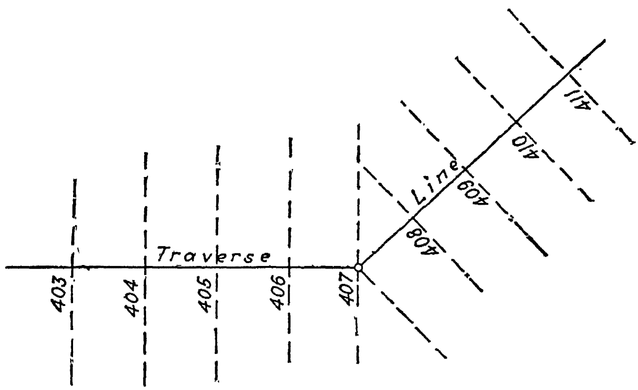


FIG. 135.

flat. If the crosslines are short their direction is laid off by eye; if they are long their direction is fixed at 90° with the traverse by means of a compass, transit, or other suitable instrument. The rod is held at breaks in the surface slope, and distances are measured from the traverse line to these points. The elevations of traverse stations are obtained from the profile levels. The center line of the right-hand page of the notebook represents the traverse, and distances and rod readings are recorded to the right or left of this line according to whether the corresponding points are on the right or left of the traverse.

136. Leveling for Earthwork.—Four general situations arise in connection with field measurements to determine volumes of earthwork.

1. A given area is to be cut or filled to a predetermined surface, for example, in excavating the basement for a building or in grading a piece of land. Cross-sections may be taken as described in Art. 134, though usually the sides of squares at the corners of which stakes are set will be less than 100 ft., sometimes as small as 10 ft. When

the grade of the finished surface has been established the cut or fill at each station will be known and the volume of earthwork can be calculated.

2. A trench is to be excavated, as when a sewer or pipe line is to be laid. Profile levels are run along the proposed line. When the grade of the bottom of the trench has been fixed, the cut at each station can be determined. Knowing the necessary width of the trench at the top and bottom and also its depth at each station, the volume of excavation can be calculated.

3. An irregular mass of unknown volume is to be excavated at a given site. For example, earth is excavated from borrow pits to furnish material for railroad and highway fills and canal banks, gravel is dug from pits and banks, and stone is blasted from quarries. It becomes necessary to determine the shape of the surface at the site both before and after the material has been removed.

4. Earth must be cut or filled to a given grade line along some route as a highway, railroad, or canal, and further, must have a prescribed shape of cross-section.

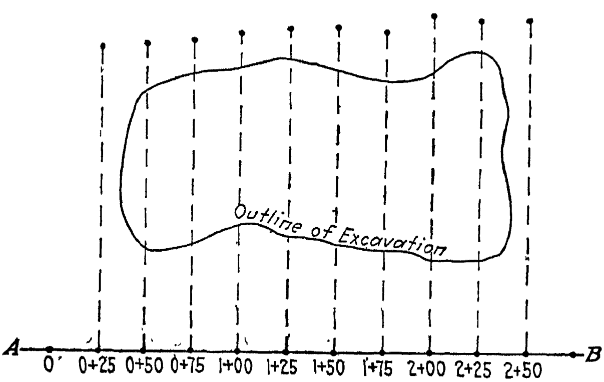


FIG. 137.—Borrow-pit cross-sections.

137. Borrow-pit Cross-sections.—Sufficient data for the calculation of the volume of a borrow pit or similar excavation may be obtained by taking cross-sections of the site before and after the material has been removed. When the site for a borrow pit has been fixed, a base line, as *AB* (Fig. 137) is established in a position where it will remain undisturbed by the future excavation. At regular intervals (such as 10, 25, or 50 ft.) along the line stakes are set and crosslines through these points are established as shown in the figure. Frequently a stake is set at the far end of each crossline to fix the extreme limits of the excavation. Levels are then run over the cross-

lines, rod readings to tenths of feet being taken at frequent intervals (so that surface irregularities will be measured accurately), and distances being measured from the base line. When the material has been removed, the crosslines are reestablished and levels are rerun over such portions as are included within the lines of excavation. A few additional measurements are necessary where the ends of the borrow fall between crosslines. The difference between the original cross-section and the final cross-section shows the area cut at each crossline, from which the volume can be determined.

138. Road Cross-sections.—Figure 138*a* illustrates typical railroad or highway cross-sections in cut and in fill. The center line

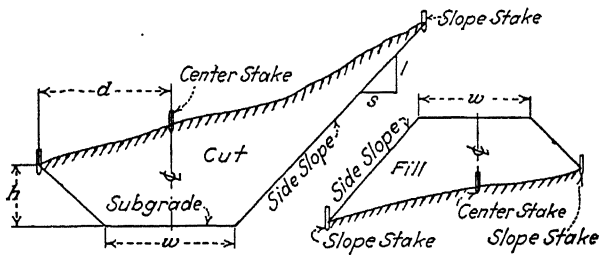


Fig. 138*a*.—Road cross-sections.

of the roadbed is located by center stakes set every 100 ft. (sometimes every 50 ft.), and profile levels are run as described in Art. 132. The profile is plotted and the grade line of the subgrade (*i.e.*, the roadbed upon which the pavement is placed in the case of the highway or upon which the ballast is placed in the case of the railroad) is established. The center cut or fill at each station is equal to the difference between the elevation of the ground line (as determined by the profile levels) and the elevation of the grade line (as established on the profile).

Prior to actual construction, cross-sections are taken at full stations, at points where the line runs from cut into fill, and at such plusses as are necessary to provide reliable data upon which calculations of volume may be based; also, as a guide to those who are to do the grading, *slope stakes* are set opposite each center stake at the points marking the intersection of the side slopes with the natural ground surface, and both center and slope stakes are marked with the cut or fill (the distance above or below subgrade) at the point where the stake is driven.

The subgrade is usually a plane surface, transversely level, and on a given road is of uniform width in cut and a uniform but not usually the same width in fill. The side slopes are plane surfaces of constant

slope for a given material of excavation. The rate of the side slope is stated in terms of the number of units measured horizontally to one unit measured vertically. Thus a 2 to 1 slope indicates a slope which in a horizontal distance of 2 ft. rises (or falls) 1 ft. The slope most commonly employed for cuts or fills through ordinary earth is 1½ to 1. For coarse gravel, the slope is often made 1 to 1; for loose rock, ½ to 1; for solid rock, ¼ to 1; for soft clay, 2 or 3 to 1.

138a. Level readings for cross-sections are usually taken with the engineer's level, and distances to the right or to the left of the center stake to points where observations are taken are measured with a metallic tape. Rod readings and distances are observed to tenths of feet. Prior to going to the field the leveler secures a record of elevations of ground points as obtained from the profile levels, and also the elevation of the established grade at each station. In the field, the instrument is set up in any convenient location and the H.I. is obtained by a backsight on a bench mark. As a check on the profile levels the rod is held on the ground (sometimes on a short wooden peg driven flush with the ground) at a given station, a foresight is taken, and the cut or fill is calculated and marked on the back of the stake. A crossline through the station is established by estimation or by laying off a 90° angle with the tape (or with the transit if the cut is to be deep or the fill is to be high). On each side of the center line, rod readings and distances from the center are taken to points of marked change in slope along the cross-section until the estimated location of the slope stake is reached. Here the rod is moved up or down the slope until by trial the measured distance from the center stake is made equal to the computed slope-stake distance for the particular cut or fill indicated by the rod reading; then the slope stake is set at this point.

If the ground is level in a direction transverse to the center line, the only rod reading necessary is that at the center stake, and the distance to the slope stake can be calculated once the center cut or fill has been determined; such a cross-section is called a *level section.* If the ground slopes in both directions from the center line, rod readings must be taken at each slope stake, in addition to the one taken at the center; the cross-section is called a *three-level section.* When rod readings are taken to the center stake, the slope stakes, and at points on each side of the center at a distance of half the width of the roadbed the cross-section is called a *five-level section.* A cross-section for which observations are taken to points between center and slope stakes at irregular intervals is called an *irregular section.* Where the cross-section passes from cut to fill, it is called a *side-hill section*, and an additional observation is made to determine the

distance from center to the grade point; that is, the point where the subgrade will intersect the natural ground surface. A peg is usually driven to grade at this point and its position is indicated by a witness stake marked "grade."

138b. Figure 138*b* illustrates a suitable form of cross-section notes. The left-hand page is seen to be essentially the same as for profile leveling with the exception that a column is added for grade elevations. The notes are for a portion of the line for which profile level notes are shown in Fig. 133. The cross-section levels are seen to check the elevations of stations as determined by the profile levels within 0.1 or 0.2 ft., which is as close as can be expected. The cross-section notes on the right-hand

Cross-Sections
Cox Brook to Big Forks
Road Bed 20 ft. in Cut, 16 ft. in Fill
Slope 1½:1

I. N. R. Final Location 21
June 4, 1928 — F. F. Smith ⊼
Cloudy — J. Richie Rod
O. Byram Tape

Sta.	B.S.	H.I.	F.S.	Elev	Grade	Left	Ctr.	Right	Remarks
B.M.28	2.67	566.98		564.31					50 ft. Lt. Sta 605+10
605			1.9	565.1	556.00	$\frac{C8.6}{22.9}$	C9.1	$\frac{C11.2}{26.8}$	Gravel in this hill
606			4.0	563.0	555.60	$\frac{C5.0}{17.5}$	C7.4	$\frac{C8.4}{22.6}$	
607			4.5	562.5	555.20	$\frac{C4.6}{16.9}$	C7.3	$\frac{C8.0}{22.0}$	
608			8.0	559.0	554.80	$\frac{C1.8}{12.7}$	C4.2	$\frac{C5.1}{17.7}$	
+25			9.2	557.8	554.70	$\frac{0.0}{10.0}$	C3.1	$\frac{C2.6}{13.9}$	
T.P.	1.94	557.19	11.73	555.25					On plug
+90			2.8	554.4	554.44	$\frac{F1.8}{10.7}$	0.0	$\frac{C2.4}{13.6}$	
609			3.4	553.8	554.40	$\frac{F3.2}{12.8}$	F0.6	$\frac{0.0}{4.0}$ $\frac{C1.0}{11.5}$	
+20			5.6	551.6	554.32	$\frac{F3.6}{13.4}$	F2.7	$\frac{0.0}{8.0}$	
610			11.1	546.1	554.00	$\frac{F8.2}{20.3}$	F7.9	$\frac{F6.3}{17.5}$	
+40			11.2	546.0	553.84	$\frac{F8.0}{20.0}$	F7.8	$\frac{F7.8}{19.7}$	Top of bank Cox Brook
+45			14.6	542.6	553.82	$\frac{F11.2}{24.8}$	F11.2	$\frac{F11.2}{24.8}$	In brook
+60			14.5	542.7	553.76	$\frac{F11.1}{24.7}$	F11.1	$\frac{F11.1}{24.7}$	" "
+65			11.6	545.6	553.74	$\frac{F8.2}{20.3}$	F8.1	$\frac{F8.0}{20.0}$	Top of bank
611			10.9	546.3	553.60	$\frac{F7.8}{19.7}$	F7.3	$\frac{F7.2}{18.8}$	
612			9.7	547.5	553.20	$\frac{F7.6}{19.4}$	F5.7	$\frac{F3.1}{12.7}$	
613			10.8	546.4	552.80	$\frac{F7.8}{19.7}$	F6.4	$\frac{F2.0}{11.6}$	
T.P.	11.96	559.69	9.46	547.73					On stump
	16.57	564.31	21.19						
		559.69	16.57						
		4.62 = 4.62 ck.							

Fig. 138*b*.—Railroad cross-section notes.

page are in line with the station to which they refer. They illustrate a portion of a line passing from cut into fill where the slope is such that three-level sections are adequate. It is to be noted that cross-sections are taken at 608 + 25 where the left edge of the roadbed passes from cut to fill, at 608 + 90 where the center line passes from cut to fill, and at 609 + 20 where the right edge of the roadbed passes from cut to fill. The cross-sections at 608 + 90 and 609 are side-hill sections. It is seen that cuts or fills are shown, rather than elevations, and that the cut or fill and the distance out for each point is expressed in a form resembling that of a fraction; the numerator of the fraction (cut or fill) and the denominator (the distance out) are the coordinates for which the origin is the midpoint of the roadbed at subgrade. The values in columns marked "right" or "left" are for points at which the slope stakes are driven

138c. Canal Cross-sections.—Canal cross-sectioning is carried out in a manner similar to that for highways and railroads. Three cases commonly arise:

1. *Canal All in Cut.*—Requires no artificial banks. The field work is the same as for a road, slope stakes being set at the intersections of the side slopes with the actual ground surface.

2. *Canal with Two Banks.*—Cross-sections partly in cut and partly in fill. Figure 138c illustrates this form of section. Stakes marking

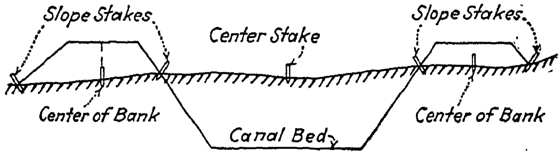


FIG. 138c.—Canal cross-section.

the center line of the canal are placed, and profile levels are run as in railroad or highway work. The cross-section party takes cross levels and sets center stakes and slope stakes for each bank. The distance from canal center to bank center is a constant so long as the canal section remains unchanged. The distances from center to slope stakes depend upon the cut or fill and must be determined by trial.

3. *Canal on Side Hill.*—The case is similar to a railroad or highway on a side hill, except that on the downhill side an artificial bank is required. The center and slope stakes for this bank are set as in 2. The canal bed is (or should be) always in cut for its full width.

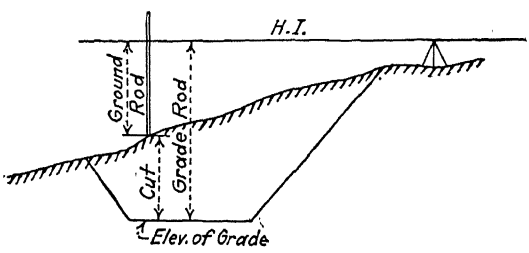


FIG. 138d.—Road cross-section in cut.

138d. Cuts and Fills.—Figure 138*d* shows the level in position for taking rod readings at a section *in cut.* The height of instrument (H.I.) has been determined; the elevation of grade at the particular station is known. The leveler calculates the difference between the H.I. and grade elevation, which difference is known as the *grade rod;* that is, H.I. − El. of grade = grade rod. The rod is held at any point for which the cut is desired, and a reading is taken, which reading is called the *ground rod.* The difference between the grade rod and the ground rod is equal to the cut. Ordinarily,

when the grade rod has been computed, the cut at any point is determined by mental calculation.

Figure 138e is a similar illustration for a cross-section *in fill.* It is clear that if the H.I. is *above* grade, the fill is the *difference* between the ground rod and the grade rod; if the H.I. is *below* grade the fill is the *sum* of the grade rod and the ground rod.

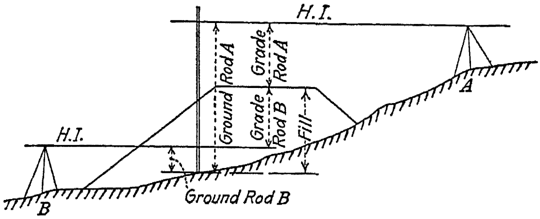


Fig. 138e.—Road cross-section in fill.

139. Setting Slope Stakes.—The process of setting slope stakes requires some additional explanation.

If w is the width of roadbed or canal bed, d is the measured distance from center to slope stake, s is the side-slope ratio (ratio of horizontal distance to drop or rise), and h is the cut (or fill) at the slope stake, then by Fig. 139 when the slope stake is in the correct position

$$d = \frac{w}{2} + hs \tag{1}$$

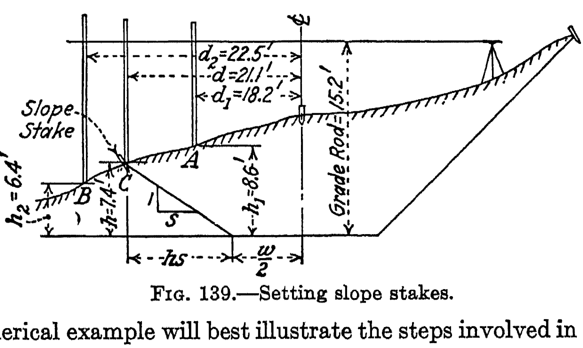


Fig. 139.—Setting slope stakes.

A numerical example will best illustrate the steps involved in securing the equality designated in Eq. (1). Let w = 20 ft.; side slope, $1\frac{1}{2}$ to 1; grade rod, 15.2 ft. Suppose a slope stake is to be set on the left of the center stake (Fig. 139). As a first trial the rod is held at A, ground rod = 6.6 ft.; h_1 = grade rod − ground rod = 15.2 − 6.6 = 8.6. The calculated distance for this value of h_1 is $\frac{w}{2} + h_1s$ = $10 + 8.6 \times \frac{3}{2} = 22.9$ ft. Measurement from the center stake shows d_1 to be 18.2. Hence the rodman should go farther out. For

a second trial the rod is held at B; ground rod = 8.8; h_2 = grade rod − ground rod = 15.2 − 8.8 = 6.4 ft. $\frac{w}{2} + h_2s = 10.0 + 9.6 =$ 19.6 ft. The measured value of d_2 is 22.5 ft. Hence the rod is too far out. As a third trial the rod is held at C, ground rod = 7.8; h = 15.2 − 7.8 = 7.4 ft. $\frac{w}{2} + hs = 10.0 + 7.4 \times 3/2 = 21.1$ ft. The measured value of d is 21.1 ft., hence this is the correct position of the slope stake. In the notes the coordinates of the slope stake are given by the expression $\frac{c7.4}{21.1}$, but the trial observations are not recorded. While the above example is for a cut, the same procedure is followed in fill.

Slope stakes are set side to the line. On the back of the stake is marked the station number. On the front (side nearest the center line) is marked the cut or fill at the stake, and sometimes the distance from center to slope stake. The numbers read down the stake.

140. Use of Ward Tape and Tape Rod.—The Ward tape is a metallic tape specially designed to facilitate the work of setting slope stakes by mechanically solving the expression $\frac{w}{2} + hs$. The face of the tape is graduated in feet and tenths in the customary manner. On the back the zero point is at the distance $\frac{w}{2}$ from the end of the tape, and the graduations are s times as far apart as those on the face. Thus a Ward tape designed for a 20-ft. roadbed and 1½ to 1 slope would have its "back" zero point 10 ft. from the end and a "back" reading of 4.0 would be 16.0 ft. from the end of the tape. Hence a given reading on the back of the tape indicates the cut or fill for which the corresponding reading on the front of the tape is the calculated distance from center to slope stake.

If the cross-section is in cut, the tape rod (Art. 109*f*) is held with figures wrong side up at the center stake, and the ribbon is moved until the rod reading equals the cut at the center. When the rod is held at any point in the cross-section its reading gives the cut directly, and no calculations are required. If the cross-section is in fill the rod is held with figures right side up. For a side-hill section passing from cut to fill the rod is held with figures wrong side up until the grade point is established. It is then reversed, held on the grade point, and the ribbon is turned until the rod reading is zero. Fills for the remaining points in the cross-section are read directly, as before.

When a slope stake is to be set, the rod is held at a trial point and the cut or fill is observed. The leveler calls out this value to the rodman, who observes the "cut" or "fill" distance on the back of the tape, at which the rod is held. The rod is moved up or down the slope until the rod reading agrees with the "cut" or "fill" distance, when the correct position of the slope stake is determined. The rodman then turns the

tape over and reads the actual distance from the center, which is likewise the computed distance.

The use of the tape rod eliminates subtracting the ground rod from the grade rod, and the use of the Ward tape eliminates adding one-half the width of the roadbed to the cut or fill multiplied by the slope. Their use greatly reduces the chances of mistake and speeds up the work. While a given Ward tape is designed primarily for a particular width of roadbed it may be employed for other widths without great inconvenience. Thus, if designed for $\frac{w}{2} = 10$ ft. it may be used when $\frac{w}{2} = 12$ ft. by fastening to the ring a string 2 ft. long and measuring from the end of the string, but adding 2 ft. to all distances out, observed by the rodman.

141. Setting Grades.—Typical works of construction where grades are required to be established in the field are as follows:

1. *Buildings.*—For the excavation, the batter boards which are commonly set up to give line may be set some whole number of feet above the bottom. A pair of posts is driven at each end of each outside building line, the posts being transverse to the line, about 3 feet apart, and about 3 feet outside the excavation. The leveling rod is held at each post, a sight is taken, and the grade (plus the chosen distance) is marked; a board is then nailed across the two posts, with the top of the board at this level. A nail is driven in the top of the board on the building line, which is given with the transit. Carpenter's lines stretched between nails of opposite batter boards define both the line and the grade, and measurements can be made conveniently by the workmen for excavation, setting forms, and alining masonry and framing.

Grades for footings are given by ground pegs driven to the elevation either of the top of the footing or of the top of the floor. Lines for footings are given by batter boards set in the bottom of the excavation. Column bases and wall plates are set to grade directly by the leveler. The position of each column is marked in advance on the footing, and when either the concrete form, the steel member, or the first course of masonry has been placed, its alinement and grade are checked directly. Similarly, at each floor level the governing lines and grade are set and checked, except that for pre-fabricated steel framing, the structure as a whole is plumbed by means of the transit at every second or third story level. Throughout the construction of large buildings, selected key points are checked in order to detect settlement, excessive deflection of forms or members, errors, or mistakes.

2. *Sewers and Pipe Lines.*—Grades may be given either by batter boards set some whole number of feet above the bottom of the trench or top of pipe, or by ground pegs or other suitable reference marks

set just outside the trench. For example, on a paved street nails or crayon marks may be used. Distances below such points to the actual grade are marked on stakes or pavement. On account of the fact that many foremen are unfamiliar with tenths of feet, these distances are usually given in feet, inches, and eighths of inches.

3. *Pavements.*—For city streets, ground pegs are usually driven to the grade of the top of the curb and are placed just outside the curb line on both sides of the street, ordinarily at 50-ft. intervals. The curb is built first, and the grade for the edge of the pavement is then marked on the face of the curb. Ground pegs are set on the center line of the pavement, either at the grade of the finished subgrade (in which case holes are dug when necessary to place stakes below the ground surface) or with the cut or fill indicated on the peg or on an adjacent stake. Where the street is wide, an intermediate row of stakes may be set between center line and curb, conforming to the crown of the pavement. It is usually necessary to reset the pegs after the street is graded.

For highways, the procedure is similar to that for streets, except that at each edge the pegs are driven to the grade of the edge of the pavement. For concrete highways, pegs are set so that the side forms may be placed directly upon them, and a line of stakes is set near one edge, from which to aline the forms. Excavation and width are subsequently checked by means of templates conforming to the cross-section of the subgrade or of the finished pavement.

4. *Railroad Track.*—Track is usually laid on the subgrade and is lifted in position after the ballast has been dumped. On one side of the track and perhaps 3 ft. from the rail, stakes are usually driven to the elevation of the grade of top of rail.

141a. The operation of leveling for grades is similar to profile leveling. After the grade line has been established on the profile, the grade elevation for each station is known. Leveling for grades is started from a bench mark and is carried forward by turning points. The grade rod to be employed in setting a given grade stake to grade is calculated by subtracting the grade elevation from the H.I. The rodman starts the stake and holds the rod on its top. The leveler reads the rod and calls out the approximate distance the stake must be driven to reach grade. The rodman drives the stake nearly the desired amount, and a second rod reading is taken; and so the process is continued until the rod reading is made equal to the grade rod. The top of the stake is usually marked with crayon to indicate that it is at grade. Sometimes the rod is moved up or down the side of the stake until the grade rod is read, when the position of grade is indicated by a crayon mark or a nail driven into the stake at the

foot of the rod. When points are established a given distance above or below grade the process is the same, except that the distance to grade is indicated either on the grade stake or on a witness stake nearby. Usually grade elevations are determined to hundredths of feet.

The notes are kept as in profile leveling except that the right-hand column of the left-hand page is for grade elevations. Also when the stakes are not driven to grade a record is kept of the cuts or fills (that is, the distances from established points to the actual grade line). The intermediate foresights recorded are the calculated grade rods, if the stakes are driven to grade; in any case the last reading taken on the grade stake at each station is shown in the foresight column opposite the station to which it refers.

The distance between points at which grade is established depends upon the character of the work and upon whether the grade is uniform (straight line in profile) or on a vertical curve. On track construction, grades are usually given at each 100-ft. station but are sometimes given at every 50 ft. on vertical curves. For road pavements and sewers, grade is generally established every 50 ft. if the grade is uniform and every 25 ft. or even every 10 ft. if the grade is on a vertical curve.

142. Shooting-in Grade.—Unless the grade is level the methods described in the preceding article necessitate calculating the grade rod for every station which is established at grade. Likewise if grade is established at any intermediate point the chainage of which has not been previously determined, the plus of the new station must be measured before the grade rod can be calculated.

When the line is tangent (straight in plan) and the grade is uniform for a considerable distance, the work of setting grade stakes may be facilitated by "shooting-in" the grade in the manner now to be described. Let A and B be two stations some distance apart (say 800 to 1,000 ft.) on tangent, between which stations a uniform grade is to be established. A line of differential levels is run to include both A and B, and stakes are driven to grade (or to a fixed distance above or below grade) at these points. The level is then set up close to B (with one pair of foot screws in line with A), and with the rod held on the stake, a reading is obtained by sighting through the objective end of the telescope (as in the two-peg test of the dumpy level). The rodman next holds the rod on A, and the leveler moves the telescope in a vertical plane (by means of the foot screws if a level is used, or by means of the altitude tangent screw if the transit is used) until the horizontal cross-hair appears to cut the rod at the reading previously obtained with rod at B. Neglecting the effect

of the earth's curvature and atmospheric refraction (which will be of no consequence for the distances suggested above) the line of sight is now a uniform distance above the uniform grade for all points between A and B. Hence the grade at any intermediate station is established by observing the same rod reading, the instrument, of course, remaining undisturbed for the interval during which the intermediate stakes are being set. As a check, it is well to sight again to A before the instrument is moved, in order to detect any displacement of the line of sight.

When the grade is nearly level, as is often the case for drainage works, the process described above may be simplified. The sensitiveness of the bubble may be determined as described in Art. 103*a*, and the number of spaces on the level tube corresponding to various rates of grade may be calculated. By means of the bubble the line of sight may then be inclined at the same slope as the grade line. Employing this method, differential levels are run, each set-up of the instrument being on line, and close to a station. When the H.I. is determined by a backsight to a turning point, the grade rod for that station near which the instrument is set up is calculated, and the line of sight is brought parallel with the grade line by means of the bubble, as just described. Stakes between the turning point and the instrument are then set to this reading. For the stations in advance, the line of sight is, of course, inclined in the opposite direction (*i.e.*, the bubble moves to the corresponding position at the other end of the tube when level is reversed). Finally the rodman chooses a new turning point in advance, and a foresight is taken with the instrument leveled. It is necessary to exercise considerable care in placing one pair of foot screws on line, otherwise the horizontal cross-hair is likely to be appreciably displaced from the true horizontal during the process of bringing the line of sight parallel with the grade line.

143. The Gradienter.—Some levels are equipped with a micrometer screw at one end of the crossbar, by means of which the telescope may be turned through small vertical angles without moving the foot screws. When the screw has such a pitch that one full turn (sometimes two full turns) will cause a change of 1 per cent in the inclination of the line of sight, and further, the screw has an attached drum the rim of which is divided into 100 parts (sometimes into 50 parts), it is called a *gradienter*. When grades are to be established with a level equipped with gradienter the elevations are carried forward by direct leveling, as described in the preceding article, the instrument always being set up near a station on the line. After the H.I. for a given set-up has been determined, the gradienter drum is set to read zero when the instrument is level. The line of sight is then made parallel with the grade line by turning the gradienter until it reads the desired grade. (Thus, if one turn of the screw inclines

the line of sight 1 per cent and the gradienter drum is divided into 100 parts, a 1.2 per cent grade would be laid off by one full turn and 20 spaces as indicated by the drum graduations.) Since the line of sight is parallel with the grade line, the grade rod for the station at which the level is set up is also the grade rod of any other station in the direction in which the instrument is pointed. For points in advance of the level, the gradienter must be set to a reading equal and opposite to that for points in the rear.

The altitude tangent screw of the engineer's transit is often designed to be used as a gradienter. It is employed in the manner just described.

144. Contour Leveling.—In topographic surveying the engineer's level, in conjunction with other instruments, is sometimes employed for the direct location of contours.[1] Also in connection with the surveys for reservoir sites, levels are usually run to establish the proposed shore line. The process of establishing lines of this character consists in carrying a line of levels forward by turning points and in finding, by trial, a series of ground points at the required elevation.

Proposed shore lines are usually defined by stakes set at long intervals where the shore contour is straight and at short intervals where it is irregular or curved. A line of levels is run approximately along the contour. At each set-up the contour rod (the difference between the elevation of the H.I. and the elevation of the contour) is computed. The rodman proceeds along the line giving rod readings at critical points. At each point he is directed up or down the slope until the leveler reads the contour rod. Here a stake is set. In topographic surveying the process is essentially the same, except that no stakes are set (see Art. 471*b*, p. 692).

145. Establishing Grade Contours.—In connection with the preliminary surveys for highways and canals in hilly or mountainous country where the general route lies along a side hill, levels are frequently run to establish points along some required grade. The irregular line joining such points is termed a *grade contour*. If a level (or transit) with gradienter is available, the simplest method is to follow the same procedure as in contour leveling (Art. 144), except that for both turning points and ground points the gradienter is set so that the line of sight is at the required grade.

Generally rod readings are taken only to points where there is a noticeable change in the direction of the grade contour. The level should be in such a location that the ground points will not deviate greatly from the straight line joining the instrument and the adjacent turning points.

[1] A contour is an imaginary line connecting points of equal elevation on the surface of the earth (see Art. 431, p. 634).

Often in rough work the only intermediate ground point between turning points is at or near the level, and the rod readings for turning points are taken with rod held on the ground.

In more careful work, as when the grades are nearly level, the elevations may be carried forward as in differential leveling (that is, bubble centered when backsights and foresights are taken to turning points) and distances may be measured by stadia; at the same time the grade contour is located as described above. The stadia distances and the elevations determined by differential leveling offer a means of checking the grade contour at any point.

If the level is not equipped with a gradienter, unless the grade is nearly level so that the required slope can be laid off with the bubble, the grade contour must be established by the more laborious method of direct leveling, distances being measured by stadia or by pacing and a new grade rod being calculated for each point at which a stake is driven, as described in Art. 141*a*.

146. Vertical Curves.—On both railways and highways, in order that there may be no abrupt change in the vertical direction of moving vehicles, adjacent segments of differing gradient are connected by curves in a vertical plane. These are termed *vertical curves*. Usually the vertical curve is the arc of a parabola, this form being well adapted to gradual change in direction and elevations along the curve being readily calculated. The length of vertical curve depends upon conditions, being in general greater for railroads than for highways, and increasing with the difference in grade between adjacent segments. The maximum allowable rate of change in grade per station is usually governed by specifications. The length of the vertical curve can not be less than the algebraic difference in gradient between the two segments connected, divided by the maximum allowable rate of change per station. Usually in railroad work the length of vertical curve is an even number of stations and in highway work some convenient whole number of feet.

When two segments each of uniform but differing gradient intersect, the station and plus of the point of intersection and the elevations of stations along the grade lines are determined. The length of the vertical curve to connect the two segments is then assumed and the stations and elevations of the beginning and end of curve are calculated. Following this the offsets from the uniform gradient to the curve are computed, making use of the property of the parabola, that the offset from a parabola to its tangent varies as the square of the distance from the point of tangency. These offsets furnish sufficient data so that elevations of points along the curve may be calculated. In the field the vertical curve is laid out by setting grade stakes at the computed elevations, just as along a uniform

grade. The calculations for a vertical curve are illustrated by the following example:[1]

Example: On a railroad a +0.8 per cent grade meets a −0.4 per cent grade at Sta. 90 + 00 and at Elev. 100.0. The maximum allowable rate of change of grade per station is 0.2. It is desired to establish a vertical curve connecting the two grades.

The algebraic difference in gradient is +0.8 − (−0.4) = 1.2 per cent. The maximum rate of change is 0.2 per cent. Hence the length of curve is $\frac{1.2}{0.2} = 6$ stations or 600 ft. The length on either

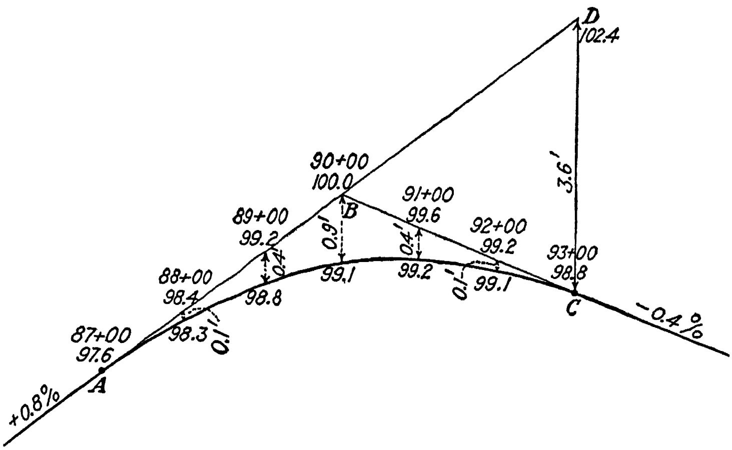


Fig. 146.—Vertical curve.

side of the vertex ($AB = BC$, Fig. 146) is $\frac{600}{2} = 300$ ft. The station at A is therefore $90 - 3 = 87$ and the station at C is $90 + 3 = 93$. The elevation of A is $100.0 - 3 \times 0.8 = 97.6$ and the elevation of C is $100.0 - 3 \times 0.4 = 98.8$.

The elevation of the point D on the +0.8 per cent grade opposite C is $100.0 + 3 \times 0.8 = 102.4$. The line AD is tangent to the curve at A. Hence the offset at C between tangent and curve is $102.4 - 98.8 = 3.6$ ft. Making use of the property of the parabola that offsets vary as the squares of their distances from the point of tangency A (Sta. 87),

the offset at Sta. 88 is $\frac{(1)^2}{(6)^2} \times 3.6 = 0.1$ ft.

at Sta. 89 is $\frac{(2)^2}{(6)^2} \times 3.6 = 0.4$ ft., and

at Sta. 90 is $\frac{(3)^2}{(6)^2} \times 3.6 = 0.9$ ft.

[1] In practice, grade elevations are measured to the nearest 0.01 ft.

For similar reasons the offset from the tangent *CB*
at Sta. 91 is 0.4 ft., and
at Sta. 92 is 0.1 ft.

The elevations of points on curve are then determined as shown in the following tabulation.

Station	87 = *A*	88	89	90	91	92	93 = *C*
El. of tangent	97.6	98.4	99.2	100.0	99.6	99.2	98.8
Tangent offset	0.0	0.1	0.4	0.9	0.4	0.1	0.0
El. of curve	97.6	98.3	98.8	99.1	99.2	99.1	98.8

A convenient check is afforded by calculating the "second differences" between the elevations of consecutive points on the curve, since for a parabola these should be a constant. For the example just given, the following tabulation illustrates the computations.

Stations	Elevation, ft.		Diff., ft.	Second diff., ft.
	Back sta.	Forward sta.		
87–88	97.6	98.3	+0.7	
				0.2
88–89	98.3	98.8	+0.5	
				0.2
89–90	98.8	99.1	+0.3	
				0.2
90–91	99.1	99.2	+0.1	
				0.2
91–92	99.2	99.1	−0.1	
				0.2
92–93	99.1	98.8	−0.3	

146a. An alternate method of computing the elevations of points along a vertical curve is as follows:

The elevation of the mid-point of the long chord is found by computing the average of the elevations of the beginning and ending of the vertical curve (points of tangency). The elevation of the mid-point of the vertical curve is the mean of the elevation of the mid-point of the long chord and the elevation of the vertex, or point of intersection of the tangents. The tangent offsets to various points along the curve are then computed, employing the known property of a parabola, that the tangent offset varies as the square of the distance from the tangent point.

Thus, in Fig. 146, the elevation of the mid-point of the long chord is the average of the elevations A and C.

$$\tfrac{1}{2}(97.6 + 98.8) = 98.2 \text{ ft.}$$

The mid-point of the vertical curve is midway between the vertex B and the mid-point of the long chord

$$\tfrac{1}{2}(100.0 + 98.2) = 99.1 \text{ ft.}$$

The offset from vertex to curve is

$$100.0 - 99.1 = 0.9 \text{ ft.}$$

The tangent offsets at stations 89 and 91 are

$$\frac{(2)^2}{(3)^2} \times 0.9 = 0.4 \text{ ft.}$$

and the offsets at stations 88 and 92 are

$$\frac{(1)^2}{(3)^2} \times 0.9 = 0.1 \text{ ft.}$$

147. Numerical Problems.

1. Complete the profile level notes shown below.

Sta.	B.S.	H.I.	F.S.	El.
B.M. 10	6.32	....	....	836.76
179	....	....	10.1	
180	....	....	7.8	
+35	....	....	12.6	
181	....	....	4.7	
182	....	....	3.4	
T.P. 38	7.32	....	2.11	
183	....	....	8.5	
+40	....	....	4.6	
184	....	....	7.2	
185	....	....	10.6	
T.P. 39	5.93	....	11.49	
186	....	....	4.2	

2. The width of roadbed of a proposed railroad is 24 ft. in cut and the side slopes are 1½ to 1. At a given station the elevation of grade is 515.75. For obtaining the cross-section at the station the H.I. of the level is 528.32. The ground rod at the center stake is 6.5. Compute the grade rod and the center cut. The rod reading at the right slope stake is 1.2 and at the left slope stake is 10.9. Compute the cut and the distance out to each slope stake.

3. Make a page of cross-section notes for a highway running from cut into fill. The grade of the highway is 4.0 per cent, width of roadbed 24

ft. in cut and 18 ft. in fill, side slopes $1\frac{1}{2}$ to 1. Show observations at center and slope stakes and at grade points.

4. Make a page of notes for establishing grades of top of rail from station 750 to station 762. Elevation of grade at station 750, 381.6; −0.6 per cent grade stations 750 to 758; −0.4 per cent grade stations 758 to 762; elevation of bench mark near station 750, 378.47.

5. The radius of curvature of the level tube of a level is 75 ft. and one space on the tube is equal to $\frac{1}{10}$ in. How many spaces will the bubble have to be displaced from the center to make the line of sight parallel with a grade rising 1.5 ft. per mile?

148. Field Problems.

PROBLEM 1. PROFILE LEVELING FOR A RAILROAD

Object.—To determine the elevations necessary for plotting the profile of a line (see sample notes, Fig. 133).

Procedure.—(It is assumed that the line has been laid out and stakes, marked with consecutive station numbers, have been driven every 100 ft.) (1) If no bench mark is given, select some permanent point as a bench mark, assuming an elevation such that no station will fall below the datum.

(2) Adapt the procedure indicated in Arts. 132 and 133 to the field conditions encountered.

Hints and Precautions.—(1) Read the rod carefully to the nearest 0.01 ft. on bench marks and turning points, and quickly to the nearest 0.1 ft. on ground points. (2) To be consistent, ground elevations should be recorded only to the nearest 0.1 ft. (3) Rod readings should be taken at all full stations and at such other points (plus stations) as are necessary to obtain a sufficiently accurate profile. These plus stations in general will be at points where the slope of the ground changes noticeably and at highways, railroads, and streams. (4) Bench marks should be established every 1,500 or 2,000 ft. if the line is long. These should be placed at some distance to one side of the line, in such positions that they are not likely to be disturbed during construction. All bench marks should be well described in the notes. (5) It is customary to mark the number and elevation on each bench mark at the time it is established. It is important, therefore, that the computations for turning points be performed as the work progresses, and that each page of notes be checked as soon as it is filled.

PROBLEM 2. PROFILE LEVELING FOR A PIPE LINE

Object.—To prepare the line of a proposed sewer or water main for excavators. It is supposed that the line has already been run and that center stakes marked with station and plus have been set every 25 or 50 ft.

Procedure.—(1) Opposite each stake on the line and far enough from the line to insure its not being disturbed by the excavation, drive a short

peg (or spike) flush with the ground, and beside this peg drive a stake marked with the station number of the center stake and the offset of peg from center stake. (2) Start from a bench mark as in problem 1, and take profile readings on the ground pegs to the nearest 0.01 ft. Complete the level work as in problem 1. (3) Roughly plot the profile, fix the grade of the bottom of the trench, and determine the amount of cut at each station. (4) Mark the cut in plain figures to the nearest ⅛ in. on the side stakes.

Hints and Precautions.—(1) Take rod readings on the turning points with greater care than on the ground pegs. (2) The form of notes will be similar to Fig. 133, except that columns must be added for offsets of pegs from center line, grade elevations, and cuts. (3) All stakes should be marked to read down the stake. Center stakes should be driven with the marked side toward the beginning of the line. Side stakes should be driven side to the line in order that they will not be confused with center stakes. On the back or the side farthest from the line should appear the station number and offset; on the front or side nearest the line, the cut. (4) In paved streets or hard roads it is impossible to drive stakes or ground pegs; and spikes, chisel marks, or paint marks are used instead. The spikes are driven flush with the road surface. In order that they may be found without difficulty, their position with respect to more prominent objects is carefully recorded. (5) The purpose of the ground pegs is to furnish grade and line (after the center stakes are removed) to ditchers and pipelayers.

Problem 3. Setting Slope Stakes. Cross-sections

Object.—To prepare a proposed railroad or highway for grading, and to obtain data for calculating earthwork (see sample notes, Fig. 138*b*).

Procedure.—(1) From the level notes of problem 1 plot a profile and fix a grade such that the amount of cut will approximately balance the amount of fill. (2) Drive short pegs flush with the ground beside the center stakes, run profile levels over the line, checking the elevations obtained with those of problem 1; and mark on the back of each center stake the cut or fill at that point, as C 3.9 or F 4.7. (3) Opposite each center stake, at right angles to and on both sides of the line, set slope stakes at points where the side slope of the cut or fill will intersect the surface of the ground. These stakes should be driven side to the line, leaning toward the center line if in cut and away from the center line if in fill. They should be marked on the side facing the center line with the cut or fill and distance from the center stake, as $C\,\frac{6.2}{19.3}$; and on the side farthest from the center line they should be marked with the station number (of the center stake) and whether on the right or left, as L 17 + 00. The numbers should read down the stake. (4) Drive ground pegs at "grade points" or points where the grade passes from cut to fill, and mark the position of such pegs by stakes marked "grade." Grade pegs should be driven at the center and on either side at a distance of one-half the width of the roadbed from the center.

CHAPTER X

PLOTTING PROFILES AND CROSS-SECTIONS; VOLUMES OF EARTHWORK

PROFILES AND CROSS-SECTIONS

149. The Profile.—Usually the profile is plotted on regular profile paper. The general practice is to begin the profile at the left end of the sheet; the station numbers thus increase from left to right. The horizontal and vertical scales to be employed depend upon the purpose of the profile. If it is for fixing grades, as for a railroad or highway, a scale of 1 in. = 400 ft. (horizontal) and 1 in. = 20 ft. (vertical) is frequently used; if it is to be made the basis of earthwork calculations, as for a sewer or other pipe line, a scale as large as 1 in. = 40 ft. (horizontal) and 1 in. = 4 ft. (vertical) may be required.

Figure 149*a* illustrates, to reduced scale, a portion of the profile for a proposed railroad. In this case only the accentuated horizontal lines are shown, and the distance between them represents a difference in elevation of 5 ft. and the space between vertical lines represents a horizontal distance of 100 ft. The numbered heaviest horizontal lines indicate multiples of 50 ft. in elevation, and the heaviest vertical lines indicate multiples of 10 stations counting from the beginning of the line.

The profile is plotted from the profile level notes. The ground line is formed by drawing a line through the plotted points. Usually this is done as the elevations are plotted. The profile should not be a succession of straight lines between adjacent points, for this does not represent the actual variation in the ground; on the other hand the profile at the summits and depressions should not be unduly rounded, for on the drawing such points are exaggerated in sharpness on account of the exaggerated relation between horizontal and vertical scales. Figure 149*a* illustrates the irregular form of the profile.

Notes on the profile show the station and plus of important objects, as streams, roads, etc. crossed by the line. Such notes are placed directly above the points on the profile to which they refer. Generally an alinement diagram near the bottom of the sheet is so placed that points on the diagram are directly below corresponding points on the profile. In this way a ready comparison between profile

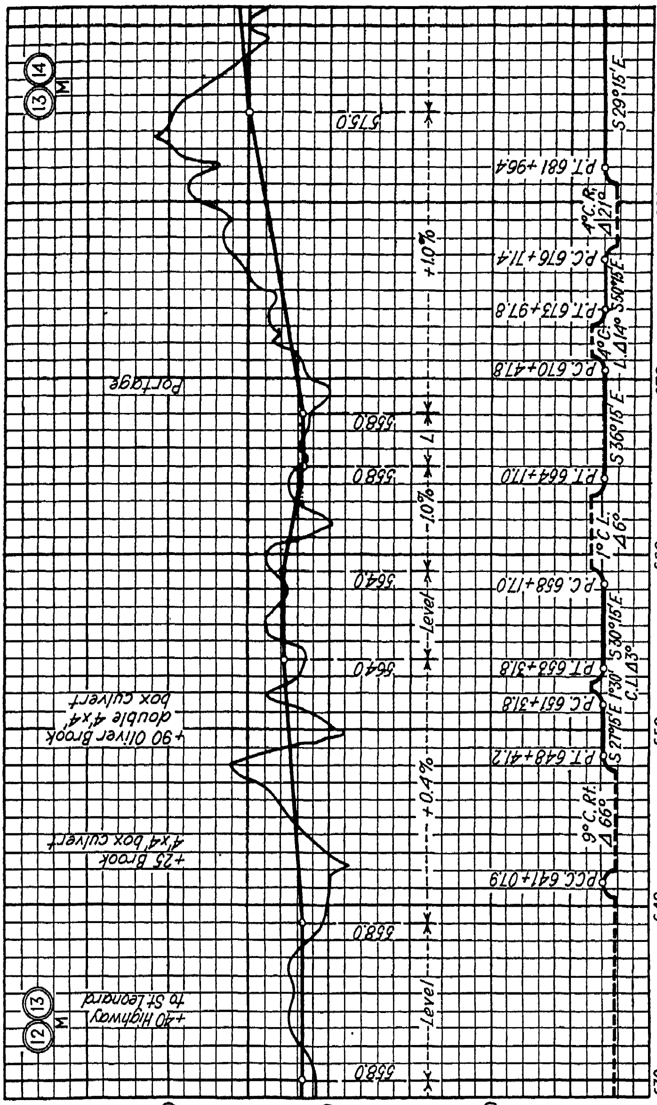


FIG. 149a.—Profile of railroad. (Only accentuated lines of paper shown.)

and plan may be made without referring to the map. The alinement diagram indicates the location of tangents, curves, and changes in direction of the line, but is not a true plan view, except when the line is straight. The diagram at the bottom of Fig. 149*a* is a suitable form for a railroad, highway, or similar line. Sometimes the

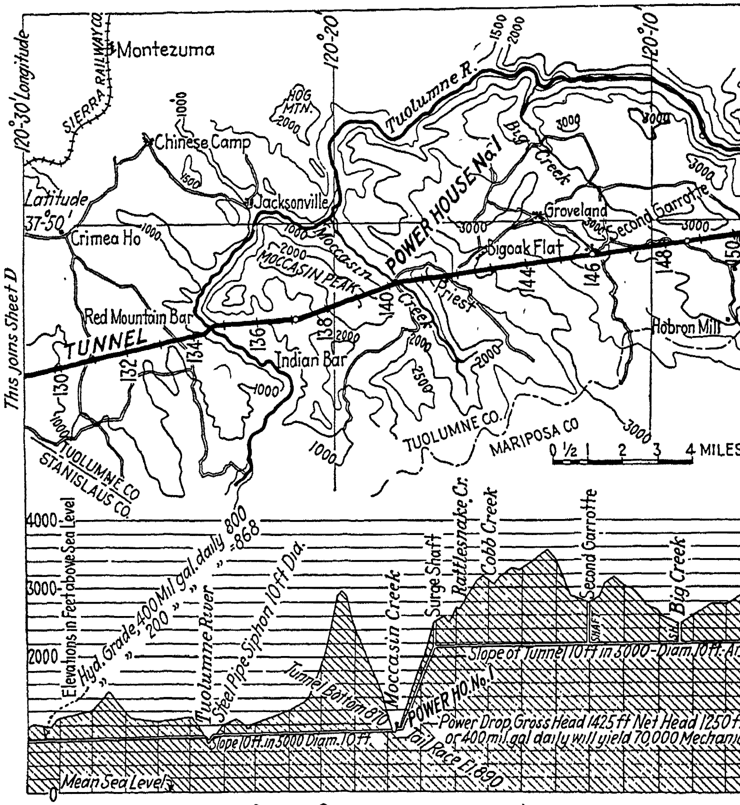


FIG. 149*b*.—General map and profile.

directions of land lines, streams, and other objects crossed by the line are indicated on the alinement diagram.

Sometimes general drawings show the profile on the same sheet with the map or plan of the line, as is illustrated in Fig. 149*b*. Such an arrangement is convenient for purposes of comparison in that the general relation between plan and profile may be seen at a glance.

150. Fixing Grades.—While the factors influencing the choice of grades will not be discussed here in detail, it is pertinent to mention some

of them. In the location of highways or similar routes, maximum permissible rates of grade are usually established (Chap. XXI). Likewise the elevation of grade is fixed within narrow limits at certain governing points as at terminals and at stream, highway, and railway crossings. Other things being equal, between such governing points the grade is fitted to the ground until the volume of earthwork in cuts will balance that in adjacent fills. For sewers and drains certain minimum permissible grades are established, and these with the profile of the ground and elevations of governing points (such as connections with mains, depths of basements, etc.) fix the grade. In any case the ground profile furnishes the basis for the study of economic grade elevation.

Between the points at which the elevation of grade is fixed, grade lines are established on the profile until by trial a satisfactory solution is obtained. Rates of grade and stations of points of change in grade are then fixed, and the corresponding elevations of points of change are calculated. Field and office work are simplified if rates of grade are expressed in an exact decimal, as 2.5 per cent or 0.65 per cent, and not fractionally, as $2\frac{1}{3}$ (2.333+) per cent or $\frac{1}{6}\!/\!_{71}$ (0.225+) per cent. The grade line may be a succession of straight lines abruptly changing direction at grade intersections, or if the changes in grade are considerable, it may be a series of straight lines connected by vertical curves at the summits and depressions. The selected grade line is shown on the profile as illustrated by Fig. 149*a*. The points of change are marked by small circles, and on vertical lines through these points are shown their elevations (and plusses if they do not fall at full stations). The rates of grade are shown just above the grade line or on dimension lines, as illustrated in the figure.

151. Finishing the Profile.—The profile is finished in ink. If it is to be blueprinted all lines are shown in black; otherwise the grade line and the numerical notes pertaining to it are generally shown in red and the remainder of the sheet is inked in black. Sometimes the alinement diagram with its accompanying notes is shown in a contrasting color, as blue or orange. The station numbers of the heavy vertical lines are placed at the bottom of the sheet. The elevations of the heaviest horizontal lines are written at each end of the sheet, and at intermediate points if the profile is long. Numbers and explanatory notes which are written vertically on the profile are ordinarily placed so as to be read from the right end, though some engineers prefer them to read from the opposite end so that as the profile is unrolled with the zero end towards the body the notes will appear right side up.

152. Other Profiles.—The profiles discussed in the preceding articles are of value in showing the relation between grade and the natural ground and are useful in fixing grades and estimating volumes of earthwork. In connection with construction, profiles are also frequently employed

to portray graphically the progress of the work. For example, in railroad and highway construction an estimate is made of work done during each month; that is, the volume of earthwork moved, the amount of track or pavement laid, etc., are calculated from field measurements. When the monthly estimate has been completed, the progress profile is brought up to date by tinting the portions completed with a particular color assigned for the month, also by showing within or adjacent to each tinted area the volume of earthwork or other quantity involved. Pavement or track laid, right-of-way fences completed, culverts, bridges, etc., constructed may be designated by appropriate colored symbols on the alinement diagram.

For some kinds of work, such as for subways and foundations, profiles showing not only the surface line but also the various subterranean strata are made. Points for the subsurface profiles are plotted from boring records, and full lines joining corresponding points indicate the upper and lower limits of the several subsoils. The various strata are then indicated in the profile either by tinting with contrasting colors or by using appropriate symbols.

153. Plotting Cross-sections.—Irregular cross-sections for earthwork are frequently drawn to scale. They are plotted on regular cross-section paper, usually with 10 divisions to the inch in both directions. Governing points of the cross-section are plotted from the cross-section notes. The surface may be indicated either by an irregular line drawn through the points or by a broken line connecting the several points. The scale to be used depends upon the accuracy with which it is desired to calculate the cross-sectional area. For large cross-sections a scale of 1 in. = 10 ft. both horizontally and vertically is appropriate. If the cross-sections are shallow, sometimes the vertical scale is exaggerated, as for profiles.

The cross-section for the first station on the base or center line is placed in the upper left-hand corner of the sheet, and successive cross-sections are placed one below the other. Below each cross-section is shown its station number. Areas of cross-sections in square feet and volumes of earthwork between successive cross-sections in cubic yards are also recorded on the sheet. The cross-sections may be inked, but since they are generally used only in the office they are commonly shown in pencil.

Sometimes cross-sections are plotted on the same sheet with the profile, in which case the horizontal scale of the cross-section is made larger than that of the profile.

Profiles and cross-sections may also be constructed from topographic maps, employing contours, as explained in Chap. XXIV, "Topographic Mapping."

EARTHWORK CALCULATIONS

154. Areas of Regular Cross-sections.—Areas of regular cross-sections[1] are readily determined by numerical calculations without plotting. For purposes of computing earthwork the areas are calculated in square feet. For a trench the cross-sectional area at any point is determined by multiplying the average of the top and bottom widths by the depth. The same method of calculation may be applied to level cross-sections for railroads and highways, or if d is the distance to either slope stake from the center, w is the width of the roadbed, and c is the center cut or fill, then the area of the level section is

$$A = c\left(d + \frac{w}{2}\right) \tag{1}$$

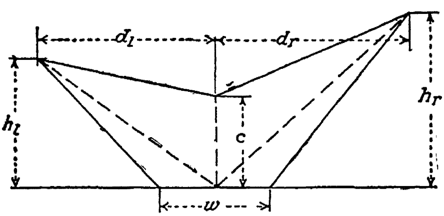


FIG. 155.—Three-level section.

155. Area of Three-level Section.—In Fig. 155 the three-level section may be divided into four triangles, as shown. Then from the figure the area is

$$A = \frac{1}{2} \cdot \frac{w}{2}(h_l + h_r) + \frac{1}{2}c(d_l + d_r),$$

or

$$A = \frac{w}{4}(h_l + h_r) + \frac{c}{2}(d_l + d_r) \tag{2}$$

Rule.—Multiply the sum of the distances to the slope stakes by one-half the center cut or fill; to this add the product of one-fourth the width of the roadbed and the sum of the side heights. The result is the cross-sectional area.

156. Areas of Irregular Road Cross-sections.—Figure 156 represents an irregular road cross-section. The cross-section notes give C the cut (or fill) at the center stake A, and distances from center to

[1] Regular cross-sections are cross-sections for which levels are taken at one point on each side of the center line. Level and three-level sections are regular.

and cuts at points E, F, G, and H. The method indicated here is an adaptation of the general method for computing areas by means of coordinates (Art. 274, p. 395). In this case the cross-section notes provide the x and y coordinates for each vertex of the section (the origin being at 0) if the expression $\frac{0}{w/2}$ is supplied for the two vertices M and N, and if algebraic signs, plus and minus, are used to designate directions to the right and left of the origin, respectively.

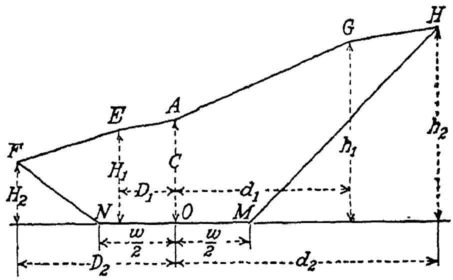


FIG. 156.—Irregular section.

In the usual form the notes are recorded as follows:

$$\frac{H_2}{D_2} \qquad \frac{H_1}{D_1} \quad \frac{C}{0} \quad \frac{h_1}{d_1} \qquad \frac{h_2}{d_2}$$

Then, as stated above, if the algebraic signs and the coordinates of M and N are supplied, the coordinates of the section appear as follows:

$$\frac{0}{-\frac{w}{2}} \quad \frac{H_2}{-D_2} \qquad \frac{H_1}{-D_1} \quad \frac{C}{0} \quad \frac{h_1}{+d_1} \qquad \frac{h_2}{+d_2} \quad \frac{0}{+\frac{w}{2}}$$

The calculation of the area will be made more convenient if now the opposite algebraic sign is placed on the opposite side of each lower term. The coordinates then appear thus:

$$\frac{0}{-\frac{w}{2}+} \quad \frac{H_2}{-D_2+} \quad \frac{H_1}{-D_1+} \quad \frac{C}{0} \quad \frac{h_1}{+d_1-} \quad \frac{h_2}{+d_2-} \quad \frac{0}{+\frac{w}{2}-}$$

The area may now be computed by the following rule:

Rule.—Multiply each upper term by the algebraic sum of the two adjacent lower terms, using the signs facing the upper term. The algebraic sum of these products is double the area of the cross-section.

Example: Below are the notes for an irregular cross-section; the width of the roadbed is 20 ft.; the cross-sectional area is to be calculated by the

above rule. The coordinates $\frac{0}{w/2}$ are recorded, and the double opposite algebraic signs are supplied as shown:

$$\frac{0}{-10+} \quad \frac{2.4}{-13.6+} \quad \frac{4.2}{-10.0+} \quad \frac{6.1}{0} \quad \frac{8.3}{+15.0-} \quad \frac{10.2}{+25.3-} \quad \frac{0}{+10-}$$

$$\begin{aligned}
2.4 \times (+10.0 - 10.0) &= \ \ 0.0 \text{ sq. ft.} \\
4.2 \times +13.6 &= +\ \ 57.1 \text{ sq. ft.} \\
6.1 \times (+10.0 + 15.0) &= +152.5 \text{ sq. ft.} \\
8.3 \times +25.3 &= +210.0 \text{ sq. ft.} \\
10.2 \times (-15.0 + 10.0) &= -\ \ 51.0 \text{ sq. ft.} \\
\hline
\text{Double area} &= +368.6 \text{ sq. ft.} \quad \text{Area} = 184.3 \text{ sq. ft.}
\end{aligned}$$

For side-hill sections, where the road is partly in fill and partly in cut, the cross-sectional areas may be conveniently calculated by dividing the section into partial areas consisting of trapezoids and triangles.

156a. When cross-sections are bounded by curved lines or are very irregular they are usually plotted as described in Art. 153, and the areas are determined either by subdividing into rectangles and triangles and calculating the area of each, or with greater facility by traversing the perimeter of each cross-section with a polar planimeter.

157. Polar Planimeter.—Figure 157 shows a polar planimeter with adjustable tracing arm. The planimeter is supported at three

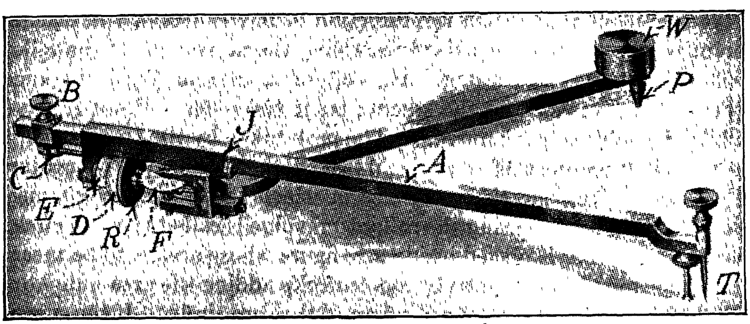


Fig. 157.—Polar planimeter.

points, namely, the anchor point or pole P, the roller R, and the tracing point T. The arm carrying the anchor point is hinged or pivoted to the frame of the planimeter. The tracing arm A slides through a sleeve in the frame of the planimeter and is clamped in position by the screw at B. On the arm are graduations which, when set to the index J, give known relations between the reading of the planimeter and area. Fine settings of the arm are made by

means of the tangent screw C. The circumference of the drum D of the roller R is graduated into 100 parts. At E is a vernier by means of which revolutions of the roller may be read to thousandths. By means of a worm, the revolutions of the roller turn the disk F in the ratio 10 to 1. The disk is graduated by ten radial lines marked from 0 to 9. Hence each space on the disk is the equivalent of one complete revolution of the roller. By means of an index mounted on the frame the whole number of revolutions of the roller is read on the disk; decimals of a revolution are given to hundredths by the drum reading opposite the index of the vernier, and are estimated to thousandths by reading the vernier. The anchor is a needle point which is pressed into the paper. The anchor arm is held down by the weight W.

When a plotted area is to be determined, the anchor point is fixed in the desired position on the paper, the tracing point is placed at some point on the perimeter of the figure, and either the roller is set to read zero or preferably an initial reading is taken. The perimeter is then completely traversed until the tracing point is brought to its original position. As the tracing arm is moved, the accompanying movement of the roller is partly sliding and partly rolling. By means of the calculus it can be shown that the amount the planimeter rolls is a direct function of the area traversed, provided the pole or anchor point is outside the figure; or in other words, the planimeter is a mechanical integrator.

It is evident that, when the direction of movement of the tracing point is parallel with the axis of the roller, the movement of the roller will be entirely by sliding. Hence, for a certain fixed distance between the anchor and the tracing point the roller will not revolve while the point is being moved. The path taken by the tracing point is the circumference of a circle of which the anchor is the center. This circle along the circumference of which the tracing point may be moved without causing the roller to revolve is called the *zero circle*. When the anchor is placed inside any figure whose area is to be measured, the area is determined by *algebraically* adding the area indicated by the planimeter reading to the area of the zero circle. If the size of the figure is less than that of the zero circle and if the motion of the tracing point is clockwise, the planimeter reading will be negative; if the size of the figure is greater than that of the zero circle, the planimeter reading will be positive. The area of the zero circle for various settings of the tracing arm is usually shown either on top of the arm or in the planimeter case. It can readily be determined by traversing a given figure first with the pole outside the figure and then with the pole inside. The algebraic difference between the two planimeter

readings, when multiplied by the constant for the given setting of the tracer arm, determines the area of the zero circle.

Just to the right of each graduation on the tracing arm is indicated the number of units of area (for a given scale) per revolution of the roller when the graduation is set at index *J*. The accuracy of the planimeter for a given setting of the tracing arm is tested by traversing a figure of known area. Usually it is desirable that a simple relation exist between revolutions and square inches or square feet at the scale of the drawing. If the planimeter reading is too large to give a simple constant of multiplication, the length of the tracing arm is increased, and *vice versa.*

Many planimeters have a fixed length of tracing arm. For this type the common relation between revolutions of roller and square inches is 1 to 10.

When the horizontal and vertical scales are dissimilar, as for profiles, areas with the planimeter may be determined as readily as when the scale is the same in each direction. For example, if a profile were plotted to the scale of 1 in. = 400 ft. (horizontal) and 1 in. = 20 ft. (vertical), then each square inch on the paper represents $20 \times 400 = 8{,}000$ sq. ft.

158. Areas with the Planimeter.—The sheet on which the area is plotted is stretched flat and free from wrinkles. If the tracing arm is adjustable, it is set so that some convenient relation exists between area and revolutions of roller. For example, if a cross-section were plotted to the scale of 1 in. = 10 ft., the arm might be set at the $10\square''$ graduation, indicating an area 10 sq. in. or at the above scale, 1,000 sq. ft., per revolution of roller. Hence the least count of the vernier would be the equivalent of 1 sq. ft. of area. The accuracy of the tracing-arm setting is then tested by carefully tracing the boundary of a figure of known area, say a 2 by 5-in. rectangle, and if necessary the length of the arm is adjusted until by trial the desired relation is established. The anchor point is placed outside the test figure, and the boundary is traced in a clockwise direction, thus increasing the reading of the planimeter. If the test figure is a rectangle, as above suggested, the tracing point is guided by a triangle or straight edge placed along the sides. The planimeter is read with tracing point at the point of beginning of the traverse and again after the figure has been traced. The difference between the two readings gives the number of revolutions of the roller. Particular care should be exercised in returning the tracing point exactly to the point of beginning before taking the final reading.

An accessory furnished with some planimeters consists of a flat bar at one end of which is a needle point, and at the other end of which is a

small hole into which the tracing point may be set. The distance between needle point and hole is equal to the radius of a circle whose area is 10 sq. in. The needle point is fastened to the paper, and the planimeter is quickly tested by tracing the circumference of the circle, the tracing point being held in the hole of the bar as it is revolved about the needle point.

After the planimeter has been tested, the figures whose areas are desired are traversed and the areas are computed by multiplying the difference between initial and final readings by the constant. Often the roller is set at zero with the tracing point at the point of beginning, in which case the final reading gives revolutions directly. In order to avoid consideration of the area of the zero circle, the anchor is generally placed outside the figure; to produce an increase in the reading of the planimeter the tracing point is moved in a clockwise direction.

If the figure is larger than can be traced in one operation it may be subdivided into smaller figures each of which may be traversed in the usual manner. When a considerable number of areas of large figures are to be determined, however, faster progress may be made by placing the anchor point *inside* the figure and algebraically adding the area of the zero circle to the area indicated by the planimeter. The area of the zero circle is determined as described in the preceding article. If the area of the figure is less than that of the zero circle the area represented by the difference between initial and final readings of the planimeter is subtracted from that of the zero circle, and *vice versa*.

Some care must be exercised to avoid confusion between positive and negative results. For example, in traversing clockwise an area less than that of the zero circle, with the anchor point inside the figure, if the initial reading is 0.000 and the final reading is 8.525, the number of revolutions is $8.525 - 10 = -1.475$, and not 8.525. Thus, it is necessary to note whether the net movement of the roller is forward or backward. Whether or not the area of the figure is less than that of the zero circle can usually be determined by inspection. If it is smaller and the direction of movement of the tracing point is made *counterclockwise*, the final reading will be larger than the initial reading. Thus for the example just cited had the direction of movement been counterclockwise the final reading would have been 1.475 instead of 8.525. Generally it is preferable to traverse the figure in a direction such that the difference in readings will be positive.

Example 1: The area of a cross-section is to be measured with the anchor point of the planimeter inside the figure. The scale of the drawing is 1 in. = 10 ft. The tracing arm is so set that one revolution of the roller measures 10 sq. in. on the paper. The area of the zero circle is found to be 164.31 sq. in. The area of the figure appears to be larger than that of the zero circle. The initial reading is 0.111, and after tracing

clockwise around the figure the final reading is 8.763. The number of revolutions is $8.763 - 0.111 = +8.652$ corresponding to $10 \times 8.652 = +86.52$ sq. in. Adding 164.31 sq. in. (the area of the zero circle) there is obtained $164.31 + 86.52 = 250.83$ sq. in., which is the actual area of the figure. Since the scale is 1 in. = 10 ft., 1 sq. in. on the paper is the equivalent of $10 \times 10 = 100$ sq. ft. Hence the area of the cross-section is $250.83 \times 100 = 25{,}083$ sq. ft.

Example 2: Conditions as in example 1, except that the area of the figure is less than that of the zero circle. The figure is traced counter-clockwise.

$$\begin{array}{lrl}
\text{Initial reading} & 1.234 & \\
\text{Final reading} & \underline{3.765} & \\
\text{Difference} & 2.531 \text{ rev.} = & -25.31 \text{ sq. in.} \\
 & \text{Area of zero circle} = & \underline{164.31} \\
 & \text{Area of figure} = & 139.00 \text{ sq. in.} \\
 & = & 13{,}900 \text{ sq. ft.}
\end{array}$$

Attention is drawn to the fact that the constant by which revolutions of the roller must be multiplied to give area can readily be determined for any setting of the tracing arm. All that is necessary is to determine the difference in planimeter readings for a known area. Then by proportion, *any required area is to the corresponding difference in readings as the figure of known area is to its difference in readings.*

Example 3: A planimeter registers 0.876 revolutions of the roller for an area of 10 sq. in. With the same setting of the tracing arm, a figure whose area is desired is traversed and, with the anchor outside, the difference in readings is 2.731. Let A be the desired area; then

$$\frac{A}{2.731} = \frac{10.00}{0.876} \text{ or } A = \frac{10 \times 2.731}{0.876} = 31.2 \text{ sq. in.}$$

158a. Errors in Planimeter Measurements.—If the relation between revolutions and area is accurately established, the errors involved in planimeter measurements are accidental in character and are due principally to the inability of the observer to follow exactly the boundary of the figure with the tracing point. For the same care and skill on the part of the observer, the smaller the area, the larger the relative error of measurement. Hence it is desirable that the areas be plotted to a scale consistent with the relative accuracy with which it is desired to determine areas. Ordinarily, planimeter measurements of small areas may be expected to be correct within 1 per cent, and measurements of figures of considerable size may be correct within perhaps 0.1 or 0.2 per cent. Few, if any, cases arise where the areas of plotted cross-sections for earthwork

can not be determined with a precision as great as is justified by the nature of the field measurements.

159. Volumes of Earthwork.—Volumes of earthwork are calculated by a variety of methods, depending upon the nature of the excavation and of the data. Where cross-sections have been taken along a route their areas are determined as described in preceding paragraphs, and the volumes of the prismoids between successive cross-sections are calculated either by the method of average end areas or by the prismoidal formula. The same procedure may be followed for borrow pits and other excavations of similar nature, or if observations of elevation after excavation has been completed are made at the same points as those obtained on the original ground the volume may be calculated by dividing it into vertical truncated prisms. Estimates for grading are frequently based upon a topographic map showing the contours for the undisturbed ground and also contours for the ground as it will appear when grading has been completed. The volume is conveniently determined by dividing it into prismoids with horizontal bases and sloping sides.

Methods of computing volumes of earthwork by the use of contours are described in Art. 436*b*, p. 643.

Total volumes are almost invariably expressed in cubic yards.

160. Volume of Borrow Pit.—Figure 160*a* illustrates the plan view of a borrow pit, observations having been taken at the intersections of full lines. The numbers written diagonally are the cuts in feet. The full lines are seen to divide the pit into volumes of triangular, rectangular, and trapezoidal cross-sections.

Actually the upper and lower surfaces are warped, but for the purpose of computing volumes they are assumed to be plane and thus the volumes are assumed to be truncated prisms. The volume of a triangular truncated prism (as *abc*) is $V = \frac{A}{3}(h_1 + h_2 + h_3)$ in which A is the horizontal sectional area and h_1, h_2, and h_3 are the corner heights.

Obviously any rectangular prism (as *defg*) may be divided into triangular prisms by either of two diagonal lines; but unless there are considerable variations in the corner heights, the error introduced by assuming the volume as a rectangular truncated prism is inconsiderable as compared with errors due to undetected or neglected irregularities in the ground surface. Under these conditions the volume is determined by multiplying the average of the corner heights by the horizontal sectional area. Where it is clear that the method just mentioned would introduce an error of consequence in the computed volume the triangular prisms to be considered are indicated in the

field notes. Thus in the volume *jklm* the dotted line *km* is shown on the sketch to indicate that the shape of the ground warrants calcu-

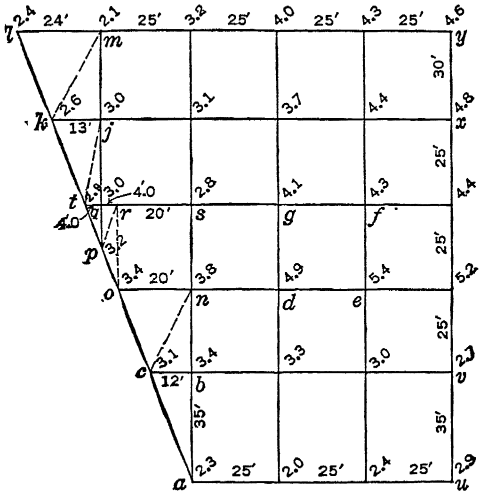


FIG. 160*a*.—Borrow pit.

Figure	Σ Cor. ht., ft.	Area, sq. ft.		Volume, cu. ft.
uvba	32.7	25 × 35	×¼	7150
vxjqsb	161.9	25 × 25	×¼	25300
xymj	59.9	25 × 30	×¼	11230
nors	13.0	25 × 20	×¼	1630
klm	7.1	12 × 30	×⅓	850
jkm	7.7	6.5 × 30	×⅓	500
kjt	8.5	6.5 × 25	×⅓	460
jqt	8.9	2 × 25	×⅓	150
tqp	9.1	2 × 12	×⅓	70
pqr	9.2	2 × 12	×⅓	70
opr	9.6	2 × 25	×⅓	160
con	10.3	10 × 25	×⅓	860
bcn	10.3	6 × 25	×⅓	520
abc	8.8	6 × 35	×⅓	620
				49,570 cu. ft.
			or	1,836 cu. yd.

FIG. 160*b*.—Volume of borrow pit.

lating volumes of the triangular prisms *jkm* and *klm*. For similar reasons the volume *nopqrs* is divided into one rectangular and two triangular prisms.

If several adjacent rectangular prisms have the same horizontal section (that is, the same horizontal dimensions) they are computed as one solid, as follows: Multiply each corner height by the number of prisms of the same horizontal section in which it occurs; sum up the values thus determined, and multiply by the horizontal sectional area of a single prism. The product divided by four gives the volume of the solid.

Figure 160*b* illustrates a suitable form for calculations and shows the computations for the borrow pit of Fig. 160*a*. The computations demonstrate the rule stated in the preceding paragraph. Thus *uvba* is seen to be made up of three 25- by 35-ft. rectangular prisms. Adding the corner heights by starting at *a*, and proceeding clockwise around the figure, we have $2.3 + 3.4 + 2(3.3 + 3.0) + 2.7 + 2.9 + 2(2.4 + 2.0) = 32.7$ ft. as shown in the first line of the second column (Fig. 160*b*). In the figure the individual volumes are shown to the nearest 10 cubic feet, and the final volume is given to the nearest cubic yard. To compute volumes of earthwork to decimals of a cubic yard, as is sometimes done, is absurd when one considers that small irregularities in the ground surface between points at which elevations are taken would doubtless make a difference of several cubic yards between the actual and the computed volume. When volumes are large, calculations to the nearest 10 or even to the nearest 100 cu. yd. may be as exact as the nature of the field measurements will justify.

161. Volumes by Average End Areas.—The most common method of determining volumes of excavation for railroads, highways, canals, and similar works is the method of *end areas*. It assumes that the volume between successive cross-sections is the average of their end areas multiplied by the distance between them, or expressed as a formula

$$V = \frac{l}{2}(A_1 + A_2) \tag{3}$$

in which V is the volume (cubic feet) of the prismoid between end bases or cross-sections having areas (square feet) A_1 and A_2, and l is the length (feet) of the prismoid, or in other words, the distance between cross-sections. If cross-sections are taken at full 100-ft. stations the volume in cubic yards between successive cross-sections A_1 and A_2 (square feet) is

$$V_y = 1.85(A_1 + A_2) \tag{4}$$

The above formulas are exact when $A_1 = A_2$, but are approximate for $A_1 \gtrless A_2$. As one of the areas approaches a point, as on running from cut to fill on side-hill work, a maximum error of $16\frac{2}{3}$ per cent would occur if the formulas were followed literally. In this case,

however, the volume is usually calculated as a pyramid, *i.e.*, *volume = ⅓ area of base times length.* Considering the fact that cross-sections are usually a considerable distance apart, and that minor inequalities in the surface of the earth between sections are not considered, the method of average end areas is sufficiently exact for ordinary earthwork.

162. Volumes by the Prismoidal Formula.—It can be shown that the volume of a prismoid is

$$V = \frac{l}{6}(A_1 + 4A_m + A_2) \qquad (5)$$

in which l is the distance between end bases or sections, A_1 and A_2 are the areas of the end sections, and A_m is the middle area or area halfway between the end sections. A_m is determined by averaging the corresponding linear dimensions of the end sections and not by averaging the areas A_1 and A_2. The use of the formula is best illustrated by an example.

Example: Following are shown the three-level cross-section notes for two stations 100 ft. apart. The width of the roadbed is 20 ft. The volume of earthwork between the two stations is to be calculated by the prismoidal formula. Below the regular cross-section notes are shown those for the mid-section obtained by averaging the values given for sections at stations 115 and 116. In the column headed "Area, square feet," are areas of cross-sections computed by formula (2), Art. 155. Then by the prismoidal formula given above

$$V = {}^{100}\!/_{6}(212.0 + 4 \times 154.0 + 103.0) = 15{,}520 \text{ cu. ft. or } 575 \text{ cu. yd.}$$

Station	Cross-section			Area, square feet	Volume, cubic yards
	L	*C*	*R*		
115	$\frac{c4.0}{16.0}$	$\frac{c6.0}{0}$	$\frac{c12.0}{28.0}$	212	
					575
116	$\frac{c2.0}{13.0}$	$\frac{c3.0}{0}$	$\frac{c8.0}{22.0}$	103	
Mid-section	$\frac{c3.0}{14.5}$	$\frac{c4.5}{0}$	$\frac{c10.0}{25.0}$	154	

While the prismoidal formula gives the true volume of a prismoid, the difference between results obtained through its application and

values obtained by the method of average end areas is not large except where the change in cross-section is abrupt.

For the above example the volume calculated by average end areas is 583 cu. yd. and the difference between the results obtained by the two methods is 8 cu. yd. or about 1.4 per cent. As an example of the magnitude of the error in volume introduced by apparently insignificant variations of the surface, suppose that between the two cross-sections given in the above example a sag takes place gradually until at station 115 + 50 it amounts to 0.5 ft. over a width of 20 ft., thus forming two wedges of error with a base of 10 sq. ft. and a length of 50 ft. The volume of these two wedges is $2 \times \frac{1}{2} \times 10 \times 50 = 500$ cu. ft. $= 18$ cu. yd., and the error is more than twice as great as the error due to calculating the volume by average end areas instead of by the prismoidal formula. To one familiar with field conditions it is evident that much larger surface irregularities than those cited above are likely to go unnoticed unless more than the usual care is taken in field measurements.

It may be concluded that, so far as volumes of earthwork are concerned, the use of the prismoidal formula is justified only when cross-sections are taken at short intervals, when the observations are so conducted that small surface deviations will be measured, and when the areas of successive cross-sections differ widely.

163. Prismoidal Correction.—It can be shown that the difference between the two volumes, calculated by the two methods, for the prismoids defined by three-level sections, is given by the equation:

$C_v = 0.309\,(H_0 - H_1)(D_0 - D_1)$ in which

C_v is the difference in volume, or the correction, in cubic yards, for a prismoid 100 ft. long,

H_0 is the center height at one end section,

H_1 is the center height at the other end section.

D_0 is the distance between slope-stakes at the end section where the center height is H_0, and

D_1 is the distance between slope-stakes at the other end section.

C_v is known as the *prismoidal correction;* it is *subtracted from* the volume as determined by the average-end-area method to give the more nearly correct volume as determined by the prismoidal formula.

164. Volumes from Road Profiles.—Preliminary estimates of earthwork for highways, railroads, and canals made prior to the location of the route are based upon the preliminary profile. If the side slopes were vertical, the volume of any cut or fill would be a direct function of the area between grade line and ground line on the profile. Since the side slopes are inclined, the volume in cut or fill increases at a relatively greater rate than does the depth, hence, except for the purpose of rough estimates, the area representing cut or fill on

the profile cannot be directly taken as a measure of volume, as would be the case for a trench. For very rough estimates the profile area of any given cut or fill may be measured with the planimeter, and the volume may be calculated by multiplying this area (to the scale of the profile) by the volume per foot of depth per foot of length for a level section whose depth is the average cut (or fill) for the portion of the profile under consideration.

Example 1: The length of a given cut is 1,650 ft., and the area between ground line and grade line is 18,500 sq. ft. The roadbed is 20 ft. wide and the side slopes are 1½ to 1. It is desired to determine roughly the volume of earthwork.

$$\text{Average depth of cut is } \frac{18{,}500}{1{,}650} = 11.2 \text{ ft.}$$

For a level section the distance to slope stake is

$$d = \frac{w}{2} + cs = 10 + 1\tfrac{1}{2} \times 11.2 = 26.8 \text{ ft.}$$

The cross-sectional area of the average section is

$$A = c\left(\frac{w}{2} + d\right) = 11.2\ (10.0 + 26.8) = 412 \text{ sq. ft.}$$

For the average section the volume in cubic yards per linear foot is

$$V_1 = 11.2\ \frac{(10 + 26.8)}{27} = 15.3 \text{ cu. yd.}$$

The total volume is therefore

$$V = 15.3 \times 1{,}650 = 25{,}200 \text{ cu. yd.}$$

For less approximate calculations the cut or fill at each full station is scaled from the profile and the corresponding volume in cubic yards per 100 ft. is calculated for a level section whose depth is the scaled cut or fill. The level section at the station is assumed to exist over a length of 100 ft. (*i.e.*, 50 ft. in advance and 50 ft. back of the station). The total volume for a given cut or fill is obtained by summing up the volumes per station obtained in the manner just described. Tables of volumes per 100 ft. for various widths of roadbed, side slopes, and cuts or fills, are given in texts on railroad surveying. If such a table is not available, volumes may be conveniently determined by constructing a diagram showing cuts or fills as ordinates and volumes in cubic yards per 100 ft. as abscissas. Volumes may also be determined by means of a scale graduated in cubic yards per 100 ft. of length for various depths of cut or fill, the scale being applied to the profile.

The above method may be modified by constructing an earthwork diagram, either on the sheet with the profile or on a separate sheet, the ordinates of the diagram being in cubic yards per foot of length and the abscissas being distances along the line in feet. When the diagram is on the profile sheet any convenient horizontal line is chosen as the base from which ordinates are measured, those above the base representing volumes in cut and those below representing volumes in fill. The horizontal scale is made the same as for the profile. The vertical scale is some convenient scale, as 1 in. = 20 cu. yd. per foot, depending upon the magnitude of the volumes involved. At each full station and at each plus where the direction of the profile changes abruptly, the distance between grade line and ground line is scaled, the volume per foot of length for a level section of corresponding depth is determined from tables or from a diagram, and this volume is plotted as an ordinate to the earthwork diagram. The diagram is completed by drawing an irregular line through the points thus plotted. The area under the diagram (at the scales used in plotting) gives the volume in cubic yards. Thus, if the horizontal scale were 1 in. = 400 ft., and the vertical scale were 1 in. = 20 cu. yd. per foot, then 1 sq. in. on the paper is the equivalent of 400 ft. × 20 cu. yd. per foot = 8,000 cu. yd.

165. Errors in Earthwork Quantities.—It is instructive to consider the probable errors which affect the determination of earthwork quantities. These may be discussed in relation to each of the three general methods commonly used; namely (1) volumes computed from data obtained in setting slope stakes; (2) volumes computed from irregular cross-sections; and (3) volumes estimated from contour maps (Art. 436*b*, p. 643).

The measurements taken to compute earthwork quantities include horizontal measurements, usually taken with a metallic tape, and vertical measurements, taken with a level and rod. These measurements are subject to accidental errors due principally to marking the ends of the tape, to reading the rod, and to variations in the elevation of the ground surface where the rod is held. The size of these errors will vary greatly under various field conditions, but to illustrate the principles involved, a probable error of ± 0.05 ft. will be assumed for each measurement, *i.e.*, for each horizontal (tape) and vertical (rod) reading. It is believed that this value is a fair assumption for average field conditions.

Since the horizontal distances are usually much greater than the vertical, it is evident that the per cent error in horizontal measurements is much less than in the vertical. Hence, errors in computed volumes result for the most part from errors in cuts and fills. And since the magnitude of the errors is independent of the magnitude of the distances themselves, the per cent of error in the final result is greater for small volumes than for large ones.

1. *Volumes from Slope-stake Data.*—The principles stated above may be illustrated in the case of volumes computed from slope-stake data by the following table. The roadway is assumed to be 20 ft. wide, with side slopes of 1½ to 1. Volumes are computed for sections 100 ft. long.

Average cut or fill, feet	Area, square feet	Probable error		Volume, cubic yards	Probable error	
		Value, square feet	Per cent of area		Value, cubic yards	Per cent of volume
2.0	46	±0.7	1.5	170	±1.8	1.1
4.0	104	±0.8	0.8	385	±2.1	0.5
5.5	155	±1.0	0.6	574	±2.6	0.5
12.5	485	±1.5	0.3	1,794	±3.9	0.2

An inspection of this table shows: (1) that the per cent of error in the area and in the volume varies inversely with the depth of the cut or fill; (2) that the magnitude of the probable error is not important as compared with the probable errors due to variations over the ground surface; and (3) that the probable errors show an uncertainty of one or more in the last unit of the computed quantities. Hence it will be consistent to carry one decimal place in intermediate computations of areas and volumes; but it is absurd to record values beyond the last whole unit, either of areas or of volumes.

2. *Volumes from Irregular Cross-sections.*—The remarks of the preceding paragraph regarding roadway volumes apply equally well to borrow-pit volumes. Since the shapes of borrow pits are more irregular than those of roadways, however, and since two rod readings are required at each point, it may be expected that the computed volume for a shallow borrow pit will be affected by a larger per cent of error than a corresponding volume in a roadway. On the other hand, the readings are usually taken at small intervals (25 to 50 ft.), hence the errors due to irregularities in the ground surface are not so great as in the case of roadways; and since many readings are taken, the law of accidental errors tends to reduce the per cent of error in the total volume.

Assuming a probable error of ±0.05 ft. for a single rod reading, the total probable error for the borrow pit shown in Fig. 160*a* is ±2.4 cu. yd. The volume is 1,836 cu. yd. and the probable error is 0.1 per cent, which is about one-half as large as the error in the roadway volume of 1,794 cu. yd. given above.

3. *Volumes from Contour Maps.*—The errors of determining volumes from contours depend upon the scale of the map, the contour interval, and the accuracy with which contours are shown. The larger the scale and the smaller the contour interval, the more reliable are the calculated volumes. Under the usual conditions, scales of 50 to 100 ft. to the inch and contour intervals of 1 or 2 ft. may render estimates correct within

5 or 10 per cent, depending upon the magnitude of the grading operations. Rough estimates are sometimes made from maps showing 5-ft. contours, but unless the cuts and fills are deep and the grading is on a large scale, the relative error involved is likely to be great. Where the ground is gently sloping and the cuts and fills are shallow, reliable estimates of volume cannot be made unless the contour interval is very small. A contour interval of ½ ft. is often employed for such work.

166. Office Problems.

Problem 1. Plotting Profile

Object.—To plot a profile from level notes, and to fix the grade line for a railroad, highway, pipe line, or similar work.

Procedure.—(1) Choose a horizontal and vertical scale in keeping with the purpose of the profile. If it is for a railroad or highway, and if later field measurements will give data for calculating volumes of earthwork, a scale of 1 in. = 400 ft. (horizontal) and 1 in. = 20 ft. (vertical) may be conveniently used with Plate *A* profile paper; if the profile is to be made the basis of earthwork computations, as it may be when it is for a sewer, a scale as large as 1 in. = 40 ft. (horizontal) and 1 in. = 4 ft. (vertical) may be required. (2) Examine the field notes to determine the range between points of maximum and minimum elevation. Number each of the heaviest horizontal lines with its elevation. Near the foot of heavy vertical lines record the 100-ft. station numbers, these numbers increasing from left to right, and being multiples of ten for small scales. (3) From the profile level notes plot the profile. Through the plotted points draw a freehand curve. Show the names of streams and roads crossed, directly above their crossings on the profile. Check the profile, and ink it with *black* ink. (4) Fix the grade line. At each change of grade draw a small circle, and indicate the elevation (and plus if it does not fall at a full station) of each of these points on a vertical line directly under or over the small circle. Draw a continuous line through the foot of these verticals, and on it record the rates of grade, as shown in Fig. 149*a*. (If desired, the rates of grade may be noted directly on the grade line.) Ink, in *red*, the grade line and all lines and numbers referring to it. (5) Near the bottom of the paper indicate in *black* ink the horizontal alinement, using a scheme similar to that shown in the figure. (6) Make a title showing the name and location of the work, the horizontal and vertical scale, the date, and the name of the draftsman.

Hints and Precautions.—(1) The profile should be checked by reading elevations and stations from the profile, not from the level notes. Two men can work together to good advantage—one reading the notes, while the other plots the profile; when checking the profile, they should exchange places. (2) It is a common mistake to read the elevations of turning points and bench marks as ground elevations. This may be avoided by enclosing each of these elevations in the field notebook with a circle. (3) A more uniform width of line can be obtained if the profile

is inked with a ruling pen rather than with a lettering pen. The draftsman should not endeavor to make it a series of straight lines, for this will not be the condition of the actual profile; on the other hand, he should not round off the summits and depressions an undue amount, for such points will naturally be greatly exaggerated in sharpness on account of the relation between horizontal and vertical scales.

Problem 2. Area with Planimeter

Object.—To determine an area with the polar planimeter. It is assumed that the figure, the area of which is to be found, is plotted to scale.

Procedure.—(1) Set the tracing arm so that a complete revolution of the measuring wheel will bear some simple relation to the given scale and unit of measurement. For example, it may be convenient because of the plotted size of the figure to have a complete revolution represent 10 sq. in.; if so, set the index to that mark on the tracing arm. (2) Test the accuracy of the setting by constructing a figure of known dimensions, say a 2 by 5-in. rectangle, and with the pole or fixed point in position *outside* the figure, set the tracing point over one corner. Record to four places the reading of the roller and disk. (3) Carefully and slowly move the tracing point completely around the boundary *in a clockwise direction* to the point of starting. Again record the reading of the roller and disk. (4) It is evident that for the example stated above, the difference between the initial and final readings should equal one complete revolution. If this difference is too small, slightly reduce the length of the tracing arm by means of the slow motion screw, and *vice versa.* (5) In like manner repeat the tests until the desired relation is obtained. (6) Having adjusted the planimeter, traverse the perimeter of the figure the area of which is to be determined, using the same method of procedure as with the figure of known area. Check the operation. (7) Transform the difference between readings into area. (8) Determine the area of the zero circle as described in Art. 158. (9) Measure the given figure with pole *inside,* and calculate the area. (10) Set the tracing arm so that the relation between revolutions of roller and area is unknown. Determine the difference in planimeter readings for the 2 by 5-in. rectangle and also for the given figure. By proportion determine the area of the figure.

Hints and Precautions.—The planimeter should be manipulated very carefully, and no result should be accepted as correct until checked. The paper should be stretched on a level surface so that it is free from wrinkles, the contact edge of the measuring wheel should be bright and free from dirt, and the mechanism of the planimeter should be so adjusted that it will move with the utmost freedom, yet without lost motion. The position of the pole should be so chosen that the measuring wheel will stay on the paper as the tracing point is moved about the figure.

PROBLEM 3. PLOTTING CROSS-SECTIONS; QUANTITIES OF EARTHWORK

Object.—To plot cross-sections from field notes and to calculate quantities of earthwork. It is assumed that the cross-section notes are for railroad, highway, or similar work, and give cut or fill rather than elevations.

Procedure.—(1) Beginning near the top and left-hand end of the sheet or roll of cross-section paper, choose convenient heavy horizontal and vertical lines as grade and center lines. With these as coordinates plot the cross-section notes of the first station, counting the number of spaces out from the center and up from the grade line corresponding to these distances in the notes. Usually the scale used on such work is 1 in. = 10 ft. or one space equals 1 ft., but it may be larger or smaller. Mark the plotted points with dimensions identical with those of corresponding points in the notes. Draw straight lines showing roadbed and side slopes of cut or fill and the original ground, thus enclosing the section. Outside and just below the cross-section and near the center line, mark the station number. (2) At a convenient distance below and on the same center line, plot the next section in similar manner. (3) When the bottom of the sheet is reached, plot the next section a little farther to the right and at the top of the sheet; and in this way continue until all plotting is done. (4) Calculate the area of each section as explained in Arts. 155 and 156, and show its value within the section (as 123 sq. ft.). (5) Compute volumes by both prismoidal and average-end-area methods and show the volume of each prismoid between its end sections (as 97 cu. yd.). (6) By each method find the total yardage in each cut and fill, and mark these totals conspicuously. (7) Make an appropriate title.

167. Numerical Problems.

1. From the notes of Fig. 138*b* calculate by the method of average end areas the volume in cut and in fill between station 605 and station 613. The roadbed is 20 ft. wide in cut and 16 ft. wide in fill and the side slopes are $1\frac{1}{2}:1$. Tabulate data in the following form: "Sta." "Cross-section," "Area," "Volume."

2. From the data of problem 1, compute volumes by the prismoidal formula. Note the discrepancy in per cent between volumes as determined by the prismoidal and average-end-area methods.

3. What error in volume between station 605 and station 606 (Fig. 138*b*) would be introduced if the recorded cuts at centers and slope stakes are 0.1 ft. too great? What is the error in terms of per cent of the volume by average end areas?

4. Suppose that between the above two stations a sag gradually takes place over a width of 24 ft. becoming a maximum of 1 ft. at 605 + 50. What error is introduced in the calculated volume?

5. In plan a borrow pit is 75 by 135 ft. Before and after excavation levels are run and offsets are measured from stations along one of the 135-ft. sides. The calculated cuts in feet are in the table on page 229.

Offsets	Stations						
	0	0 + 30	0 + 50	0 + 75	1 + 00	1 + 15	1 + 35
0	0.0	1.5	0.0	4.5	6.2	4.7	0.0
25	1.2	2.9	10.6	9.7	7.9	8.4	2.5
50	2.5	3.7	8.7	8.7	9.4	8.4	3.6
75	0.0	0.0	1.9	7.6	6.8	6.3	0.0

Compute the volume by the method described in Art. 160.

6. Following are the notes for an irregular cross-section. The width of the roadbed is 24 ft. and the side slope is 1½:1. Calculate the cross-sectional area by the methods of Art. 156.

$$\frac{c4.2}{18.3} \quad \frac{c6.8}{12.0} \quad \frac{c11.2}{0} \quad \frac{c14.4}{10.0} \quad \frac{c16.8}{25.0} \quad \frac{c18.4}{39.6}$$

7. The tracing arm of a planimeter is set so that the roller reads 0.583 rev. for 10 sq. in. A figure for which the vertical scale is 1 in. = 8 ft. and the horizontal scale is 1 in. = 20 ft. is traversed clockwise with pole outside, and the difference in planimeter readings is 1.932 rev. What is the actual area of the figure in square inches? What area in square feet does it represent?

8. With the tracing arm of the planimeter set as in problem 7, the perimeter of an area is traversed clockwise first with pole outside and then with pole inside. The corresponding differences in readings are 2.095 and −7.786. What is the area of the zero circle in square inches?

9. The perimeter of a figure is traversed clockwise with the pole inside and the tracing arm of the planimeter set as in problems 7 and 8. The difference in readings is −3.781. What is the area in square inches?

10. A given fill for a railroad is 1,350 ft. long. The profile is plotted at the horizontal scale of 1 in. = 400 ft. and at the vertical scale of 1 in. = 20 ft. The perimeter of the area between ground line and grade line is traversed with a planimeter set so that 1 rev. of the roller is equal to 10 sq. in. on the paper, and the difference in readings is 0.269. What is the average depth of the fill in feet? Estimate the volume of the fill in cubic yards assuming a level section of the average depth, roadbed 18 ft. wide, and side slopes 1½ to 1.

CHAPTER XI

MEASUREMENT OF ANGLES AND DIRECTIONS

GENERAL METHODS

168. Location of Points.—As has been stated, the purpose of a survey is to determine the relative positions of points on or near the surface of the earth. The location of a point is fixed if measurements are made of (1) its direction and distance from a known point, (2) its direction from two known points, (3) its distance from two known points, or (4) its direction from one known point and its distance from another. If the relative position of points as seen in horizontal projection is desired, the field operations involve the measurement of horizontal distances, as described in Chap. VI, and the determination of direction in the horizontal plane. In addition, if the relative elevations of points are required, they are determined by one or another of the methods of leveling described in Chaps. VII to X.

For horizontal projection or plan, the direction of any line (as defined by two points) is determined by horizontal angular measurements between the line and some reference line. For vertical projection, the direction of one point with respect to another is defined by the vertical angle between the plane of the horizon and the line joining the two points. In general, therefore, the angular measurements of surveying are either horizontal or vertical, or approximately so.

When the angle between two points is mentioned, it is understood to mean the horizontal angle between lines passing through the two points and converging at a third. Thus at O (Fig. 169a) the angle between B and C is the horizontal angle BOC. The vertical angle to a point is its angle of elevation or depression from the horizontal; as measured from some point of reference the angle is positive or negative according as the observed point is above or below the horizontal plane passing through the point of reference. Thus the vertical angle to a point B as measured from A is positive if B is higher than A. Measurement of vertical angles as applied to indirect leveling was briefly considered in Chap. VII, and will be described more in detail in Art. 198, p. 266. The present chapter treats only of angles and directions in the horizontal plane.

169. Angles and Directions.—The relative directions of lines connecting survey points may be obtained in a variety of ways. Figure 169a shows lines about a point. The direction of any line (as OB) with respect to an adjacent line (as OA) is given by the horizontal angle between the two lines (as a) and the direction of rotation (as clockwise). The direction of any line (as OC) with respect to a line not adjacent (as OA) is not given by any of the measured angles but may be computed by adding the intervening angles (as $a + b$).

Figure 169b shows the same system of lines but with angles measured from a line of reference OM. The direction of any line (as OA) with respect to the line of reference (as OM) is given by the

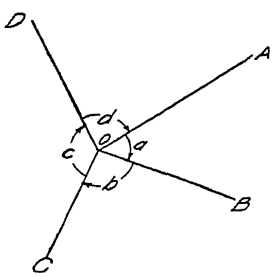


FIG. 169a

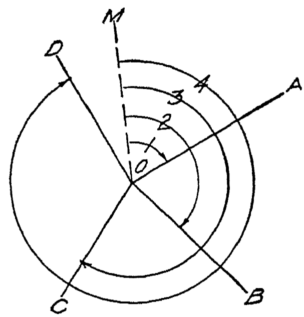


FIG. 169b.

angle between the lines (as 1) and its direction of rotation (as clockwise). The angle between any two lines (as AOC) is not given directly, but may be computed by taking the difference between the direction angles of the two lines (as $\angle 3 - \angle 1 = \angle AOC$).

In the first case it will be noted that a given angle denotes the direction of a line with respect to an adjacent line. In the second case a given angle denotes the direction of a line with respect to a fixed line of reference. In surveying, angular measurements may fall under either of the above general cases. The fixed line of reference may be any line in the survey or it may be purely imaginary. It is termed a *meridian*. If it is arbitrarily chosen without special reference to the points of the compass, as is often the case, it is called an *assumed meridian;* if it is a true north and south line passing through the geographical poles of the earth it is called a *true meridian;* if it lies parallel with the magnetic lines of force of the earth as indicated by the direction of the magnetic needle it is called a *magnetic meridian.*

170. True Meridian.—The true meridian is determined by astronomical observations to be described in a later chapter. For any given point on the earth its direction is always the same, and hence directions referred to the true meridian remain unchanged regardless of time. The lines of most extensive surveys, particularly those of higher precision, are referred to the true meridian, as are generally also the lines marking the boundaries of landed property.

171. Magnetic Meridian.—The direction of the magnetic meridian is that taken by a freely suspended magnetic needle. Unlike the geographic poles, the magnetic poles are constantly varying in position; moreover, they are at some distance from the true poles. Hence the direction of the magnetic meridian is by no means constant nor is the magnetic meridian in general parallel with the true meridian. The magnetic meridian is employed as a line of reference on the rougher surveys where one or another of the several forms of magnetic compass is used, or in connection with more precise surveys where angular measurements are approximately checked by means of the compass needle.

172. Magnetic Needle.—Any slender symmetrical bar of magnetized iron when freely suspended at its center of gravity takes up a position parallel with the lines of magnetic force of the earth. In horizontal projection these lines define the magnetic meridian. In elevation, the lines are inclined downward toward the north in the Northern Hemisphere, and downward toward the south in the Southern Hemisphere. Since the bar takes a position parallel with the lines of force, it becomes inclined with the horizontal. This phenomenon is called the *magnetic dip.* The angle of dip varies from 0° at or near the equator to 90° at the magnetic poles. The needle of the magnetic compass rests on a pivot. To counteract the effect of dip, so that the needle will take a horizontal position when directions are observed, a counterweight is attached to one end (the south end in the Northern Hemisphere). The counterweight usually consists of a short piece of fine brass wire wound about the needle and held in place by spring action. So long as the needle is used in a given locality and so long as it loses none of its magnetism, it will remain balanced. When for any reason it becomes unbalanced, it is adjusted to the horizontal by sliding the counterweight along the needle. At the mid-point of the needle is a jewel which forms a nearly frictionless bearing for the pivot.

173. Magnetic Declination.—The angle between the true meridian and the magnetic meridian is called the *magnetic declination.* If the north end of the compass needle points to the east of the true meridian, or in other words, if the direction of rotation from the

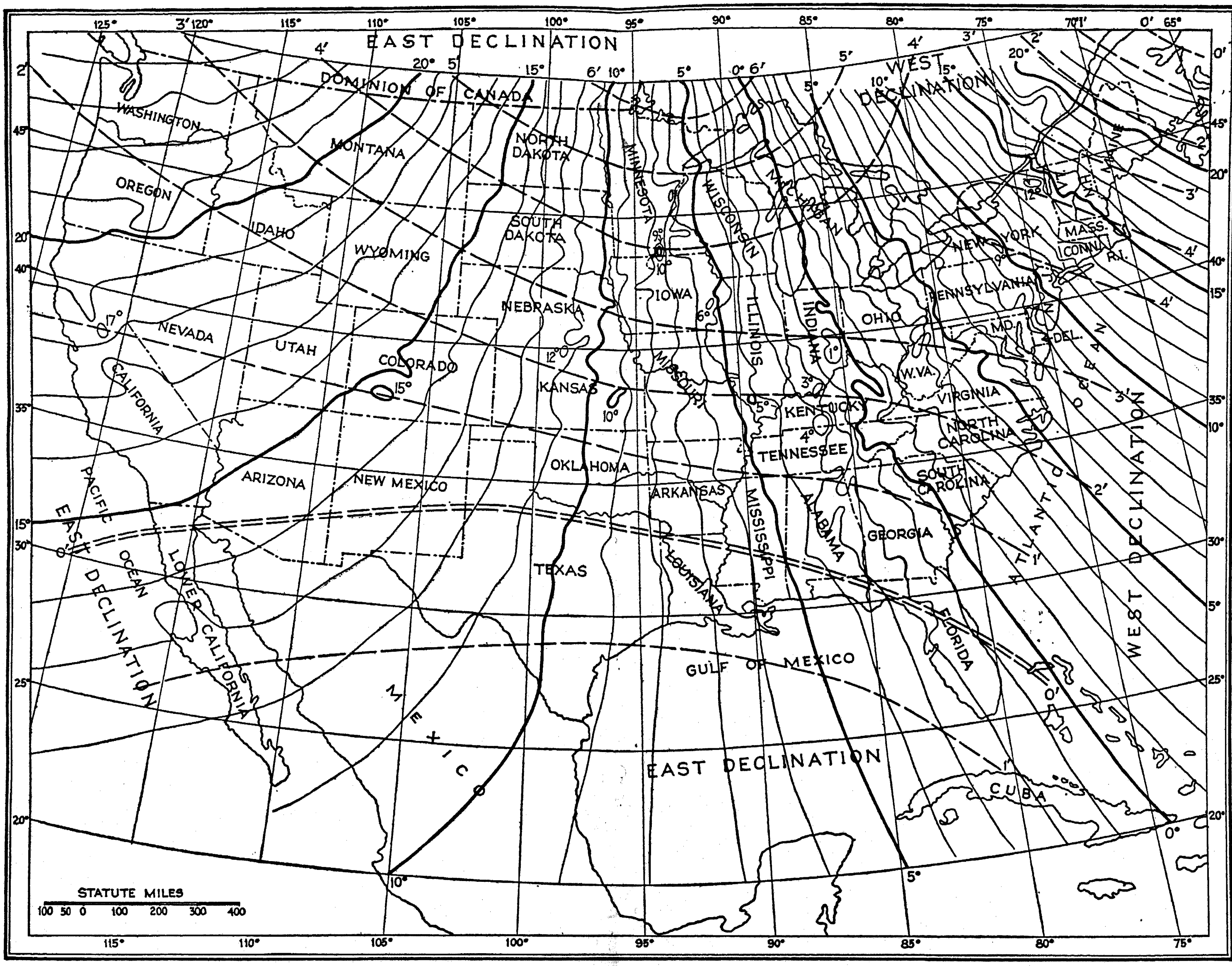


FIG. 173.—Isogonic chart of the United States for 1930. (*From the U. S. Coast and Geodetic Survey.*)

The lines of equal magnetic declination (full lines) apply to January 1, 1930. East of the line marked 0° (agonic lines) the north end of the compass needle points west of north, west of that line it points east of north.

The north end of the compass needle is moving to the westward for places north of the line of no change and to the eastward for places south of that line at an annual rate indicated by the lines of equal annual change (dash lines).

(*Facing page* 232.)

true meridian to the magnetic meridian is clockwise, the declination is said to be east; if the north end of the needle points to the west of the true meridian, the declination is said to be west.

If a true north and south line is established, the mean declination of the needle for a given locality can be determined by compass observations extending over a period of time. The declination may be estimated with sufficient precision for most purposes from the isogonic chart of the United States shown in Fig. 173.

173a. Isogonic Chart.—This chart shows lines of equal declination for the date Jan. 1, 1930, as based upon observations made by the U. S. Coast and Geodetic Survey at stations widely scattered throughout the country. The *agonic line*, or line of zero declination, is the heavy, full, irregular line extending in a southeasterly direction from the Great Lakes. The *isogonic lines* or light, full lines when east of the agonic line mark the paths where the declinations were on the above date 1° west, 2° west, etc.; similarly, those west of the agonic line show the routes along which the declinations were 1° east, 2° east, etc. Thus in the northern part of Maine the declination is seen to be 20° west, and in the northern part of Washington, 24° east.

174. Variations in Magnetic Declination.—Changes in the direction of the magnetic meridian cause corresponding variations in the magnitude of the declination. Chief among these is the *secular variation.*

The secular variation is a gradual change extending over a long period of years. Although its causes are not well understood, the indications are that it is more or less periodic in character, and in some localities in the United States, it amounts to several degrees in a cycle. Like a pendulum, the magnetic meridian swings in one direction for perhaps a century and a half until it gradually comes to rest, and then swings in the other direction; and as with a pendulum the velocity of movement is greatest at the middle of the swing. In Fig. 173 are shown by dash lines the annual rates of change in the secular variation for the year 1930. North of the line marking no change, the west declinations are increasing and east declinations are decreasing at the annual rates indicated on the chart; south of the line of no change the east declinations are increasing. On account of the magnitude of the secular variation, an kowledge of its behavior is of considerable importance to the surveyor, particularly in retracing lines whose directions are referred to the magnetic meridian as it existed years previously. When *variation* is spoken of without further qualification it is taken to mean the secular variation.

The solar-diurnal variation is a periodic swing of the magnetic needle occurring during each day. For points in the United States the north end of the needle reaches its extreme easterly swing at about 8 a.m., and its extreme westerly swing at about 1 p.m. The needle reaches its mean position between 10 and 11 a.m. and between 6 and 7 p.m. In Table VI are given for each hour of the day the average yearly values of the variation for several places in North America. These values change slightly from year to year and are greater in summer than in winter. It will be noted from the table that, in a general way, the higher the latitude, the greater the range in the solar-diurnal variation; the average for points in the United States is less than 07′, a quantity so small as to need no consideration for most of the work on which the compass needle is employed.

Other variations are the *annual variation,* a periodic annual swing, quite distinct from the secular variation, amounting to less than 01′ for most places in the United States, and *irregular variations* due to magnetic disturbances. The magnitude of the latter variations can not be predicted, but may be a degree or more, particularly at high latitudes. The variations are most likely to occur during magnetic storms, when the Aurora Borealis is to be seen.

175. Local Attraction.—Objects of iron or steel, some kinds of iron ore, and electric wires alter the direction of the lines of magnetic force in their vicinity and hence are likely to cause the compass needle to deviate from the magnetic meridian. The deviation arising from such local sources is called *local attraction.* In certain localities, particularly in cities, its effect is so pronounced as to render the magnetic needle of no value for determining directions. Local attraction is not likely to be the same at one point as at another, even though the points be but a short distance apart. The steel tape, chaining pins, axe, and small objects of iron or steel that are on the person are sources of local attraction which may be avoided but which when overlooked frequently introduce serious errors. By methods later to be described, the magnitude of local attraction may usually be determined, and directions observed with the compass may be corrected accordingly.

176. Bearings.—The direction of any line with respect to a given meridian may be defined by the *bearing.* In Fig. 176*a* let *SN* represent a meridian, either true, magnetic, or assumed, and let *OA*, *OB*, *OC*, and *OD* be lines whose directions with respect to the meridian are desired. The bearing of a line is indicated by the quadrant in which the line falls and the acute angle which the line makes with the meridian in that quadrant. Thus the line *OA* is in the northeast quadrant and makes an angle of 37° in that quadrant with the

meridian. The bearing of OA is read as North 37° East and is written N37°E. The bearings of OB, OC, and OD are respectively S62°E, S50°W, and N20°W. In all cases values of bearing angles lie between 0° and 90°. If the direction of the line is parallel with the meridian and north, it is written as N0° or *Due North.* If perpendicular to the meridian and east, it is written as N90°E or *Due East.*

When the bearing of a line is referred to the true meridian, it is called a *true bearing;* when referred to the magnetic meridian it is called a *magnetic bearing;* and when referred to an assumed meridian it is called an *assumed bearing.*

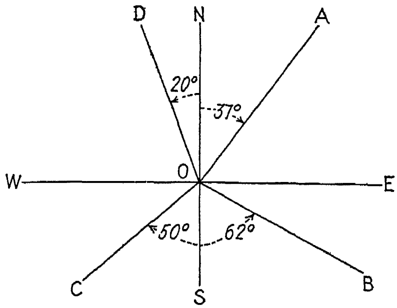


Fig. 176*a*.

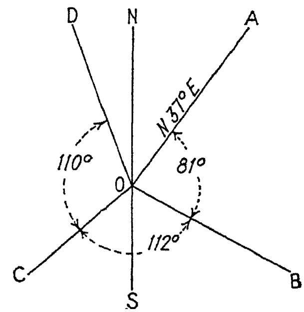


Fig. 176*b*.

Observed bearings are those for which the actual bearing angles are directly obtained by field observations. *Calculated bearings* are those indirectly obtained by computation. Thus if in Fig. 176*b* the observed bearing of OA is N37°E and the angle $AOB = 81°$, $BOC = 112°$, and $COD = 110°$, then the calculated bearings are

$$\text{Bearing } OB = 180 - 37 - 81 = \text{S62°E}$$
$$\text{Bearing } OC = 112 - 62 = \text{S50°W}$$
$$\text{Bearing } OD = 180 - 50 - 110 = \text{N20°W}$$

177. Azimuths.—The azimuth of a line is its direction as given by the angle between the meridian (either true, magnetic, or assumed) and the line, always measured in a clockwise direction from the south point (or from the north point) of the meridian. In astronomical observations azimuths are always reckoned from the true south; in surveying, some surveyors reckon azimuths from the south point and some from the north point of whatever meridian is chosen as a reference, but on any given survey the direction of zero azimuth is either always south or always north. Azimuths are called *true azimuths*, *magnetic azimuths*, or *assumed azimuths* according as the meridian to which they are referred is true, magnetic, or assumed. Azimuths may have values between 0° and 360°.

An *observed azimuth* is one which is directly obtained by a field observation. Thus in Fig. 177*a* azimuths measured from the south point are Az. $OA = 217°$, Az. $OB = 298°$, Az. $OC = 50°$, and Az. $OD = 160°$; or in Fig. 177*b* in which are shown the same lines with azimuths measured from the north point, Az. $OA = 37°$, Az. $OB = 118°$, Az. $OC = 230°$, Az. $OD = 340°$.

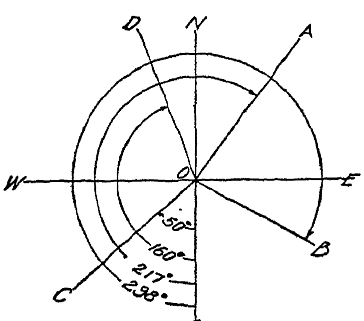


Fig. 177*a*.—Azimuths from south.

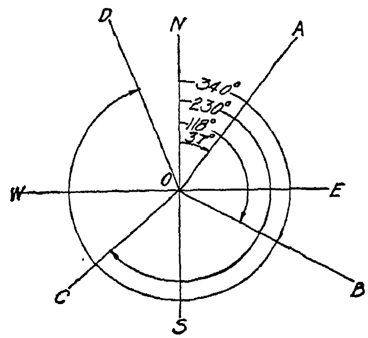


Fig. 177*b*.—Azimuths from north.

Calculated azimuths are those obtained by computation. Thus in Fig. 177*c* if the azimuth of OA as reckoned from the south is 217° and the angle AOB is 81°, the calculated azimuth of OB is $217 + 81 = 298°$. Similarly the calculated azimuth of OC is $298 + 112 - 360 = 50°$, and the calculated azimuth of OD is $50 + 110 = 160°$.

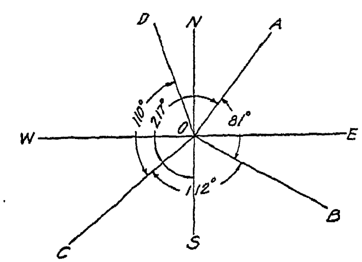


Fig. 177*c*.

Since both azimuths and bearings give directions with respect to a meridian, the operation of calculating azimuths from observed bearings, or *vice versa*, is a simple matter if a sketch is drawn to show the existing relations.

Example 1: Calculate azimuths (reckoned from south) corresponding to the following bearings: OB, N16°E; OC, S75°E; OD, S38°W; OE, N89°W. The azimuths are OB, $180 + 16 = 196°$; OC, $360 - 75 = 285°$; OD, 38°; OE, $180 - 89 = 91°$.

178. Deflection Angles.—The angle between a line and the prolongation of the preceding line is called a *deflection angle*. Thus in Fig. 178 if AB is the preceding line, the deflection angle to the line BC is as indicated. Deflection angles are recorded as *right*

or *left* according as the angle as measured from the preceding line is clockwise or counter-clockwise, or in other words, according as the line to which measurement is taken lies to the right or left of the prolongation of the preceding line. Thus in Fig. 178 the deflection angle at B is clockwise 22° and is recorded as 22°R; the deflection angle at C is counter-clockwise 33° and is recorded as 33°L. Deflection angles may take any value between 0° and 180° but are not usually employed for angles greater than 90°. In any closed polygon the algebraic sum of the deflection angles (considering right deflections as of sign opposite to left deflections) is 360°.

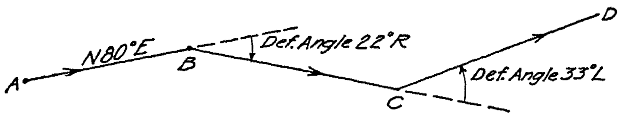


FIG. 178.—Deflection angles.

If the bearing of any line is known and the deflection angles are observed, the bearings of other lines may be calculated. Thus in the figure the bearing of AB is given as N80°E, hence the bearing of BC is $180 - 80 - 22 =$ S78°E.

179. Other Kinds of Angles.—In a closed polygon the angles inside the figure between adjacent lines are called *interior angles.* If n is the number of sides in a closed polygon, then the sum of the interior angles is $(n - 2)180°$.

Sometimes angles are determined by clockwise measurements from the preceding to the following line, as illustrated by Fig. 179. Such

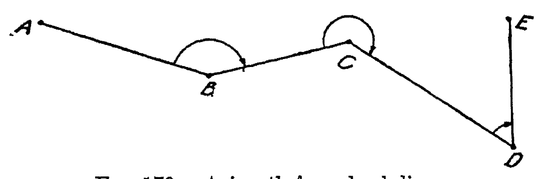


FIG. 179.—Azimuth from back line.

angles are often termed *azimuths from back line*, or *angles to the right*, though these designations are by no means universal.

180. Traverses.—The broken line connecting a succession of established points along the route of a survey is called a *traverse* or a *traverse line.* Distances along the line between successive points are determined by direct measurement. At each point where the traverse changes direction an angular measurement is taken. If the traverse forms a closed figure, as for example, the boundary of a parcel of land, it is called a *closed traverse;* if it does not form a closed figure, as for example, the survey for a highway, it is called a *continuous*

traverse or an *open traverse*. Traverses are also designated according to the purpose of the survey, the field instrument or methods employed, or the kind of angular measurements observed. Thus we speak of a *preliminary traverse* meaning a traverse forming the basis of the preliminary survey, a *transit-stadia traverse* meaning a traverse for which the angles are measured with the transit and distances with the stadia, and an *azimuth traverse* meaning one for which the observed angles are azimuths. The points defining the traverse line are called *traverse stations* or *traverse points*.

Generally deflection angles are employed on continuous traverses where the change in direction of the line is less than 90° at each traverse station. For the location of railroads, highways, canals, etc., angular measurements are nearly always taken by deflections. Azimuths are widely used on topographic surveys and similar surveys where a large number of details are located by angular measurements from the traverse stations. Ordinarily the interior angles of traverses run to establish the boundaries of land are observed. Traverses by bearings are rarely run except on rough surveys where the magnetic compass is employed, though magnetic bearings are generally observed as a rough check on deflection angles, azimuths, or interior angles determined by more precise methods.

For details of the methods of traversing, see Chap. XIII.

181. Triangulation.—Where the lines of a survey form triangular figures whose angles are measured and whose distances are determined by trigonometric computations, the operation of making the necessary field observations is called *triangulation* (Chap. XXVIII). The simplest case is that of a single triangle, one of whose sides is of known length. If any two angles of the triangle are measured, sufficient data are obtained for calculating the lengths of the other two sides. Further, if the third angle is measured, the angular measurements may be checked. When the survey is made up of a series of triangles so connected that, having measured the angles of the triangles and the length of one line, the lengths of other lines may be calculated, the series is called a *triangulation system*. The side of known length, upon which all calculated distances are based, is called a *base line*.

Triangulation is often necessary in connection with traversing where the direct measurement of one or more lines is impossible. Simple triangulation is also employed for the location of tunnel shafts and bridge piers. A simple chain of triangles or quadrilaterals affords a convenient means of locating points on opposite sides of a stream. Groups of polygons are suitable for the survey of an area. Generally an extensive triangulation system, such as for a large city

or a state, is composed of a combination of simple triangles, polygons, and quadrilaterals. The advantage of triangulation over traversing lies in the small number of linear measurements which are necessary; the disadvantage lies in the increased labor of computing.

For details of the methods of triangulation, see Chap. XXVIII.

182. Angles with Tape.—If the sides of a triangle are measured, sufficient data are obtained for computing the angles. While for most surveying work the angles are measured directly, there are occasions where angles may be determined with sufficient precision by tape measurements. Figure 182 illustrates the simplest method and the one commonly employed. Points a and b are established on lines AB and AC by swinging the 100-ft. tape about A as a center. The chord length ab is then measured. If α is the angle, then

$$\sin \frac{1}{2}\alpha = \frac{ab}{200}$$

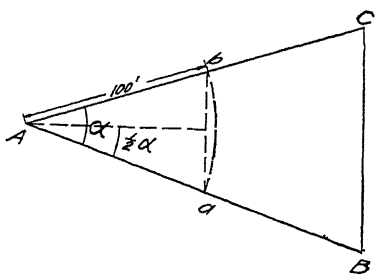


Fig. 182.—Angle with tape.

The error in the computed value of the angle depends upon the care with which the points a and b are established and the precision with which the measurements are taken. On level ground for angles of less than 90° the error need not exceed five or ten minutes. For angles greater than 90° the chord distance for the deflection angle gives a more reliable determination. It is obvious that surveying by this method is too slow to be employed except in emergency when surveying instruments are not available. The method is in general utilized for rough estimates and for checks of more precise measurements.

183. Angles and Directions with Transit.—On most surveys angles are measured with the engineer's transit, the use and adjustment of which will be described in the following chapter. The essential features of the transit here to be considered are (1) a horizontal circle, graduated in degrees, which may be rotated and which may be clamped in any position, (2) a plate which may be rotated inside the graduated circle and which carries verniers for reading the graduated circle, and (3) a telescopic line of sight which is attached the vernier plate and rotates with it, and which may be rotated in altitude. By means of the verniers the graduated circle on most instruments can be read to the nearest minute of arc, and on some precise transits to the nearest 10 seconds.

If the angle ABC in Fig. 183a is to be measured, the transit is set at B. The index of the vernier is set at zero on the graduated circle and a sight is taken to A. The graduated circle is clamped in position and the line of sight is turned to C. Since the vernier plate is moved with the line of sight, it is rotated through the angle ABC and hence the vernier reads the angle. If azimuths are to be observed (and the circle is graduated from 0° to 360° in a clockwise direction) the backsight to A is taken with the vernier set to read the azimuth

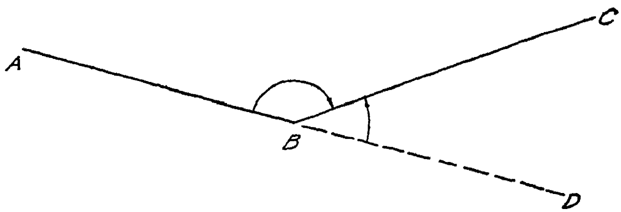


Fig. 183a.

of the line BA. When the line of sight is rotated to C, the vernier reading will be the azimuth of the line BC. If the deflection angle at B (Fig. 183a) is to be observed, a backsight is taken on A with the vernier set at 0°, the line of sight is rotated first in altitude (plunged) to point in the direction of BD, and then is rotated in azimuth until the point C is sighted; the vernier then reads the deflection angle DBC.

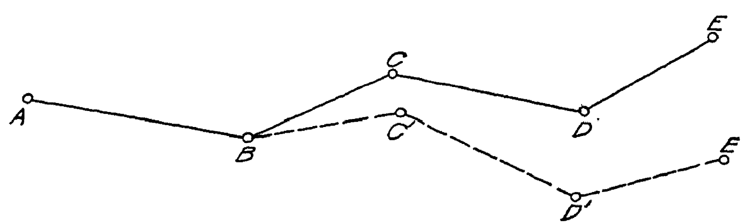


Fig. 183b.

In running a traverse, as $ABCDE$ (Fig. 183b) angular measurements are made at successive stations. If an error occurs in any angle, as ABC, then the observed or calculated directions of all succeeding lines in the traverse, as BC, CD, and DE, will be affected by the amount of the error. If the error at B is CBC', then BC', $C'D'$, and $D'E'$ indicate the observed or calculated directions of the following lines.

184. Direction with Magnetic Compass.—The essential features of the compass used by the surveyor are: (1) a compass box with

circle graduated from 0° to 90° in both directions from the N and S points and having the E and W points interchanged as illustrated in Fig. 184*a*; (2) a line of sight in the direction of the SN points of the compass box; and (3) a magnetic needle which may be raised from its pivot and clamped in position. When the line of sight is pointed in a given direction the compass needle (when brought to rest) gives the magnetic bearing. Thus in the figure the bearing of *AB* is N60°E. If the N point of the compass box is nearest the object sighted, the bearing is read by observing the north end of the needle.

The varieties exhibiting the three features just mentioned are: (1) various *pocket compasses* which are generally held in the hand when bearings are observed, and which are used on reconnaissance or other rough surveys; (2) the *surveyor's compass* which is mounted on a light tripod and which was formerly much used on all kinds of land surveying, but is now little employed except for forest surveys; and (3) the *transit compass*, a compass box mounted on the upper or vernier plate of the engineer's transit, a description of which is given in the following chapter.

184a. Pocket Compasses.—Figure 184*b* illustrates one pattern of the pocket compass for which the line of sight is given by a line on the inside of the cover. An observation is taken by laying the cover back and holding the compass so as to sight along the line inside the cover. When this line of sight is in the proper direction the needle is given time to come to rest. It is then raised and clamped in position by depressing the pin at *A*, the compass is lowered, and the bearing is read.

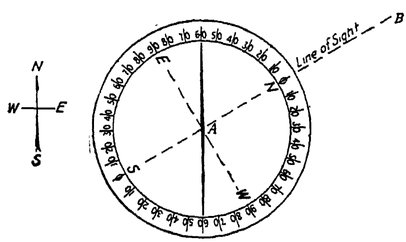


FIG. 184*a*.

The Brunton *pocket transit* shown in Fig. 184*c* is designed primarily as a hand instrument but may also be used on a tripod or a Jacob's staff (a stick about 5 ft. long). Inside the cover is a mirror *A*; at *B* is a folding peep sight. The cover is tilted until with the eye at *B* the reflected image of the compass circle is visible. At *C* a portion of the mirror glass is without silver, and a line is etched on the glass in line with the N and S points of the compass and the peep sight *B*. The peep sight and the etched line define the line of sight of the instrument. An observation is taken by holding the peep sight to the eye and viewing the distant object through the plain glass at *C*. When the proper direction is obtained the compass is leveled by centering the bubbles of the two level tubes as seen in the mirror. When the needle comes to rest the bearing is observed by means of the mirror; or if so desired the needle

may be clamped by depressing the pin at D, and the compass may be lowered and read directly. Another method of observing a bearing is to hold the pocket transit in the hand a convenient distance below the eye, viewing it as in the figure and turning it about in a horizontal plane until the image of the object defining the far end of the line is bisected by the line etched on the mirror. The bearing is then read.

The Brunton pocket transit is also used as a hand level or clinometer. When so employed it is held on edge and one of the bubbles is centered by means of a thumb nut not visible in the figure. Vertical angles are read by means of the graduated arc F and the vernier G. Its use is so nearly like that of the Abney level (Art. 108a) that no further description is necessary.

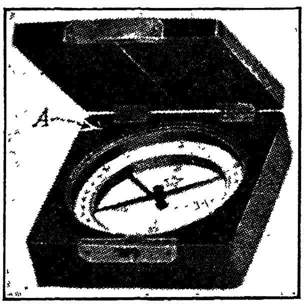


FIG. 184b.—Pocket compass.

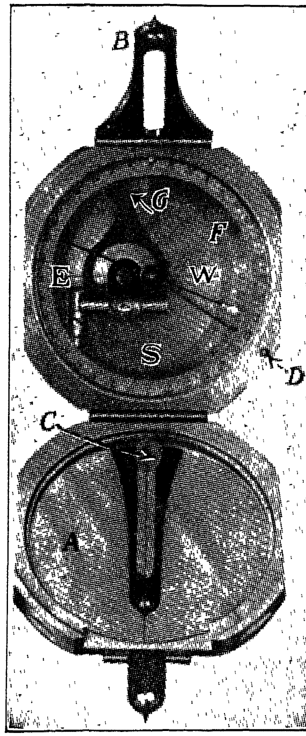


FIG. 184c.—Brunton pocket transit.

The *prismatic compass* is similar in principle to the Brunton transit, except that it has a floating card dial and that a prism is employed instead of a mirror.

184b. Surveyor's Compass.—This consists of a compass box mounted on a plate A to the ends of which are fastened the vertical sight vanes B, B (Fig. 184d). The plate is rigidly connected to a vertical spindle beneath the center of the compass, which spindle is free to revolve in a conical socket at C. At D is a leveling head consisting of a ball and socket joint, and at E are level tubes at right angles with one another by means of which the compass may be leveled. The upper portion of the socket at D is a thumb nut which is tightened until the ball is held securely by friction. F is a screw for lifting and clamping the needle, G is an "outkeeper" for

registering chainage in land surveying, H is a screw for clamping the vertical spindle, and J is a spring catch which when pulled allows the removal of the spindle from the cone-shaped socket. The compass circle is usually graduated in half-degrees, and bearings may be read to 05′ or 10′. In order that *true* bearings may be read directly, some compasses, as the one shown in the illustration, are so designed that the compass circle may be rotated with respect to the plate on which it is mounted. When the circle is in its normal position, the line of sight as defined by the vertical slits in the sight vanes is in line with the N and S points of the compass circle, and the observed bearings

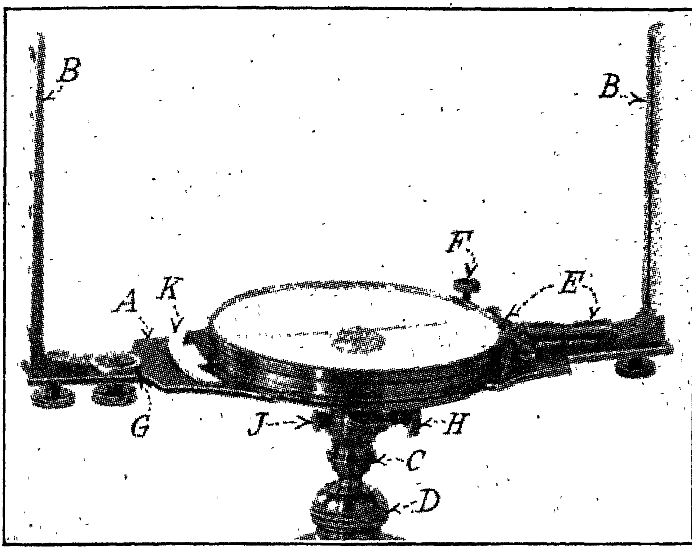


FIG. 184*d*.—Surveyor's vernier compass.

are magnetic. If the circle is turned through an angle equal to the magnetic declination, the observed bearings will be true, as is evident from Fig. 184*e*. If the declination is east, as in the figure, the circle is rotated clockwise with respect to the plate; if the declination is west, counter-clockwise. At K (Fig. 184*d*) is a graduated arc and vernier by means of which the declination may be laid off to the nearest minute. When the declination has been set off, the compass circle is clamped by a screw beneath the plate.

When the direction of a line is to be determined, the compass is set up on line and is leveled. The needle is released, and the compass is rotated about its vertical axis until a range pole or other object on line is viewed through the slits in the two sight vanes. When the

needle comes to rest the bearing is read. Ordinarily the sight vane at the S end of the compass is held next to the eye, in which case the bearing is given by the north end of the needle.

When a traverse is run, only alternate stations need be occupied; but a check is secured and local attraction is detected if both a backsight and a foresight are taken from each station. Unlike the transit traverse, where an error in any angle affects the observed or calculated directions of all following lines, an error in the observed bearing of one line in a compass traverse has no effect upon the observed *directions* of any of the other lines. This is an important advantage, especially in the case of a traverse having many angles.

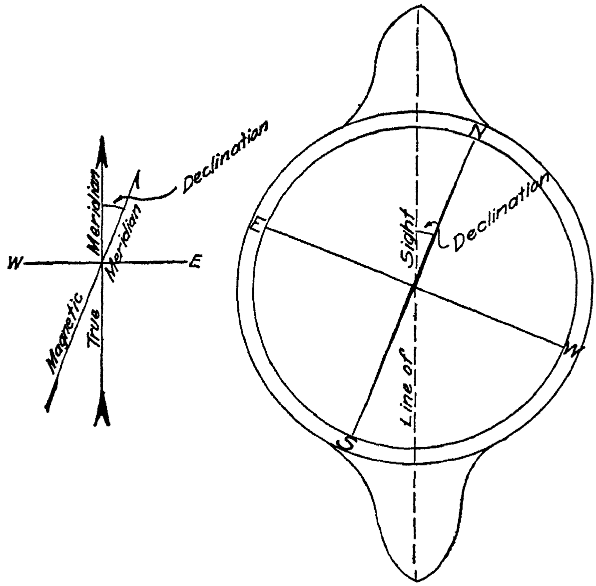


FIG. 184*e*.—Magnetic declination.

184c. Corrections for Local Attraction.—If local attraction exists at any station in a traverse, both the back and forward bearings taken from that station will be affected by the same amount. Disregarding for the time being the accidental errors in observing, it is probable that the terminal points of any line, as AB, are free from local attraction if the back bearing from B is the reverse of the forward bearing from A. Keeping in mind that the calculated *angle* between the forward and back lines from any station may be correctly determined from the observed bearings taken from that station regardless of whether or not the needle is locally affected, the direction of a line

free from local attraction may be chosen as a basis, the traverse angles may be computed from observed bearings, and starting from the unaffected line, the correct bearings of successive lines may be calculated.

Thus from the observed bearings tabulated below it is seen that points A and B are free from local attraction since the back and forward bearings of AB are opposite. Hence the correct bearing of BC is S60°E. The back bearing of BC shows an attraction of 2°. The angle at C as given by the bearings taken at that point is $180 - 62 - 31 = 87°$. The correct bearing of CD is therefore $180 - 60 - 87 =$ S33°W.

Station	Line	Observed bearing	Calculated angle	Corrected bearing	Correction
A					
	AB	N45°E		N45°E	0°
	BA	S45°W		S45°W	0°
B			105°		
	BC	S60°E		S60°E	0°
	CB	N62°W		N60°W	2°
C			87°		
	CD	S31°W		S33°W	2°
	DC	N30°E		N33°E	3°
D			100°		
	DE	N70°W		N67°W	3°
	ED	S67°E		S67°E	0°
E					

Most surveyors find it more expedient to consider the magnitude and direction of the error due to local attraction and then to make corrections to observed bearings without computing the traverse angles.

Thus, for the observed bearings tabulated above it is seen that the correct bearing of CB is N60°W and the observed bearing is N62°W. The local attraction at C is, therefore, 2° clockwise, or the correction to any observed bearing taken with the compass at C is 2° clockwise. The observed bearing of CD is S31°W and the corrected bearing of CD is therefore $31 + 2 =$ S33°W.

Owing to errors of observation there are likely to be discrepancies between the observed back and forward bearings of lines, even though no local attraction exist. If the discrepancies are small and are apparently not of a systematic character, it is fair to assume that the errors are due to other causes than local attraction.

The amount of metal about the person of the instrumentman is ordinarily not large enough to deflect appreciably the compass needle. However, it is good practice to avoid a change of position between two readings.

When the compass traverse forms a closed figure, the interior angle at each station may be calculated from the observed bearings. The sum of the interior angles should equal $(n - 2)\ 180°$ in which n is the number of sides in the traverse. Since the error of observing a bearing is accidental in character, the error of closure of the traverse (as indicated by the sum of the calculated interior angles) is assumed to be distributed equally, and the interior angles are corrected accordingly. The bearings are then adjusted by starting from some line whose observed bearing is assumed to be correct and by calculating the bearings of successive lines by means of the corrected interior angles, after the manner described in the first paragraph in this article.

184d. Sources of Error in Compass Work.

1. *Needle Bent.*—If the needle is not perfectly straight a constant error is introduced in all observed bearings. The error may be eliminated by reading both ends of the needle and averaging the two angular values; or the glass cover of the compass box may be removed and the needle may be straightened with a pair of pliers.

2. *Pivot Bent.*—If the point of the pivot supporting the needle is not at the center of the graduated circle there is introduced a variable systematic error, the magnitude of which depends upon the direction in which the compass is sighted. For one direction the error is zero; for a direction 90° thereto it is a maximum. It may be eliminated by reading both ends of the needle and averaging the values thus obtained, or by bending the pivot until the end readings of the needle are 180° apart for any direction of pointing.

3. *Plane of Sight Not Vertical, or Graduated Circle Not Horizontal.*—This introduces a systematic error, but it is usually so small as to be of no consequence. However, with the surveyor's compass the sight vanes may become bent so that even though the instrument is leveled, an appreciable error is introduced, particularly if the line of sight is steeply inclined when taking a bearing. The vanes may be tested by leveling the compass and sighting at a plumb line. The adjustment of the level tubes may be tested by reversal, as described for the transit in Art. 209.

4. *Sluggish Needle.*—The needle is not likely to come to rest exactly on the magnetic meridian. This produces an accidental error which is often of considerable magnitude. If the needle is

"weak" it may be remagnetized by drawing its ends over a bar magnet, from the center to the ends of the magnet. The south-seeking end of the compass needle is drawn over the north-seeking half of the bar magnet, and *vice versa.* On each return stroke the needle should be lifted well above the magnet. If the pivot point is blunt it may be sharpened by rubbing it on a fine-grained oilstone. As the needle comes nearly to rest, tapping the glass with some light object will produce vibrations which tend to prevent the needle from sticking to the pivot.

5. *Reading Needle.*—The inability of the observer to determine exactly the point on the graduated circle at which the needle comes to rest is generally the source of the most important and largest accidental error in compass work. To read the needle accurately requires that its ends should be in the same plane with the horizontal circle and that the eye of the observer be above the coinciding graduation and in line with the needle. If the needle dips perceptibly its counterweight should be adjusted. Other conditions being equal, the longer the needle the smaller the error of observing. With the 6-in. needle used on many surveyor's compasses the probable error need not exceed $\pm 05'$; with the $3\frac{1}{2}$ or 4-in. needle on the engineer's transit the probable error is likely to be as much as $\pm 10'$.

6. *Magnetic Variations.*—Undetected deviations of the magnetic needle from whatever cause are the source of the largest and most important systematic errors in compass work. It is largely because of such variations that the compass, no matter how finely constructed, is not a suitable instrument for any except rough surveys. Deviations due to local attraction may be detected and corrections may be applied as described in the preceding article. Particular care should be taken to keep iron or steel objects away from the compass while it is in use, and the observer should if possible remain on the same side of the instrument. Also the needle may be attracted by static charges of electricity on the glass cover. These charges may be removed by touching the glass with a moist finger.

185. Other Methods of Determining Angles.—Angles may be graphically determined by means of the *plane table and alidade,* the use of and adjustments of which are described in Chap. XXIII. The plane table consists essentially of a drawing board mounted on a tripod in such manner that it may be leveled and may be revolved in the horizontal plane. The essential features of the alidade are a straightedge parallel to which is a line of sight. Figure 185*a* illustrates the use of the plane table for the graphical determination of the angle *AOB*. The plane table, to which is fastened a sheet of drawing paper, is set over the ground point *O*. A point *O* on the

paper is plotted over the point on the ground. With the straightedge through O, a sight is taken to A and a line is drawn. Point B is sighted and a similar line is drawn. The two lines on the paper are parallel to corresponding lines on the ground and hence define the angle AOB. The plane table is extensively employed in topo-

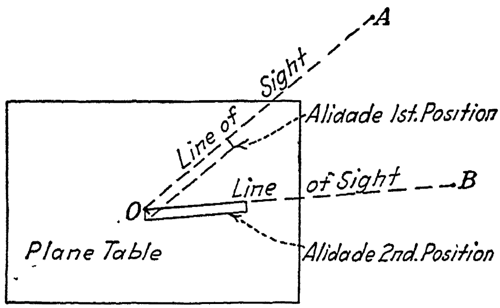


Fig. 185*a*.—Plane table.

graphic mapping, for which work the graphical representation of angles is sufficient and the numerical values of the angles are not desired.

The *sextant*, though used principally by the navigator, is sometimes employed by the surveyor, particularly on hydrographic surveys (see Chap. XXVI). The advantage of the sextant over other instru-

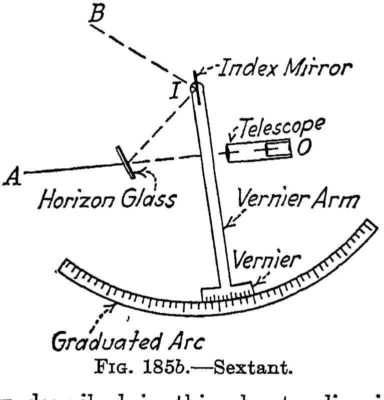


Fig. 185*b*.—Sextant.

ments previously described in this chapter lies in the fact that, through its use, an angle may be measured accurately while the observer is on a moving object. The essential features are illustrated (in plan) in Fig. 185*b*, with the instrument in position for measuring the horizontal angle AOB. It consists of a metal frame to which are rigidly attached the graduated arc, telescope, and horizon glass.

The lower part of the horizon glass is silvered, and the upper part is clear. The vernier arm is attached to a spindle which seats in a conical bearing at I. Parallel with the vernier arm and attached to it is the index mirror. With the vernier arm in one position the image of the object B is reflected from the index mirror to the lower part of the horizon glass, and thence to the telescope. Through the upper portion of the horizon glass rays of light emanating from object A enter the telescope directly. If as viewed through the telescope the image of A appears to overlie the image of B, the vernier setting on the graduated arc indicates the angle AOB; the vernier arm is moved until this condition exists. The angle measured is in the same plane as the points A, B and O at center of telescope, and hence, in general, is not a horizontal angle. In the situations where the sextant is used in surveying, however, these angles may generally be considered to be horizontal without appreciable error. When used for astronomical measurements in navigation, only vertical angles are measured.

186. Methods of Determining the Meridian.—The true meridian is established by astronomical observations, as described in Chapter XVIII. Any of the heavenly bodies whose astronomical position is known may be observed, but those commonly observed by surveyors are the sun and Polaris (the north star). For surveys of ordinary precision the instrument utilized is the transit.

On compass surveys in order to determine the magnetic declination, the true meridian is sometimes established by ranging plumb lines with Polaris, usually when the star is at elongation (farthest east or farthest west). If the time is accurately known, the observations are sometimes made when the star is at culmination (directly above or below the pole and hence on the meridian). One plumb line is suspended from some convenient high point, and a stake with tack representing the north point of the meridian is set beneath the bob. At a distance of 15 or 20 ft. south of the plumb line two stakes are set, one on either side of the estimated position of the meridian, and a piece of stout string is stretched between nails driven in their tops. A second plumb line is suspended from the stretched string. When the time of elongation or culmination approaches, the observer moves the second plumb line, keeping the two plumb lines in line with the star until the time of elongation or culmination has been reached. A stake with tack is set beneath the second plumb line. If the star is at culmination the tacked stakes define the true meridian; if the star is at elongation the true meridian is established by laying off an angle from the established line equal to the azimuth of the star as given in Table V. If the observation is made at western elongation the angle is turned clockwise; if made at eastern elongation the angle is turned counter-clockwise. The times of elongation and

culmination of Polaris may be taken from Table IV, the latitude and longitude having been approximately determined from a map.

The first plumb line will usually need to be illuminated with an artificial light. For some minutes preceding and succeeding the instant of elongation the star appears to move vertically, hence observations taken at elongation are not influenced by time errors and need not be hurried. At culmination, on the other hand, the star is moving east or west at the rate of about 06′ in 15 min. of time; hence the time must be known closely and the observation must be made quickly. If the star is at elongation and care is exercised in setting the ground points, the error in determining the meridian in this manner need not exceed 05′.

A magnetic meridian may be established by setting up the compass over any convenient point marking one end of the proposed line, and then setting a series of points on a stake or other object marking the other end of the meridian. After each setting of the line of sight, the compass should be rotated a few degrees about the vertical axis and then moved back until the needle reads zero. The mean of the points thus established is assumed to be on the magnetic meridian, provided the observations are taken at a time of day when the declination is about at its mean value, otherwise corrections for daily variation may be made as indicated in Table VI.

187. Numerical Problems.

1. The magnetic bearing of a line is S47°30′W and the magnetic declination is 12°10′W. What is the true bearing of the line?

2. The true bearing of a line is N18°17′W and the magnetic declination is 7°12′E. What is the magnetic bearing of the line?

3. In 1856 the magnetic bearing of a line was N35°15′ E and the declination was 2°10′W. At the present time the declination is 3°15′E. What is the true bearing and its present magnetic bearing?

4. Following are the observed magnetic bearings of a compass traverse: *AB*, N37°45′E; BC, N84°30′E; *CD*, S66°40′E; *DE*, S79°0′E; *EF*, N55°15′E. Calculate the deflection angles.

5. Following are deflection angles of traverse *A* to *F*: *B*, 37°21′L; *C*, 12°39′L; *D*, 63°31′R; *E*, 14°07′L. The true bearing of *AB* is S37°56′ E. Calculate the bearings of the remaining lines.

6. For the traverse of problem 4, the declination is 7°15′E. Compute the true azimuths reckoned from the north point.

7. For the traverse of problem 5 compute the true azimuths reckoned from the south point.

8. The interior angles of a five-sided closed traverse are as follows: *A*, 117°36′; *B*, 96°32′; *C*, 142°54′; *D*, 132°18′. The angle at *E* is not measured. Compute the angle at *E*, assuming the above values to be correct.

9. What are the deflection angles of the traverse of problem 8? What are the calculated bearings if the bearing of *AB* is due north?

10. The following azimuths are reckoned from the north: *AB*, 187°12′; *BC*, 273°47′; *CD*, 318°48′; *DE*, 0°48′; *EF*, 73°00′. What are the corresponding bearings? What are the deflection angles?

11. Following are the deflection angles of a closed traverse: *A*, 85°16′L; *B*, 10°11′R; *C*, 83°32′L; *D*, 63°27′L; *E*, 34°18′L; *F*, 72°56′L; *G*, 30°45′L. Calculate the error of closure. Adjust the angular values on the assumption that the probable error is the same for one angle as for any other.

12. In triangulating across a river a base line *AB* of the triangle *ABC* has a measured length of 536.27 ft. and the angles at *A* and *B* are respectively 87°32′ and 68°48′. Calculate the distance *AC*.

13. Below are bearings taken for a continuous compass traverse. Correct for local attraction.

Line	Forward bearing	Back bearing
AB	N37°15′E	S36°30′W
BC	S65°30′E	N66°15′W
CD	S31°00′E	N31°00′W
DE	S89°45′W	N89°45′E
EF	N46°30′W	S46°45′E
FG	N15°00′W	S14°45′E

14. The following are bearings taken on a closed compass traverse. Compute the interior angles and correct them for observational errors. Assuming the observed bearing of the line *AB* to be correct, adjust the bearings of the remaining sides.

Line	Forward bearing	Back bearing
AB	S37°30′E	N37°30′W
BC	S43°15′W	N44°15′E
CD	N73°00′W	S72°15′E
DE	N12°45′E	S13°15′W
EA	N60°00′E	S59°00′W

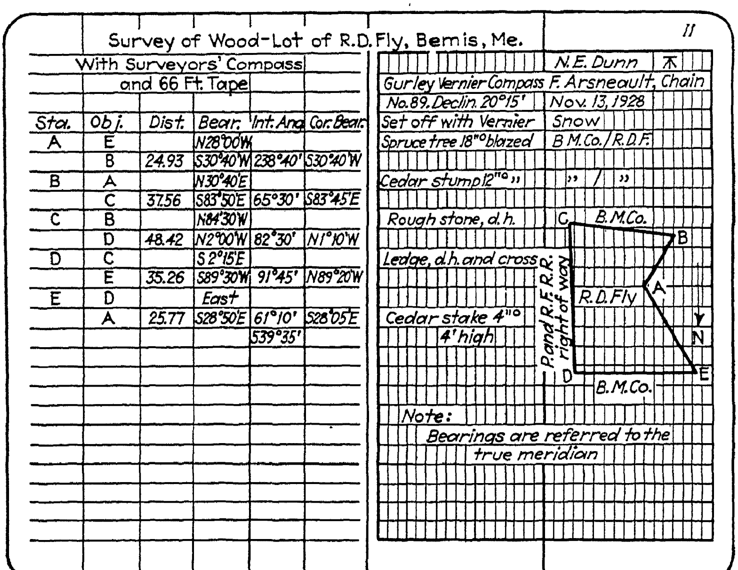
188. Field Problems.

PROBLEM 1. DETERMINATION OF MAGNETIC DECLINATION

Object.—To determine the magnetic declination with the surveyor's compass.

Procedure.—(1) See that the compass is in good adjustment. (2) Set the compass over one end of a meridian that has been determined by astronomical observations, sight along the line, and clamp the compass in that position. (3) By means of the tangent-screw, move the compass circle until the needle reads zero. (4) From the declination arc read the

declination to the nearest minute; record the declination and the time of observation. (5) Take observations as above every five minutes over a period of half an hour or more, resetting the line of sight and compass circle for each observation. (6) If possible take a series of observations at about the same time on each of several days. (7) Determine the most probable value of the declination for the hour of observation, and the probable error of a single observation and of the mean (see Art. 69). (8) Determine the mean declination by adding to or subtracting from the observed declination the average solar-diurnal variation (Table VI) for the place nearest the place of observation.



Survey of Wood-Lot of R.D. Fly, Bemis, Me.

With Surveyors' Compass and 66 Ft. Tape

Sta.	Obj.	Dist.	Bear.	Int. Ang.	Cor. Bear.
A	E		N28°00'W		
	B	24.93	S30°40'W	238°40'	S30°40'W
B	A		N30°40'E		
	C	37.56	S83°50'E	65°30'	S83°45'E
C	B		N84°30'W		
	D	48.42	N2°00'W	82°30'	N1°10'W
D	C		S2°15'E		
	E	35.26	S89°30'W	91°45'	N89°20'W
E	D		East		
	A	25.77	S28°50'E	61°10'	S28°05'E
				539°35'	

FIG. 188.—Notes for compass survey.

Hints and Precautions.—(1) Care should be taken to keep sources of magnetic disturbance away from the instrument. *Great care should be taken not to produce static charges of electricity by rubbing the glass.* A moistened finger pressed against the glass will remove such charges. (2) At each observation the compass box should be tapped lightly as the needle comes to rest so that there may be no adhesion between the jewel of the needle and the pivot. (3) Between 6 and 7 p.m. is the best time for taking magnetic observations, because at this time the magnetic declination reaches approximately its mean value for the day, as will be seen by examining Table VI. Between 10 and 11 a.m. the declination also reaches its mean value, but the rate of change of declination is more rapid at this time than between 6 and 7 p.m. (4) If the compass circle has been turned in a clockwise direction the declination is east. If the diurnal variation is positive for the time of the observation, the north end

of the needle is deflected more to the eastward than when the declination is at a mean. Hence, the mean is determined by algebraically subtracting the diurnal variation from the observed declination, east being considered as positive and west as negative.

PROBLEM 2. SURVEY OF FIELD WITH SURVEYOR'S COMPASS AND 66-FT. TAPE

Object.—To find the lengths and bearings of the sides of an assigned field.

Procedure.—(1) On the declination arc of the compass set off the magnetic declination for the place where the survey is to be made. (2) Set up at one corner of the field, A, and sight along the line AB, being sure to have the south end of the compass box nearer the eye. Lower the needle, and as it comes nearly to rest, tap the compass lightly with the end of a lead pencil. When the needle becomes stationary read the north end, estimating the bearing to the nearest 05′. (3) Take a back bearing from A. (4) Chain the line AB, and record the distance in chains, to the nearest link. (5) Set up the compass at B. Observe the back bearing of the line AB and the forward bearing of the line BC. Chain the distance BC. (6) Continue in the same manner around the field, taking both back and forward bearings from each point. (7) Calculate the interior angles of the field from the back and forward bearings measured at the vertex of each of the angles, and correct the observed bearings for local attraction, if such exists (see sample notes, Fig. 188).

Hints and Precautions.—(1) The upright sight vanes are usually unlike, the one to be attached nearer the north point of the compass box being marked for reading vertical angles, and the one nearer the south point being fitted with peep sights. It is well to bear this in mind, both when assembling the compass and when using it in the field. (2) Since the accuracy with which angles can be read depends upon the delicacy of the needle, special care should be taken to avoid any jar between the jewel bearing of the needle and the pivot point. *Never, under any condition, move the instrument without first making certain that the needle is clamped.* (3) Keep sources of attraction such as chaining pins, axe, and pocket knives away from the instrument when bearings are being observed. (4) Be sure to set off the declination in the right direction. If the declination is east, the magnetic meridian lies to the east of true north; and since the point of zero bearing must lie on the magnetic meridian when the line of sight lies in the true meridian, the graduated circle must be turned in a clockwise direction. (5) In order not to confuse the north and south ends of the needle when taking bearings, always note the position of the counterbalancing wire.

PROBLEM 3. RETRACING SURVEY WITH COMPASS AND 66-FT. TAPE. TWO ADJACENT CORNERS KNOWN

Object.—To retrace property lines from the notes of an old compass survey. The bearings given in the original notes are magnetic, and the

declination at the time of the original survey is unknown. Two adjacent corners of the plot can still be identified.

Procedure.—(1) Measure the length of the known side and compare with the original. (2) By proportionate distances compute the lengths of the other sides in terms of the re-survey tape. (3) Set up the compass at one end of the known line, sight along the line, and clamp the compass. (4) Release the needle, and as it comes to rest move the compass circle by means of the tangent-screw until the original bearing is read. (5) Proceed to lay out the field, chaining distances in terms of the re-survey tape as computed above and laying off the original bearings. Examine the ground for rotten stakes or other evidence that would have precedence over bearings and lengths of sides. (6) Reference the new corners by bearings and distances to nearby permanent objects.

*

CHAPTER XII

USE AND ADJUSTMENT OF THE ENGINEER'S TRANSIT

THE ENGINEER'S TRANSIT

189. Description.—The engineer's transit is sometimes called the "universal surveying instrument" by reason of the wide variety of uses for which it is adapted. Though the transits of the various

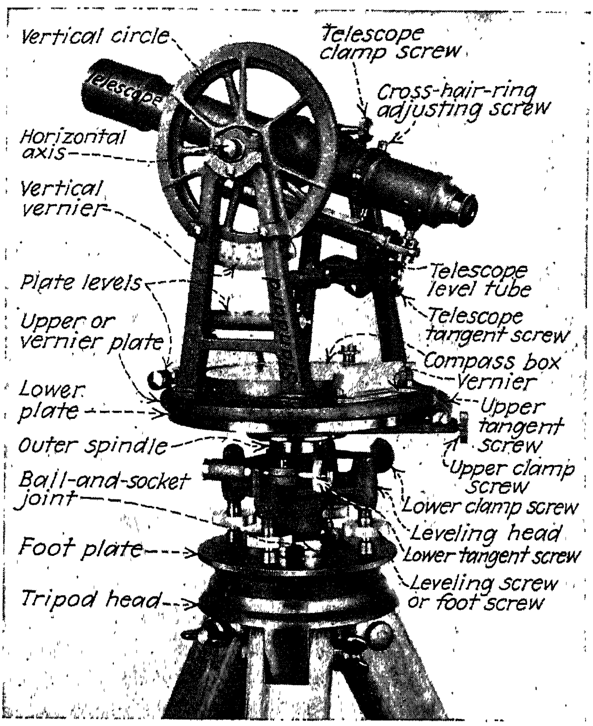


FIG. 189*a*.—Engineer's complete transit.

instrument makers differ somewhat as to details of construction, they are much alike in their essential features. Figure 189*a* is a photograph of the type of transit in most common use and Fig. 189*b* is a vertical section of the same instrument. It is seen to consist of an

upper or vernier plate to which are attached A-shaped standards supporting the telescope, and a lower plate to which is fixed a horizontal graduated circle. The upper and lower plates are fastened respectively to the inner and outer spindles, the axes of the spindles

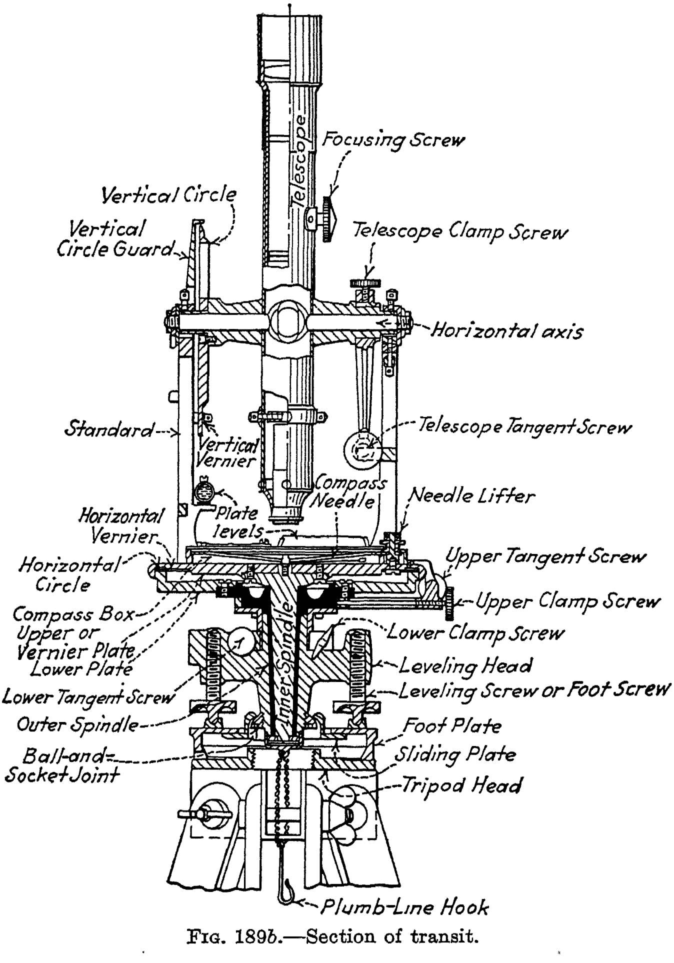


FIG. 189*b*.—Section of transit.

being concentric with the graduated circle and with the upper plate. The outer spindle is seated in the cone-shaped socket of the leveling head. Near the lower extremity of the leveling head is a ball-and-socket joint, which secures the instrument to the foot plate yet permits rotation of the instrument about the joint as a center. When the lower plate is rotated the outer spindle revolves in its socket in

the leveling head. By means of the lower clamp-screw, the outer spindle, and hence the horizontal graduated circle, may be clamped in any position. After the lower clamp has been tightened, small movements of the horizontal circle may be secured by turning the lower tangent-screw. Similarly, when the upper plate is rotated, if the lower plate is clamped, the inner spindle revolves within the outer spindle. By means of the upper clamp the inner spindle may be secured to the outer spindle, and any small movement of the verniers with respect to the graduated circle may be obtained by turning the upper tangent-screw. The axis about which the spindles revolve is called the vertical axis.

Plate levels, one of which is fastened to the upper plate and the other of which is secured to one of the standards, are provided for leveling the instrument so that the plane of the horizontal circle will be truly horizontal when observations are made. The four leveling screws, or foot screws, are threaded into the leveling head and bear against the foot plate; when the screws are turned, the instrument is rotated about the ball-and-socket joint. When all four screws are loosened, pressure between the sliding plate and the foot plate is relieved, and the transit may then be shifted laterally with respect to the foot plate. From the end of the spindle and at the center of curvature of the ball-and-socket joint a chain with hook is suspended, from which the plumb line may be hung. The instrument is mounted on a tripod by screwing the foot plate onto the tripod head.

The telescope is fixed to a transverse horizontal axis which rests in bearings at the upper extremities of the standards. The telescope may be rotated about this horizontal axis and may be fixed in any position in a vertical plane by means of the telescope clamp-screw. Small movements about the horizontal axis may be secured by turning the telescope tangent-screw. Fixed to the horizontal axis is the vertical circle, and attached to one of the standards is the vertical vernier. Beneath the telescope is the telescope level tube.

On the upper plate is placed the compass box, the details of which are the same as for the surveyor's compass described in Art. 184*b*. If the compass circle is fixed, its N and S points are in the same vertical plane with the line of sight of the telescope. The compass boxes of some transits are so designed that the compass circle may be rotated with respect to the upper plate, so that the magnetic declination may be laid off and true bearings may be read. At the side of the compass box is a screw, or needle lifter, by means of which the magnetic needle may be lifted from its pivot and clamped.

Summing up the several features: (1) the center of the transit may be brought over a given point by loosening the leveling screws and

shifting the transit laterally; (2) the instrument may be leveled by means of the plate levels and the leveling screws; (3) the telescope may be rotated about either the horizontal or the vertical axis; (4) when the upper clamp-screw is tightened and the telescope is rotated about the vertical axis, there is no relative movement between the verniers and the horizontal circle; (5) when the lower clamp-screw is tightened and the upper one is loose, a rotation of the telescope about the vertical axis causes the vernier plate to revolve but leaves the horizontal circle fixed in position; (6) when both upper and lower clamps are tightened the telescope cannot be rotated about the vertical axis; (7) the telescope may be rotated about the horizontal axis and may be fixed in any direction in a vertical plane by means of the telescope clamp- and tangent-screws; (8) the telescope may be leveled by means of the telescope level tube, and hence the transit may be employed as an instrument for direct leveling; (9) by means of the vertical circle and vernier, vertical angles may be determined and hence the transit is suitable for trigonometric leveling; (10) by means of the compass, magnetic bearings may be determined; and (11) by means of the horizontal circle and vernier, horizontal angles may be measured.

189a. There are several modifications of the instrument just described. A transit without vertical circle and telescope level tube is called a *plain transit.* One without compass and having U-shaped, one-piece standards, but otherwise the same as that illustrated in Fig. 189*a* is often called a *city transit.*

Another type employs three leveling screws, two opposite vertical verniers which are movable, a striding level, and a telescope tangent-screw with gradienter. The vertical verniers are attached to the casting forming the vertical circle guard which is so mounted that it may be rotated about the horizontal axis. Attached to the guard is the vertical vernier level. When its bubble is centered, a line through the zeros of the two verniers is horizontal. An arm of the casting projecting downwards bears against the vertical vernier tangent-screw. The vertical vernier bubble may be centered by turning the tangent-screw. When it is centered, the vertical vernier readings give correct vertical angles, regardless of whether or not the plates are leveled. The movable vertical vernier with control bubble, as just described, is a feature of considerable value in topographic surveying where a large number of vertical angles is taken from a single set-up.

The striding level is considerably more sensitive than the plate levels, and is especially useful when horizontal angles are measured between points having a large difference in elevation. At each end of the striding-level tube are wyes which rest on the horizontal axis. When the bubble is centered by means of the foot screws, the horizontal axis is truly horizontal and hence the line of sight (if in adjustment) will revolve in a vertical plane.

Grades are laid off by first leveling the telescope and then turning the gradienter screw through the required number of divisions. For the use of the gradienter see Art. 143.

The name *repeating theodolite* is often given to instruments of the general type of the one just described, but designed for surveying of high precision (Art. 561*a*, p. 837). Such instruments are generally larger and heavier in construction and have circles more finely graduated and levels more sensitive than has the ordinary transit. Usually the compass is absent, and permanently attached magnifying glasses are provided for reading the verniers.

The *mining transit* is similar to the engineer's transit except that an auxiliary telescope is attached either to one end of the horizontal axis or to the top of the main telescope. The use of the auxiliary telescope is described in Art. 393, p. 589. The vertical circles of many mining transits are graduated on the edge rather than on the side. Other modifications are a vertical arc of 180° taking the place of the full vertical circle, and the reversion telescope level, making it possible to level the telescope with the tube above, as well as in its normal position below, the telescope.

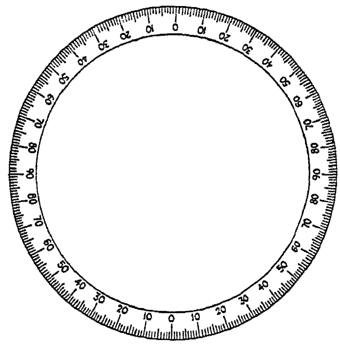
190. Level Tubes.—The sensitiveness of the several spirit levels of the transit should be such as to produce a well-balanced instrument, and hence should correspond to the fineness of graduation of the circles and the optical properties of the telescope. If the levels are more sensitive than necessary to maintain this balance, time is wasted in centering the bubbles; if less sensitive than necessary, the precision of measurements is less than it should be for the transit as otherwise designed. The plate levels of the ordinary transit reading to 01′ usually are alike in sensitiveness and have a value of about 60″ per graduation.

When horizontal angles are measured between points nearly in the same horizontal plane, it can be shown that no appreciable error is introduced even if the bubbles are some distance off center. On the other hand where there is a large difference in vertical angle between the points sighted, a small displacement of the bubble in the tube parallel with the horizontal axis causes a relatively large error in the horizontal angle. For some transits this level tube is more sensitive than the one at right angles to the horizontal axis, but instruments designed for high-grade work are often equipped with a 20″ or 30″ striding level which is employed for leveling the horizontal axis when sights are sharply inclined.

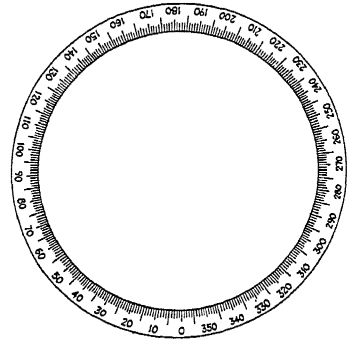
The telescope bubble has a sensitiveness of 20″ to 30″ depending upon the magnifying power of the telescope. The sensitiveness of the vertical vernier control bubble should depend upon the least reading of the vernier; for a vertical circle reading to 01′, a 30″ or

40″ level tube is commonly employed. For further details concerning level tubes see Art. 103, p. 121.

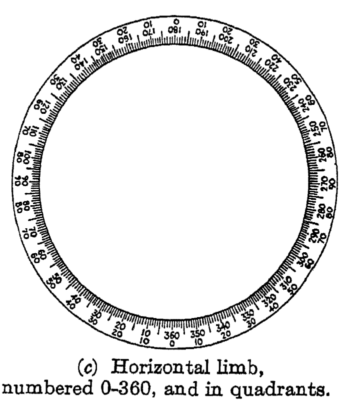
191. Telescope.—The telescope of the transit is similar to that of the engineer's level (Arts. 104 to 104*g*). When the transit is used as an instrument for direct or trigonometric leveling, the line of sight is defined by the optical center of the objective lens and a point on the



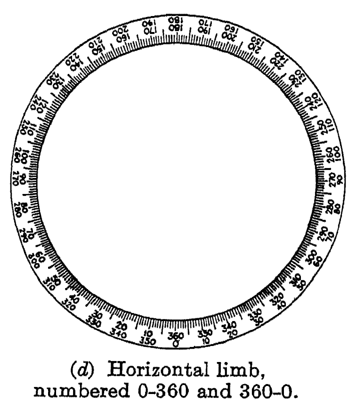
(*a*) Vertical circle,
numbered in quadrants.



(*b*) Horizontal limb,
numbered 0-360.



(*c*) Horizontal limb,
numbered 0-360, and in quadrants.



(*d*) Horizontal limb,
numbered 0-360 and 360-0.

FIG. 192*a*–*d*.—Numbering of circles.

horizontal cross-hair; when the transit is used for establishing lines, measuring angles, or taking bearings, the line of sight is defined by the optical center of the objective lens and a point on the vertical cross-hair. Most instruments of recent manufacture are equipped with stadia hairs (see Chap. XIV), which are usually mounted in the same plane with the cross-hairs. The magnifying power ranges from 18 for small instruments to 30 for larger ones designed for precise work. For the transit, as for the level, the erecting eyepiece

is generally employed, but the superior optical properties of the inverting eyepiece make it the favorite of some surveyors, and it is the type used in instruments of precision. For the relative merits of the inverting and erecting eyepieces, see Art. 104*f*.

192. Graduated Circles.—The vertical circle has two opposite zero points and is graduated usually in $\frac{1}{2}°$, the numbers increasing to 90° in either direction from the zero points, as illustrated in Fig. 192*a*. When the telescope is level, the index of the vernier is at 0°.

The horizontal circle is likewise usually graduated in $\frac{1}{2}°$ but may be numbered from 0° to 360° clockwise (Fig. 192*b*), 0° to 360° clockwise and 0° to 90° in quadrants (Fig. 192*c*), or 0° to 360° in each direction (Fig. 192*d*). Many surveyors prefer the numbering system illustrated in Fig. 192*d*.

The horizontal circles of transits designed for work of moderately high precision are graduated to 20′ or to 15′. Those for repeating theodolites are often graduated to 10′.

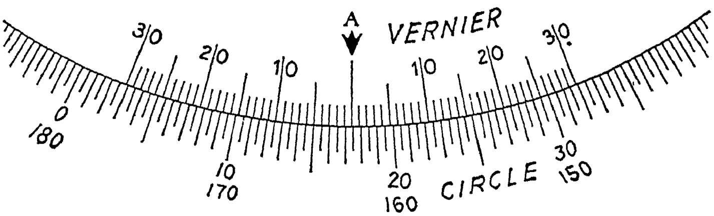


Fig. 193*a*.—Double direct vernier reading to minutes.

193. Verniers.—The verniers employed for reading the horizontal and vertical circles are identical in principle with those for the target rod described in Art. 109*d*. Practically all transit verniers of recent manufacture are of the direct type.

Figure 193*a* shows the usual type of double direct vernier. The vernier on the left of the index is for reading clockwise angles and that on the right is for reading counter-clockwise angles. Thirty spaces on the vernier are equal to 29 spaces on the circle and a single space on the circle is equal to 30′. The difference between the length of a space on the vernier and one on the circle is therefore $\frac{1}{30} \times 30 =$ 01′, which is the least reading or least count of the vernier. Considering clockwise angles, the vernier index is seen to lie between 162°30′ and 163°00′. The number of minutes greater than 162°30′, determined by observing the graduation on the left vernier that coincides with one on the circle, is seen to be 05′. Hence the angle is 162°30′ + 05′ = 162°35′. Similarly reading the right vernier, the counter-clockwise angle is 17°00′ + 25′ = 17°25′.

Figure 193*b* illustrates a double direct vernier for which 40 spaces are equal to 39 on the circle and one space on the circle is 20′. The least count is therefore $\frac{1}{40} \times 20' = 30''$. Reading clockwise, the angle is $49°40' + 10'30'' = 49°50'30''$. Reading counter-clockwise the angle is $130°00' + 09'30'' = 130°09'30''$.

Figure 193*c* represents a folded vernier reading to 20″. The full length of the vernier is employed for reading angles in either direction.

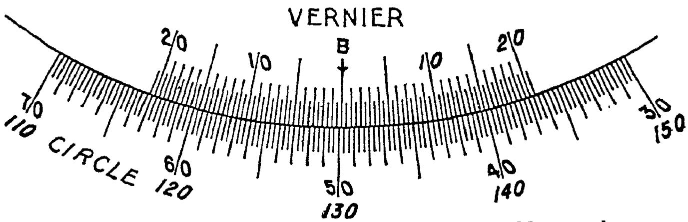


FIG. 193*b*.—Double direct vernier reading to 30 seconds.

Sixty spaces on the vernier are equal to 59 on the circle, and the circle is graduated to 20′. The vernier is read from the index towards either of the extreme divisions and then from the other extreme division in the same direction to the center. The index of the vernier and its 20′ mark are the same. Reading clockwise, the angle is $30°00' + 02'40'' = 30°02'40''$. Reading counter-clockwise the angle is $149°40' + 17'20'' = 149°57'20''$. The folded vernier is employed

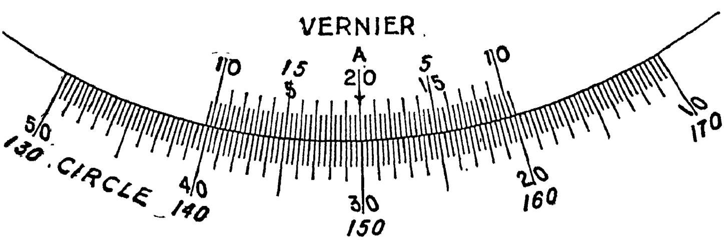


FIG. 193*c*.—Folded vernier reading to 20 sec.

where the length of the corresponding double vernier would be so great as to make it impracticable.

Figure 193*d* represents a single vernier adapted to circles numbered clockwise from 0° to 360°. Sixty spaces on the vernier are equal to 59 on the circle and one space on the circle is equal to 10′. Hence, the least count of the vernier is $\frac{1}{60} \times 10' = 10''$. In the illustration, the vernier reads 355°00′00″. This type of vernier and circle graduation is employed on some repeating theodolites.

A double vernier reading to decimals of a degree is sometimes used. Fifty spaces on the vernier are equal to 49 on the circle and one space

on the circle is equal to $\frac{1}{4}°$; hence the least count of the vernier is $\frac{1}{50} \times \frac{1}{4} = \frac{1}{200}° = 0.005°$. This and similar decimal verniers are designed to eliminate the necessity of transposing degrees, minutes, and seconds in trigonometric calculations. Obviously the decimal division is the simpler, but custom has thoroughly established the "minute-second" division and it is likely to continue in general use for many years.

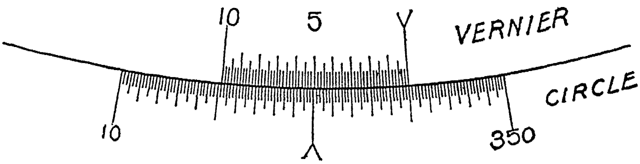


FIG. 193*d*.—Single vernier reading to 10 sec.

193a. All transits have two verniers for reading the horizontal circle, their indexes being 180° apart. The one nearest the upper clamp and tangent-screw is known as the *A* vernier, and the opposite one is known as the *B* vernier. The verniers are attached to the upper plate and are adjusted by the instrument maker so that they are much nearer to being truly 180° apart than their least count. Failure of the verniers to register identical readings, *i.e.*, not exactly 180° apart, may be due to either or both of two causes: (1) the verniers may have become displaced so that a line joining their indexes does not pass through the center of rotation of the upper plate, or (2) the spindles may have become worn or otherwise damaged so that the center of rotation of the upper plate does not coincide with the geometrical center of the graduated horizontal circle. The former condition is known as "eccentricity of verniers" and the latter as "eccentricity of centers."

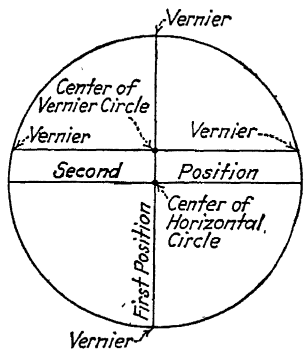


FIG. 193*e*.—Eccentricity of centers.

To correct either of these defects requires the services of an instrument maker, but neither defect limits the precision with which angles may be determined. If the verniers only are eccentric, it is clear that they will lack the same amount of being 180° apart for all parts of the graduated circle. If the centers only are eccentric there will be one setting on the graduated circle for which the indices are 180° apart (first position, Fig. 193*e*), and 90° therefrom another setting for which the verniers fail to register 180° apart by a maximum

amount (second position, Fig. 193*e*). Obviously the mean of the two vernier readings eliminates errors due to eccentricity whether it be due to verniers, to centers, or to both. Furthermore if the verniers only are eccentric no error is introduced in an angle so long as the same vernier is used for making the initial setting as for making the final reading.

USE OF THE TRANSIT

194. General.—The succeeding articles describe the elementary processes employed in running lines and in measuring horizontal and vertical angles by means of the graduated circles and verniers of the transit. Transit surveys are considered in detail in the following chapter. The process of taking magnetic bearings with the transit is the same as with the surveyor's compass. The transit may be employed for running direct levels in the same manner as when using the engineer's level, the telescope bubble being centered each time a rod reading is taken.

The operation of rotating the telescope about the horizontal axis is called "plunging the telescope." When the telescope level tube is below, the telescope is said to be in the *normal* or *direct* position; when the level tube is above, the telescope is said to be in the *reversed* or *inverted* position.

Signals generally applicable to transit work are given in Art. 23. Suggestions for the care of the transit are given in Art. 24.

195. Setting Up the Transit.—Ordinarily the transit is set over a definite point, such as a tack. For centering the transit, a plumb line is suspended from the hook and chain beneath the instrument. To set up the transit, place it approximately over the point and adjust the legs until the tripod head is nearly level. Pick up the instrument bodily without disturbing the position of the legs, set it carefully over the point, and press each shoe firmly into the ground, at the same time adjusting each leg until the plumb bob falls within ½ in. or less of the tack and the instrument is nearly level. Loosen two adjacent leveling screws and shift the instrument laterally until the plumb bob is over the tack. Tighten the screws to a firm, but not tight, bearing. Level the instrument by means of the leveling screws and the plate bubbles, first bringing each level tube parallel with a pair of leveling screws.

It is useless to center one bubble carefully until the other has been brought approximately to the level position. In the absence of any device for adjusting the height of the plumb bob, the string should be tied in a sliding bowknot. The bob should be adjusted until it just clears the tack. The operation of expeditiouslv setting up and leveling

the transit requires on the part of the instrumentman a skill that is acquired only with practice.

196. Measuring a Horizontal Angle.—If any angle, as AOB, is to be measured, the transit is set up over O as described in the preceding article. The upper motion is clamped, and by means of the upper tangent-screw one of the horizontal verniers is set at 0°. The telescope is sighted to A, the lower motion is clamped, and by means of the lower tangent-screw the line of sight is set on a range pole or other object marking the point. The upper clamp is loosened and the telescope is turned until the line of sight cuts B. The upper clamp is tightened, and the line of sight is set exactly on B by turning the upper tangent-screw. The reading of the vernier which was initially set at 0° gives the value of the angle. It is convenient to consider the lower motion of the transit as a protractor, and the upper motion as a straightedge.

Following is a list of suggestions bearing on transit work in general:

1. Make reasonably close settings by hand so that the tangent-screws will not need to be turned through more than one or two revolutions.

2. Make the last direction of movement of the tangent-screw in a clockwise direction. This compresses the opposing spring and tends to eliminate lost motion in the instrument.

3. When reading the vernier, have the eye directly over the coinciding graduation, to avoid parallax. It is also helpful to observe that the graduations on both sides of those coinciding fail to concur by the same amount.

4. The plate bubbles should be centered before measuring an angle, but between initial and final settings of the line of sight the leveling screws should not be disturbed.

5. Test the telescope for parallax by moving the head slightly from side to side, meanwhile observing the magnified image. The eyepiece slide should be focused until the cross-hairs are sharp in outline, and the objective should be moved until there is no apparent movement of the cross-hairs over the image.

6. Do not allow the hand to rest on the instrument nor the clothing to brush against it while a sight is being taken. Do not step near the feet of the tripod. The effect of violating these suggestions may be readily seen by looking through the telescope.

7. The flagman should stand directly behind the range pole, holding it lightly with the fingers of both hands, and balancing it on the tack or other mark indicating the point.

8. In sighting at a range pole the bottom of which is not visible particular care should be taken to see that it is held vertically. When the view is obstructed for a considerable distance above the point to which the sight is taken, use a plumb line behind which a white card is

held. For short sights a pencil held on the point makes a satisfactory target.

9. When a number of angles are to be observed from one point without moving the horizontal circle, the instrumentman should sight at some clearly defined object which will serve as a reference mark and should observe the angle. If occasionally the angle to the reference mark is again read any accidental movement of the horizontal circle will be detected.

10. An object may be brought into the telescopic field of view more readily if it is ranged approximately by sighting along the tube of the telescope. For this purpose some transits have peep sights on top of their telescope tubes. Some observers prefer to sight with both eyes open.

197. Common Mistakes.—In measuring horizontal angles, mistakes often made are:

1. Turning wrong tangent-screw.
2. Failing to tighten clamp.
3. Confusing numbers on the horizontal scale, as reading from the outer row when the angle turned is indicated by numbers on the inner row.
4. Reading angles in the wrong direction.
5. Dropping 30′ or 20′ by failure to take the full scale reading before reading the vernier, as for example with a circle graduated to 30′ calling the angle 21°14′ when it is actually 21°44′, the vernier reading being 14′.
6. Reading the vernier in the wrong direction.
7. Reading the wrong vernier.

198. Measuring a Vertical Angle.—The vertical angle to a point is its angle of elevation (+) or depression (−) from the horizontal. The transit is set up and leveled as when measuring horizontal angles. (For a transit having a fixed vertical vernier, greater than ordinary care should be taken in centering the plate bubbles.) The telescope is sighted approximately at the point and the horizontal axis is clamped. The horizontal cross-hair is set exactly on the point by means of the telescope tangent-screw, and the angle is read by means of the vertical vernier.

For a transit having a movable vertical vernier with control level, the telescope is sighted on the point as described above, the vernier control bubble is centered, and the angle is read.

In ordinary trigonometric leveling, vertical angles are usually taken by sighting at a leveling rod, the horizontal cross-hair being set at a rod reading equal to the height of the horizontal axis above the station over which the transit is set up. In precise trigonometric leveling, the distance between stations is usually great and vertical

angles are measured with a theodolite by sighting at points defined by signals erected at the distant stations.

If the line of sight is out of adjustment (is not truly horizontal when the vertical vernier reads zero), an error in vertical angle results. This error may be eliminated by double-sighting, once with the telescope direct and once with the telescope reversed. This error is rendered negligible for ordinary work by careful adjustment of the instrument (Art. 209, adjustment 6).

198a. Index Error.—Index error is the error in an observed vertical angle due either to displacement (lack of adjustment) of the vertical vernier or (for a transit having a fixed vertical vernier) to inclination of the upper plate.

Assuming that the axis of the telescope level is parallel to the line of sight, the true vertical angle is determined by applying to the observed angle an *index correction* equal in amount but opposite in sign to the *index error*. Thus, if the observed angle is $+12°14'$ and if the index error is determined to be $+02'$, the true value of the vertical angle is $+12°14' - 02' = +12°12'$.

Transit Having a Fixed Vertical Vernier.—Displacement of the vertical vernier introduces a constant index error which can be determined by carefully leveling the upper plate, leveling the telescope, and reading the vertical vernier. This error may be eliminated by adjustment (Art. 209, adjustment 7) or by double-sighting (Art. 198*b*).

For this type of transit, any inclination of the upper plate due to erroneous leveling of the instrument introduces an index error which varies with the direction in which the telescope is pointed, and which is equal in amount to the angle through which the fixed vertical vernier is displaced about the horizontal axis. If the plate levels are in good adjustment, this index error may be rendered negligible by careful leveling of the plate before each observation. The index error due to inclination of the plate is not eliminated by double-sighting.

The total index error due both to displacement of the vertical vernier and to inclination of the plate can be determined by leveling the telescope and reading the vertical vernier after each observation.

When a series of horizontal and vertical angles is to be measured from a given station, recentering the plate bubbles necessitates taking a new backsight (involving additional work) before additional horizontal angles can be measured, yet the plate bubbles may be considerably displaced without appreciably affecting the accuracy of *horizontal* angles. Often it involves less work to make the index correction for vertical angles than to relevel the instrument each time the plate bubbles are seen to be displaced.

Transit Having a Movable Vertical Vernier with Control Level.—Displacement of the vertical vernier introduces a constant index error which can be determined by leveling both the telescope level tube and the vernier level tube, and reading the vertical vernier. If the instrument is in good adjustment (Art. 209, adjustment 7*a*), this error is negligible. It may be eliminated by double-sighting.

For this type of transit, any moderate inclination of the upper plate does not introduce an appreciable error in vertical angles, provided the instrument is in adjustment and provided the vernier control bubble is centered each time an observation is made. In topographic surveys or similar work, the use of the movable vertical vernier with control level results in a considerable saving of time as compared with that required when the instrument is equipped with the fixed vertical vernier.

198b. Double-sighting.—*If the instrument has been carefully leveled*, the true vertical angle can be determined with a transit having a full vertical circle by reading once with the telescope direct and once with the telescope reversed, taking the mean of the two values thus obtained. This method of double-sighting is used, for example, in astronomical observations and in similar measurements of vertical angles to distant objects (see Art. 304, p. 452).

In traversing, a similar result is obtained by measuring the vertical angle of each traverse course from each end, with the telescope the *same* side up for the two observations, and taking the mean of the two values thus obtained.

These methods of averaging two readings eliminate error in vertical angle due either to displacement of the vertical vernier or to lack of parallelism between the line of sight and the axis of the level tube; but in the case of a transit having a fixed vertical vernier they do not eliminate the error due to the plate's not being level.

199. Prolonging a Straight Line.—If any straight line as AB (Fig. 199*a*) is to be prolonged to P (not already defined upon the ground), which is beyond the limit of sighting distance or is invisible from A and B, the line is extended by establishing a succession of stations, C, D, etc., each of which stations is occupied by the transit. Any of the following three methods may be employed but the second method is usually the most convenient.

Method 1.—The transit is set up at A, a sight is taken to B, and the point C is established on line. The transit is moved to B, a sight is taken to C, and point D is set on line. Thus the process is continued until point P is set.

Method 2.—The transit is set up at B and a backsight is taken to A. With both upper and lower motions clamped, the telescope is

plunged and C is set on line. If the line of sight is perpendicular to the horizontal axis, as it will be if in perfect adjustment, it will generate a vertical plane as the telescope is revolved, and the point C will lie on the prolongation of AB. The transit is then moved to C, a backsight is taken to B, the telescope is plunged and D is established; and thus the process is repeated until P is set.

If the line of sight is not perpendicular to the horizontal axis, as the telescope is plunged, say from the reversed to the direct position, the line of sight will generate a portion of a cone whose vertex is at the center of the instrument and two of whose elements are AB and BC',

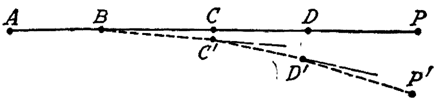


FIG. 199*a*.—Prolonging a straight line.

and C' will not lie on the prolongation of AB. If the instrument is set up at C', a backsight taken to B with the telescope in the reversed position as before, and it is plunged to the direct position, a second and similar cone is generated and D' will not lie on the prolongation of BC'. Thus, if the line is extended by the method outlined and all backsights are taken with the telescope in one position (either direct or reversed), the points established will lie along a curve instead of a straight line and each segment of the line will be deflected in the same direction (to the right or to the left) by double the error of adjustment of the line of sight. On the other hand, if say, at the even-numbered stations B, D, F, etc., backsights were taken with the telescope reversed and

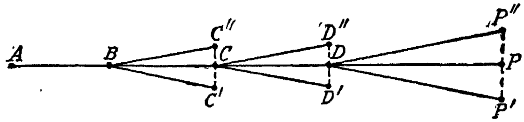


FIG. 199*b*.—Double-sighting.

at odd-numbered stations, C, E, etc. backsights were taken with the telescope direct, a zigzag line would be established, some of whose points would lie on one side of the line joining the terminals and some perhaps on the other. By the first procedure, the angular error becomes systematic in character, and by the second procedure it becomes accidental. Generally where only a few set-ups are required and the instrument is known to be in reasonably good adjustment, no particular attention is paid to the procedure to be followed, but where there is a large number of set-ups and the line is long, the latter procedure is employed.

Method 3.—This is known as the method of "double-sighting," and is employed when the instrument is in poor adjustment or when

it is desired to establish the line with high precision. If the line AB (Fig. 199*b*) is to be prolonged to some point P, the transit is set up at B and a backsight is taken to A with the telescope in, say, its *direct* position. The telescope is plunged and a point C' is set on line. The transit is then revolved about its *vertical* axis and a second backsight is taken to A with the telescope in its *reversed* position. The telescope is plunged and a point C'' is established on line beside C'. It is evident that C' will be as far on one side of the true prolongation of AB as C'' is on the other. Hence C, a point on the prolongation of AB, is set midway between the points C' and C''. In a similar manner D is established by setting up at C, double-sighting to B and setting points at D' and D''. Thus the process is repeated until the desired distance is traversed.

200. Laying Off a Horizontal Angle.—If an angle AOB is to be laid off from the line OA, the transit is set up at O, the vernier is set at 0° and the line of sight is set on A. The upper clamp is loosened, and the vernier plate is turned until the index of the vernier is approximately at the required angle. The upper clamp is tightened, and the vernier is set at the given angle by means of the upper tangent-screw. The point B is established on the line of sight.

201. Intersection of Lines.—The point of intersection of two lines as AB and CD (Fig. 201) is established as follows: By methods

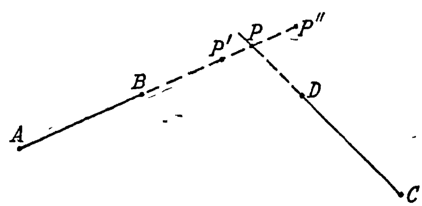


Fig. 201.—Intersection of lines.

described in Art. 199, one of the lines, AB, is prolonged and points P' and P'' are established a short distance on either side of the estimated position of the prolongation of CD. A string is stretched between P' and P''. The line CD is prolonged until it intersects the string at P. A point set at P marks the intersection of the two lines.

202. Measuring an Angle When the Transit Can Not Be Set at the Vertex.—Figure 202 illustrates a typical case where it is required to determine the angle between walls of a building. The same situation occurs when the angle between fence lines is to be measured.

The point a is established at any convenient distance from the wall. The perpendicular distance from the wall is determined by

holding the tape on point *a* and swinging the end of the tape through an arc, varying the radius until the arc becomes tangent to the wall (*i.e.*, by a swing offset). By trial the point *b* is established in such position that when it is used as a center with the same radius as at *a*, the arc will again become tangent to the wall; then line *ab* is parallel to the wall. In a similar manner points *c* and *d* are established.

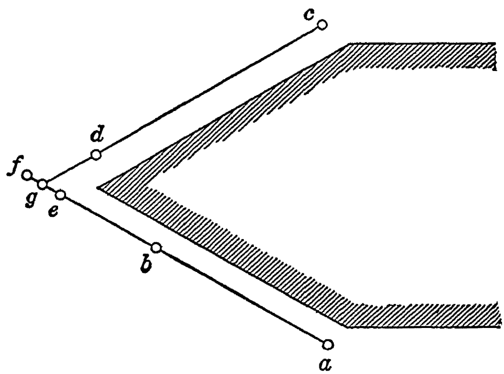


FIG. 202.—Angle of a building.

Then *e* and *f* are set by prolonging the line *ab*; and *g* the point of intersection of the lines *ab* and *cd* is determined as described in the preceding article. The transit is set up at *g* and the angle *agc*, which is equal to the angle between the walls of the building, is measured in the usual manner.

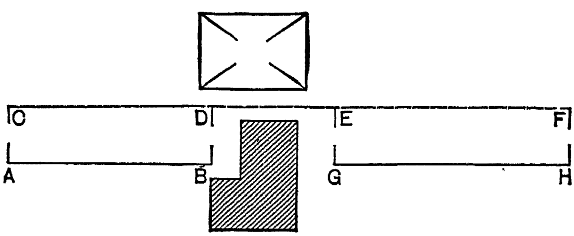


FIG. 203.

203. Prolonging a Line Past an Obstacle.—Figure 203 illustrates one method that may be employed to prolong a line past an obstacle when the offset space is limited, the line *GH* representing the prolongation of *AB*. The transit is set up at *A*, a right angle is turned, and point *C* is established at a convenient distance from *A*. Similarly the point *D* is established, the distance *BD* being made equal to *AC*. The line *CD* is now parallel to *AB*. By the methods of Art. 199 *CD*

is prolonged and points E and F are established in convenient positions beyond the obstacle. From points E and F right-angle offsets are made and G and H are set as were C and D. It is evident, if errors be neglected, that GH marks the prolongation of the line AB, and also that the distance AH is directly determined by measuring the lengths of the lines AB, DE, and GH. If the chainage is to be carried forward with precision it is necessary to erect the perpendiculars AC, BD, etc., with greater than ordinary care, and if the line is to be prolonged accurately it is essential not only that the offset distances be measured carefully but also that AB and EF, the distances between offsets, be relatively large.

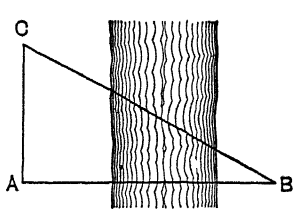


Fig. 204*a*.

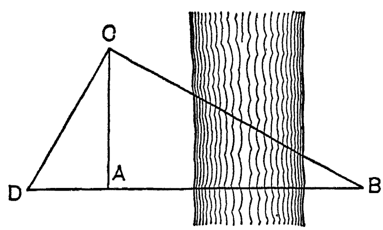


Fig. 204*b*.

204. Determination of an Inaccessible Distance.—This involves triangulation (see Art. 181). Two simple methods are described below:

Method 1.—AB (Fig. 204*a*) represents a line whose length cannot be measured directly, B being visible from A and from other points in the vicinity. The transit is set up at A, a sight is taken to B, an angle of 90° is laid off, and C is set at any convenient point from which B is visible. The length of the line AC is measured. The transit is set up at C, and the angle ACB is observed. Then $AB = AC \tan ACB$.

Method 2.—This method is applicable when trigonometric tables are not available. In Fig. 204*b*, AB represents the line whose length is to be determined. AC is established as in method 1. The transit is set up at C, a sight is taken to B and the direction of CD is fixed by laying off an angle of 90°. The point D is established at the intersection of this line and the prolongation of the line BA, as described in Art. 201. The lengths AC and AD are measured. By geometry $\triangle ABC$ is similar to $\triangle ACD$. Hence $AB = \dfrac{\overline{AC}^2}{AD}$.

For each of the above methods it is desirable that the distance AC be not less than one-half the distance AB, otherwise the errors of

measurement will produce a relatively large error in the computed distance.

205. Running a Straight Line between Two Points.—If the terminal points of a line are fixed and it is desired to establish intervening points on the straight line joining the terminals, the method to be employed depends upon the length of the line and the character of the terrain. Three common cases are considered below:

Case 1. *Terminals Intervisible.*—If the terminals are A and B, the transit is set up at A, a sight is taken to B, and intervening points are established on line.

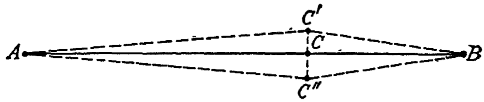


FIG. 205*a*.

If the intervening points thus established were to lie in the same plane with the center of the instrument and the terminal point B, they would define a truly straight line regardless of whether or not the horizontal axis of the transit were truly horizontal. If the horizontal axis is inclined with the horizontal, the line of sight will not generate a vertical plane as the telescope is revolved; and thus if it is necessary to rotate the telescope about the horizontal axis in order to set the intervening points, the points thus established will not lie on a truly straight line (as seen in plan view) joining the terminals.

Ordinarily the vertical angles through which it is necessary to rotate the telescope will be small and if the horizontal axis is in fair adjustment and the plate bubbles are centered, the error arising from this source is negligible. Occasionally, however, when the intervening points are to be set with high precision or when the adjustment of the instrument is uncertain and the vertical angles are large, the intervening stations are set by double-sighting.

Case 2. *Terminals Not Intervisible, but Visible from an Intervening Point.*—By trial the location of the line at the intervening point is determined. In Fig. 205*a*, A and B represent the terminals both of which can be seen from the vicinity of C. The transit is set up on the estimated position of the line, a backsight is taken to A, the telescope is plunged and the position of the line of sight at B is noted. The amount that the transit must be shifted laterally is estimated, and the process is repeated until when the telescope is plunged, the line of sight falls on the point at B, as in Fig. 205*a*. The position of the instrument so found should be tested by the method of double-sighting to eliminate possible effects of inaccurate adjustment of line of sight or horizontal axis.

If the instrument were in perfect adjustment its center would be on the true line joining AB. On the other hand, if the line of sight were not perpendicular to the horizontal axis a cone would be generated by the line of sight when the telescope was plunged, as explained in Art. 199; also if the horizontal axis were not truly horizontal the line of sight in its rotation would not generate a vertical plane, as explained under case 1. Hence, if the transit were not in adjustment, its center might be at C' and still the line of sight would bisect B when the telescope is plunged.

To locate the intermediate point truly on line, trials are made first with the telescope, say, in its direct position for backsights to A, until an intermediate point, as C', is determined. Then a second series of trials is made with the telescope in its reversed position for backsights to A until the corresponding point C'' is located. Then, for reasons previously explained, the true line is at C, halfway between C' and C''. Other intermediate points may then be established by setting up the transit at C and proceeding as in case 1.

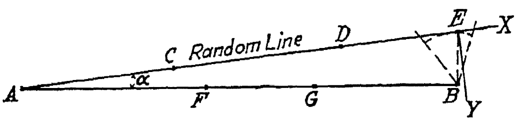


Fig. 205*b*.

Case 3. *Terminals Not Visible from Any Intermediate Point.*—AB (Fig. 205*b*) represents a straight line along which it is desired to establish intermediate points, the character of the ground being such that it is not possible to find any one intermediate position from which both terminals are visible. By one or more of the methods previously explained, a straight line, called a *random line*, is run from A in the estimated direction of B. Let AX be such a random line and let C and D be stations established during the process of extending it. The transit is set up at D and sighted towards X. The tape is swung about B as a center and the offset distance BE from the line DX to the point B is determined by sighting at the tape through the transit and taking the least reading.

The length AE is measured, and the angle which the random line makes with AB is calculated by the expression $\tan \alpha = \frac{BE}{AE}$. The transit is again set up at A, a sight is taken to C, the calculated value of α is laid off, and the line is run towards B, intermediate stations as F and G being established at desired points. With the instrument at G, a backsight is taken to F, the telescope is plunged and the linear offset error at B is noted. If the error is sufficiently large to be of importance, the points at F and G are corrected by linear measurements so as to place them on the true line, the correction being made

proportional to the distance from the point to the terminal A. That is, the offset correction at F is to FA as the error at B is to BA. In many cases, points on the random line, as C and D, can be transferred to the true line by the same method.

If the offset distance is short, compared with the length of the line AB, the degree of approximation is small and the position of E along the line AX may be estimated with sufficient precision. If the offset distance BE is large, the degree of approximation in locating the position of E is fairly large, since the tape may be swung through a considerable arc in the vicinity of the perpendicular without materially changing the offset reading; hence the position of E needs to be determined by a more exact method than simply by estimation. Usually the transit is set up at the estimated position of E, a perpendicular EY is laid off from the line EA, and the distance from this perpendicular EY to B is measured by a swing offset. This offset gives the distance that E must be moved along the line AX to place it on the perpendicular through B.

206. Measuring an Angle by Repetition.—One of the advantages of the transit not possessed by other instruments is that a horizontal angle may be mechanically multiplied and the product can be read with the same degree of precision as the single value. Thus, with the ordinary transit having verniers reading to single minutes, an angle for which the true value is between the limits $30°00'30''$ and $30°01'30''$ will be read as $30°01'$, and the limits of possible error will be $\pm 30''$. If the true angle is multiplied six times on the horizontal circle, the product, likewise read to the nearest minute, might be $180°04'$, its true value being within the limits $180°03'30''$ and $180°04'30''$, and the limits of possible error, so far as reading the vernier is concerned, being likewise $\pm 30''$. Dividing the observed product $180°04'$ by six, the single value becomes $30°00'40''$ for which the limits of possible "reading" error are $\pm 30'' \div 6 = \pm 05''$. This method of determining an angle is called "*measurement by repetition.*" While it might appear from what has been said that the accuracy with which an angle can be measured by this method varies directly with the number of times the angle is multiplied or repeated, usually the precision is not appreciably increased by more than six or eight repetitions on account of lost motion in the instrument and on account of errors due to setting the line of sight and to other causes.

To repeat an angle, as AOB, the transit is set up at O, the vernier is set at 0°, a sight is taken to A, and the lower motion is clamped. The upper clamp is loosened, a sight is taken to B, the upper clamp is tightened, and the single value of the angle is read. The vernier setting is left unaltered, the instrument is turned on its lower motion,

and a second sight is taken to A. The upper clamp is loosened, and the telescope is again sighted to B. The angle has now been doubled. In this way the process is continued until the angle has been multiplied the required number of times. The vernier is read, and the value of the angle is determined by dividing this reading by the number of times the angle was turned. To obviate mistakes, this value is compared with the angle observed at the completion of the first turn.

The exact procedure to be employed in measuring an angle by repetition depends somewhat upon the desired precision. When the method is employed primarily as a check, the angle is doubled usually without reversing the telescope between repetitions.

When it is desired to increase the precision a moderate amount, usually the angle is multiplied four or six times, half of the observations being made with the telescope in the direct position and half with it in the reversed position, and both verniers are read. Certain instrumental errors, such as those due to eccentricity and to non-adjustment of the horizontal axis, are eliminated in this manner.

When a high degree of precision is necessary, several sets of perhaps six repetitions are taken with the telescope in the direct position, and these sets are duplicated by others taken with the telescope in the reversed position. To eliminate errors of graduation, settings are so made that readings are distributed over various parts of the circle and verniers, and to eliminate eccentricity, both verniers are read. Furthermore, special care is taken so to manipulate the instrument that systematic errors due to lost motion in the clamps and to other causes will be eliminated.

If precise results are to be obtained the instrument must be manipulated with care. The plate bubbles should be kept centered, but the leveling screws should not be disturbed except between repetitions. When turning on the lower motion, the hands should be in contact with the lower plate, and when turning on the upper motion, they should be in contact with the upper plate, and not the telescope. The last motion of the tangent-screws should be clockwise or against the opposing spring. To eliminate the effect of twist in the tripod, after each repetition the instrument should be rotated on its lower motion in the same direction that it was turned on its upper motion. That is, the direction of movement is always either clockwise or counterclockwise. Due to the possibilities of unequal settlement of the tripod and of unequal expansion of the parts of the telescope, it is desirable that the observations be made as rapidly as is consistent with careful work. The instrument, so far as possible, should be protected from sun and wind.

207. Laying Off an Angle by Repetition.—If it is desired to establish an angle with a precision greater than that possible by a single obser-

vation, the methods of the preceding article may be employed in the following manner: In Fig. 207, OA represents a fixed line and AOB the angle which is to be laid off to establish the line OB. The transit is set up at O, the vernier is set at 0°, and a sight is taken to A. The vernier is set as closely as possible to the given angle and a point B' is established with the line of sight in its new position. AOB' is then repeated a sufficient number of times to fix the error within prescribed limits, and a measured value of the angle is determined by dividing the total angle turned by the number of repetitions. The length of OB' is measured.

Let AOB represent the required angle and AOB' the measured value of the angle which was laid off. Then $B'OB = AOB - AOB'$ represents the correction which must be applied to the measured angle. The correction, which is too small to be laid off by angular measurement, is applied by offsetting the distance $B'B = OB'$ tan

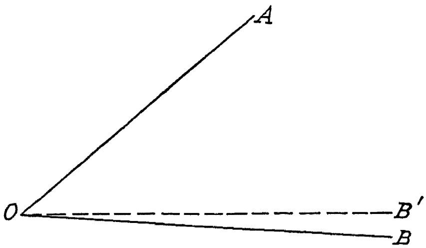


FIG. 207.

(or sin) $B'OB$, thus establishing the point B. Finally, as a check on the work, AOB is measured by repetition. It is convenient to remember that the tangent or sine of $1' = 0.0003$ (very nearly).

As an example, suppose that an angle of 30°00′ correct to the nearest 05″ were to be laid off and that the transit to be employed read to the nearest 01′. Let the total value of AOB' after six repetitions be 180°02′, correct, say, to the nearest 30″. Then the measured value of AOB' is $180°02' \div 6 = 30°00'20''$ correct to the nearest 05″ and the correction to be applied to AOB' is 20″. Suppose that $OB' = 400$ ft. Then the length of the offset $B'B$ equals tan $20'' \times 400$ ft. $= 0.0001 \times 400 = 0.04$ ft.

ADJUSTMENT OF THE TRANSIT

208. Desired Relations.—Most of what was said in Art. 115 concerning adjustments of the level applies equally well to those of the transit.

For a transit in perfect adjustment the following relations should exist:

1. The axis of each plate level should lie in a plane perpendicular to the vertical axis, so that when the instrument is leveled the vertical axis will be truly vertical and horizontal angles may be measured in a horizontal plane.

2. The optical axis, the axis of the objective slide, and the line of sight should coincide for any position of the objective, and each should be perpendicular to the horizontal axis at its mid-point. In this way when the telescope is rotated about the horizontal axis the line of sight will generate a plane regardless of whether the objective is focused for a near sight or for a far sight, and the plane will pass through the center of the instrument.

3. The vertical cross-hair should lie in a plane perpendicular to the horizontal axis so that any point on the hair may be employed when measuring horizontal angles or when running lines.

4. The horizontal axis should be perpendicular to the vertical axis so that when the telescope is plunged the line of sight will generate a vertical plane.

5. The axis of the telescope level should be parallel to the line of sight so that the transit may be employed in direct leveling.

6. If the transit has a fixed vernier for the vertical circle, the vernier should read zero when the plate bubbles and telescope bubble are centered, in order that vertical angles may be measured; or if the vernier is movable and has a control bubble, the axis of the control level should be parallel to that of the telescope level when the vernier reads zero.

7. If the transit has a striding level, its axis should be parallel to the horizontal axis. Thus when the striding level is centered and the telescope is plunged the line of sight (if in adjustment) will generate a vertical plane.

209. Adjustments.—In the description of the following adjustments it is assumed that the objective slide does not admit of adjustment, but that it is permanently fixed in the telescope tube so far as lateral motion is concerned; and that the maker has so constructed the instrument that the optical axis and the axis of the objective slide coincide and are perpendicular to the horizontal axis. This ideal construction is never exactly attained; but in most modern instruments the departure is so slight that it need not be considered in ordinary transit work, and in precise surveying the resulting errors are eliminated by methods of procedure.

For transits with adjustable objective slides, the adjustments are the same as for those with permanently fixed objective slides, except as noted in Art. 209*a*.

1. *To Make the Vertical Cross-hair Lie in a Plane Perpendicular to the Horizontal Axis.*

Test.—Sight at a well-defined point. Swing the telescope through a small vertical angle, so that the point traverses the length of the

vertical cross-hair. If the point appears to move continuously on the hair, the cross-hair lies in a plane perpendicular to the horizontal axis (see Fig. 209*a*).

Correction.—If the point does not appear to move always on the cross-hair, loosen two adjacent screws and turn the cross-hair ring

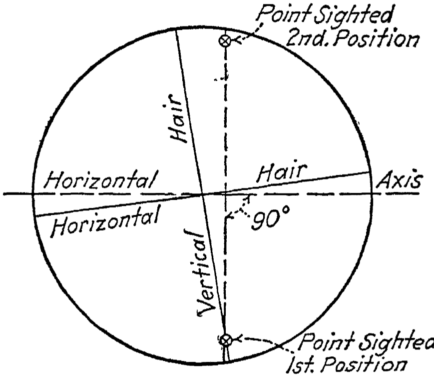


Fig. 209*a*.—Adjustment of vertical cross-hair.

in the telescope tube until the point traverses the entire length of the hair. Tighten the *same* two screws.

2. *To Make the Axes of the Plate Levels Lie in a Plane Perpendicular to the Vertical Axis.*

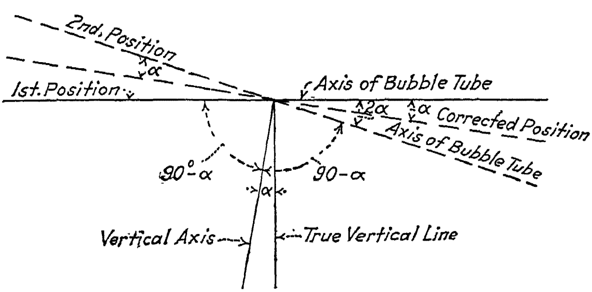


Fig. 209*b*.—Adjustment of plate levels.

Test.—Rotate the transit about the vertical axis until each level tube is parallel to a pair of opposite foot screws. Bring the bubbles to the center of the tubes by means of the foot screws. Rotate the transit 180° in azimuth. If the bubbles remain centered, the axes of the level tubes are in a plane perpendicular to the vertical axis (see Fig. 209*b*).

Correction.—If the bubbles become displaced, bring them *halfway* back by means of the adjusting screws. Level the instrument again and repeat the test to verify the results.

3. *To Make the Line of Sight Perpendicular to the Horizontal Axis.*

Test.—Set up and level the instrument. Sight on a point A (see Fig. 209c) about 500 ft. away (with telescope in direct position), plunge the telescope, and set another point B on the line of sight and about the same distance away on the opposite side of the transit. Turn the instrument in azimuth, and again sight at A (with telescope in reversed position). Plunge the telescope; if B is on the line of sight, the desired relation exists.

Correction.—If B is not on line, set a point C beside it on the line of sight. Mark a point D, *one quarter* of the distance from C to B,

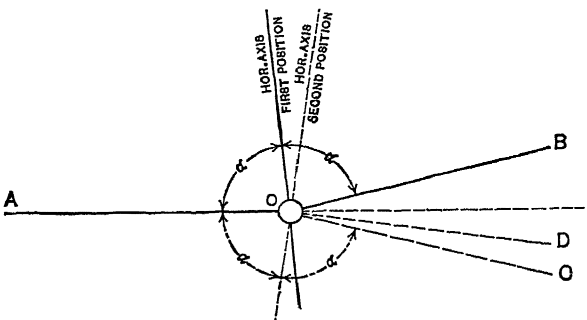


FIG. 209c.—Adjustment of line of sight.

and adjust the cross-hair ring by means of the two opposite horizontal screws, until the line of sight passes through D.

4. *To Make the Horizontal Axis Perpendicular to the Vertical Axis.*

Test.—Set the transit near a building, flagstaff, or other object upon which is some well-defined point at a considerable angle of elevation. Level the instrument very carefully, thus making the vertical axis truly vertical. Sight at the high point A (see Fig. 209d); and with the horizontal motions clamped depress the telescope and set a point B on the ground. Plunge the telescope, rotate in azimuth, and again sight on A. Depress the telescope as before; if the line of sight falls on B, the horizontal axis is perpendicular to the vertical axis.

Correction.—If the desired relation does not exist, set C on the line of sight beside B. A point D *one half* of the distance from B to C will lie in the same vertical plane with the high point. Loosen the screws of the bearing cap, and raise or lower the adjustable end of the horizontal axis until when the telescope is swung in a vertical arc the line

of sight passes through both *A* and *D*. *The high end of the axis is always on the same side of the vertical plane through the high point as the point last set.*

In readjusting the bearing cap, care should be taken not to bind the horizontal axis, but it should not be left so loose as to allow the objective end of the telescope to drop of its own weight when not clamped.

The end of the horizontal axis is thus raised or lowered by trial until when the telescope is rotated, the line of sight will pass through both the high point and the mid-point on the ground.

5. *To Make the Line of Sight, In So Far as Defined by the Horizontal Cross-hair, Coincide with the Optical Axis.*

Test.—Set two pegs, one about 25 ft. and the other 300 or 400 ft. from the transit. With the vertical motion clamped, take a rod reading on the distant point, and without disturbing the vertical motion read the rod on the near point. Plunge the telescope, turn through 180°, and set the horizontal cross-hair at the last rod reading with the rod held on the near point. Sight to the distant point. If the desired relation exists the first and last readings will be the same.

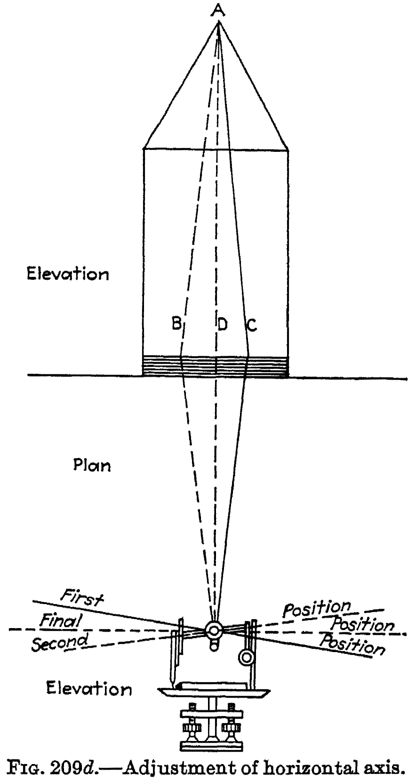


FIG. 209*d*.—Adjustment of horizontal axis.

Correction.—If there is a considerable difference between the rod readings, move the horizontal cross-hair by means of the upper and lower adjusting screws until it has apparently traversed over several times the apparent error. Repeat the process, gradually reducing the movement of the cross-hair as the rod readings to the distant point approach each other, until by successive approximations the error is reduced to zero. The rod when held on the near point should be read with great care for a small difference in the posi-

tion of the cross-hair on the near rod will be sufficient to indicate a considerable error on the distant rod.

6. *To Make the Axis of the Telescope Level Parallel to the Line of Sight.*

Test and Correction.—Same as the peg adjustment of the dumpy level (Art. 116, p. 148) except that the correction is made by moving the level tube until the axis is horizontal after having set the line of sight on the rod reading for a level line.

7. (For Transit Having a Fixed Vertical Vernier) *To Make the Vertical Circle or Arc Read Zero When the Telescope Bubble Is in the Center of the Tube.*

Test.—With the plate bubbles centered, center the telescope bubble and read the vernier of the vertical circle.

Correction.—Loosen the vernier and move it until the desired relation is established.

7*a*. (For Transit Having a Level Attached to the Vertical Vernier) *To Adjust the Auxiliary Level on the Vernier of the Vertical Circle So That Its Axis Is Parallel to the Axis of the Telescope Level When the Vernier Reads Zero.*

Test.—Bring the telescope bubble to the center of its tube, and move the vernier by means of the tangent-screw until it reads zero.

Correction.—By means of the capstan-headed screws, move the bubble attached to the vertical vernier until it is in the center of its tube.

8. *To Make the Axis of the Striding Level Parallel with the Horizontal Axis.*

Test.—By means of the leveling screws bring the striding-level bubble to the center of its tube. Lift the level from its supports and turn it end for end. If it is in adjustment, the bubble will again be centered.

Correction.—If the bubble is displaced, bring it halfway back to the center by means of the capstan-headed screw at one end of the tube (Fig. 209*b*). Relevel by means of the leveling screws, and repeat the test until the adjustment is perfected.

209a. Transit with Adjustable Objective Slide.—*To Make the Optical Axis Perpendicular to the Horizontal Axis.* As stated in Art. 104*c*, p. 124, some telescopes have objective slides which move in adjustable rings. Ordinarily objective slides of this type require no further adjustment after leaving the factory, but it is well to test the adjustment occasionally, however, and the instrumentman should be able to make corrections if necessary. Having performed the adjustment of the vertical cross-hair (adjustment 3) for an average length of sight, focus the vertical cross-hair on a distant point. Move the objective out, and bring it to a focus on

some well-defined point near the instrument. Plunge the telescope, rotate it about the vertical axis, and again set the vertical cross-hair on the near point. Sight toward the distant point. If the objective slide is in adjustment, the line of sight will strike the first point sighted. If the line of sight does not strike the distant point, move the ring controlling the objective slide by means of the screws on the sides of the telescope until by estimation the line of sight has moved one half of the apparent error at the distant point. The relation between the adjustments of the vertical cross-hair and optical axis is such that the adjustments must be repeated alternately until both are found to be correct.

The vertical adjustment of the objective slide may be performed in a similar manner with reference to the horizontal cross-hair, but it is usually best to make corrections with the horizontal cross-hair as described in adjustment 5, unless one is reasonably sure that the factory adjustment, through accident or otherwise, has been altered.

209b. To Center Eyepiece Slide.—A telescope which shows objects erect usually has an eyepiece slide, one end of which moves through an adjustable ring. When the transit has otherwise been adjusted, the cross-hairs may not appear in the center of the field of view owing to lack of coincidence of the axis of the eyepiece slide with the optical axis. The coincidence is a convenient relation but is unnecessary as far as the proper working of the transit is concerned. The adjustment is made by means of the screws which project from the telescope tube between the eye end of the telescope and the capstan screws controlling the cross-hair ring.

209c. Suggestions.—The adjustments of the transit are more or less dependent one upon another. For this reason if the instrument is badly out of adjustment, time will be saved by first making corrections roughly for those adjustments which are related, until all the tests have been tried, and by then repeating the tests and corrections (beginning at the first).

The plate levels will not be disturbed by other adjustments, and should be exactly corrected before other adjustments are attempted. Any movement of the screws controlling the cross-hair ring is likely to produce both lateral displacement and rotation of the ring; hence any considerable adjustment of the line of sight is likely to disturb the vertical hair so that it will no longer remain on a point when the telescope is rotated about the horizontal axis. The adjustment of the telescope level depends upon the unaltered position of the horizontal cross-hair and hence should not be tested until the line of sight and horizontal axis have been corrected.

If the transit has an erecting eyepiece which is permanently centered, adjustment 5 may usually be made with sufficient precision for ordinary direct or trigonometric leveling by simply moving the horizontal cross-hair until it appears in the center of the field of view. If the transit has an inverting eyepiece, the cross-hair ring limits the field of view and the

cross-hair will appear in the center regardless of whether or not it intersects the axis of the objective slide.

Care should be taken not to strain unduly the adjusting screws, but they should be brought to a firm bearing. The threads of the reticule screws are in the ring and not in the barrel of the telescope; hence the cross-hair ring is drawn toward the screw which is turned clockwise.

For those adjustments which involve sighting through the telescope, particular attention should be given to proper focusing of both the eyepiece and the objective prior to testing the adjustments.

ERRORS IN TRANSIT WORK

210. General.—The following articles are intended to point out the sources of error in transit work and to show the effect of the various errors on the precision of angular measurements.

Except in field astronomy, a measured angle is always closely related to a measured distance, and as has been previously stated (Arts. 19 and 20) there ought generally to be a consistent relation between the precision of these two classes of measurements. It is important, both from the standpoint of accuracy and from the standpoint of expediting the work, that the surveyor be able to visualize the effect of errors both in terms of angle and in terms of distance, that he appreciate what degree of care must be exercised to keep certain errors within specified limits, and that he know under what conditions various instrumental errors may be eliminated.

On surveys of ordinary precision it usually requires much more care to keep *linear* errors within prescribed limits than to maintain a corresponding degree of *angular* precision. The general tendency among surveyors is to pay undue attention to securing accuracy in angular measurements, and at the same time to overlook large and important errors in the measurement of distances.

211. Instrumental Errors.—These errors are caused by imperfections in the instrument itself. The adjustments previously described, even though carefully made, are never exact. Likewise the graduations are not perfect, and the centers are not absolutely true.

1. *Errors in Horizontal Angles Caused by Nonadjustment of Plate Levels.*—When bubbles in nonadjustment are centered, the circle is inclined and hence measured angles are not truly horizontal angles. Also the horizontal axis is inclined to a varying degree depending upon the direction the telescope is sighted. There will be one vertical plane which will include the vertical axis in its inclined position. Let this be the plane of the paper, and the vertical axis be in the position shown in Fig. 211*a*. When the line of sight is in the plane of the paper the horizontal axis is truly horizontal and the line of

sight will generate a vertical plane when the telescope is plunged; hence no error in direction is introduced regardless of the angle of elevation to the point sighted. As the transit is rotated about the vertical axis the horizontal axis becomes inclined, making a maximum angle with the horizontal when it reaches the plane of the paper. With the horizontal axis in this latter position, the line of sight generates a plane making a maximum angle with the vertical equal to the error in the position of the vertical axis, and with the line of sight inclined at a given angle, the maximum error in determining the direction of a line is introduced. The larger the vertical angle, the greater the error.

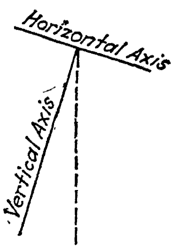


FIG. 211*a*.

The diagram of Fig. 211*b* shows for various vertical angles (values of a) the errors introduced in horizontal angles due to an inclination of 01′ in the vertical axis or one space on the plate levels of the ordinary transit. The values of H are the horizontal angles which the

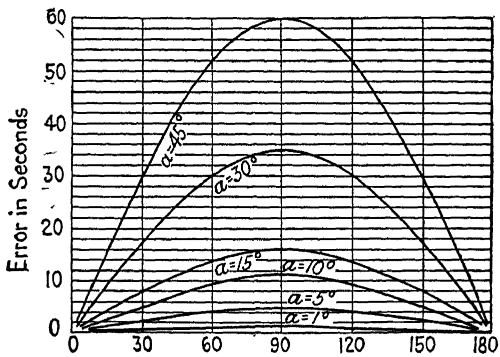


FIG. 211*b*.—Errors in horizontal angles for a 1-min. inclination of the vertical axis (a = vertical angle).

line of sight makes with the vertical plane in which lies the vertical axis in its inclined position (*i.e.*, with the plane of the paper, Fig. 211*a*). The curve for $a = 0°$ is not shown by reason of the small scale, but the maximum error occurs when $H = 45°$ or 135° and is about $\frac{1}{20}''$. Within reasonable limits the error in horizontal angle varies directly as the inclination of the vertical axis, hence a similar diagram for an inclination of 02′ would show ordinates twice as great as those of Fig. 211*b*.

While the diagram is perhaps of not much practical value, it serves to illustrate some facts that are worthy of attention:

(*a*) For observations of ordinary precision taken in flat country where the vertical angles are rarely greater than 3° and usually much less, the plate bubble may be out several spaces without appreciably affecting the precision of horizontal angular measurements. For example, if an angle were measured between $H = 0°$ and $H = 90°$, with bubble out two spaces and $a = 3°$, the error would be about 06″; or if in prolonging a straight line the telescope were plunged from the position $H = 90°$ to $H = 270°$ with both backsight and foresight taken at a vertical angle of +3°, the error would be doubled and for the bubbles out two spaces (vertical axis in error 02′), the angular error introduced in the direction of the line would be 12″.

(*b*) For angular measurements of higher precision, such as when measuring an angle by repetition, the plate levels must be in good adjustment and the bubbles must be centered with reasonable care even though the survey is conducted over fairly smooth ground. For example, if a horizontal angle were measured between the positions $H = 0°$ and $H = 90°$, $a = 5°$, and the vertical axis were inclined 30″, the error in horizontal angle would amount to 02″.2.

(*c*) In rough country where the vertical angles are large, even for surveys of ordinary precision the plate levels must be in good adjustment and the bubbles must be carefully centered if errors in horizontal angles or in the prolongation of lines are to be kept within negligible limits. For example, if a line were prolonged by plunging the telescope from the position $H = 90°$ to $H = 270°$, a for both backsight and foresight being +30° and the vertical axis being inclined 01′, the diagram shows that the error introduced is $2 \times 34''.6 = 01'09''.2$. In other words, the angle at the station at which the instrument was set instead of being a true 180° would be 180°01′09″.2 and beyond the station the established line would depart from the true prolongation about 0.1 ft. in each 300 ft.

2. *Errors in Vertical Angles Due to Nonadjustment of Plate Levels.* These errors obviously vary with the direction in which the instrument is pointed. With the fixed vertical vernier they are eliminated by observing the index error of the corresponding observed vertical angle (Art. 198*a*). With the movable vernier having a control level, errors are eliminated by keeping its bubble centered as described in Art. 198.

It may be noted further that the nonadjustment of the plate levels causes an inclination of the plane of the vertical arc. This source of error may be considered negligible.

3 *Line of Sight Not Perpendicular to Horizontal Axis.*—If the telescope is not reversed between backsight and foresight, if the sights are of the same length so that it is not required to refocus the objective, and if both points sighted are in the horizontal plane through the telescope, no error is introduced in the measurement of horizontal

angles even though this adjustment be badly out. If the instrument is plunged between backsight and foresight, the resultant error in the observed angle is double the error of adjustment. If there is a considerable movement of the objective between sights, an appreciable error may be introduced due to the fact that the line of sight does not make a constant angle with the horizontal axis for both sights.

The effect of the error depends upon the inclination (from the horizontal) of the line of sight towards the distant signal. It can be shown that the error in the direction observed is given by the equation

$$\sin E = \sin e \sec h$$

in which E is the error in the direction, e is the angle between the line of sight and the normal to the horizontal axis, and h is the actual vertical angle to the point sighted. Or if h' is the observed vertical angle, $\tan E = \tan e \sec h'$. For all ordinary cases this may be taken as

$$E = e \sec h$$

For two direct pointings (the backsight and the foresight) there will be a value of E for each, and the error in the angle is the difference between them. In the measurement of a deflection angle by the method in which the telescope is inverted between backsight and foresight, the error in angle is the *sum* of the two values of E.

The error may be eliminated by taking the mean of two angular observations, one with the telescope in the direct position and the other with the telescope reversed. In prolonging a line, errors are avoided by the method of double-sighting described in Art. 199.

4. *Horizontal Axis Not Perpendicular to Vertical Axis.*—No error is introduced in horizontal angles so long as the points sighted are in the same horizontal plane. It can be shown that the angular error θ in the observed direction of any line is given by the equation $\tan \theta = \sin e' \tan a$, or with sufficient precision $\theta = e' \tan a$ in which e' is the angle which the horizontal axis makes with the horizontal, and a is the observed vertical angle. Thus if a horizontal angle were measured between A, to which the vertical angle is $-30°$, and B, to which the vertical angle is $+15°$, the horizontal axis being inclined $02'$, then the error in horizontal angle is

$$\theta = 02'(\tan 15° - (-\tan 30°)) = 32'' + 01'09'' = 01'40''.$$

The preceding example is sufficient to show that the error in observed horizontal angle may become large. Obviously the sign of the error in horizontal angle depends upon the direction of displacement of the horizontal axis from its correct position. Hence if any angle is measured with the telescope first in the direct and then in the reversed

position one value will be too great by the amount of the error and the other will be correspondingly too small; thus the error is eliminated by taking the mean of the two values.

5. *Effect of Lack of Coincidence between Line of Sight and Optical Axis.*—Under these conditions if the line of sight is perpendicular to the horizontal axis for one position of the objective, it will not be perpendicular for other positions, but will swing through an angle as the objective is moved in or out. If an angle is measured without disturbing the position of the objective no error is introduced. For most instruments the error from this source is not sufficiently large to be of consequence in ordinary transit work. It is eliminated by taking the mean of two angles, one observed with the telescope direct and the other with it reversed.

6. *Errors Due to Eccentricity.*—With the modern transit in good condition, these errors are of no consequence in the ordinary measurement of angles. In any case, such errors are eliminated by taking the mean of readings indicated by opposite verniers.

7. *Imperfect Graduations.*—Errors from this source are of consequence only in work of high precision. They are reduced to a negligible amount by taking the mean of several observations for which the readings are distributed over the circle and over the vernier.

8. *Other Sources of Instrumental Errors.*—Other sources of instrumental errors are: (*a*) lack of parallelism between axis of telescope level and line of sight, introducing an error in leveling (see Art. 126, p. 166); and (*b*) nonadjustment of the vertical vernier, which produces a constant error (index error) in the measurement of vertical angles. If the transit is equipped with a full vertical circle, the index error may be eliminated by taking the mean of two values, one observed with the telescope direct and the other with the telescope reversed.

211a. Summing up, it is seen that:

1. Errors in horizontal angles due to nonadjustment of plate levels or of horizontal axis become large as the inclination of the sights increases.

2. Nonadjustment of the line of sight becomes of consequence only when the telescope is plunged.

3. Errors due to instrumental imperfections or nonadjustment are all systematic in character, and without exception they may be eliminated, or at least reduced to a negligible quantity, if proper methods of procedure are employed. In general, this consists in obtaining the mean of two values—one observed before and one after a reversal of the horizontal plate (by plunging the telescope and rotating about the vertical axis)—one of these values being as much

too large as the other is too small. An exception is the error in either horizontal or vertical angle due to inclination of the vertical axis, which cannot be so eliminated but which is eliminated, so far as its systematic character is concerned, by releveling the plate bubbles in addition to the reversal of the plate.

212. Personal Errors.—Personal errors arise from the limitations of the human eye in setting up and leveling the transit and in making observations.

1. *Effect of Not Setting Up Exactly over the Point.*—This produces an error in all angles measured at a given station, the magnitude of the error varying inversely with the length of sight. It is convenient to remember that 1 in. is the arc whose angle is 01′ when the radius is approximately 300 ft. Thus if the transit were offset ½ in. from the end of a line 50 ft. long the error in the observed direction of the line would be 03′, but if the line were 600 ft. long the error would be only 15″. In general the error may be kept within negligible limits by reasonable care, but many instrumentmen waste time by exercising needless care in setting up when the sights to be taken are long.

2. *Effect of Not Exactly Centering the Plate Bubble.*—This produces an error in horizontal angles after the manner described in the preceding article when the plate levels are out of adjustment. The probable error from this source is small when the sights are nearly level, but may be large for steeply inclined sights (see Fig. 211*b*). The average transitman does not appreciate the importance of careful leveling for steeply inclined sights and often uses more care in leveling than is necessary when sights are nearly horizontal. Since the error in horizontal angle is largely caused by inclination of the horizontal axis, the striding level is a necessity on precise work.

3. *Errors in Setting and Reading the Vernier.*—These are functions of the least count of the vernier and of the legibility of scale and vernier lines. For the usual 01′ transit the probable error is less than 30″; for the 30″ transit the probable error is about 15″. The use of a reading glass enables closer reading, particularly for finely graduated circles. Also in reading the vernier it is helpful to observe the position of the graduations on both sides of the ones that appear to coincide, and to note that the unmatched graduations appear to lack coincidence by the same amount.

4. *Not Sighting on the Exact Point.*—This is likely to be a source of rather large error on ordinary surveys where sights are taken on the range pole, of which often only the upper portion is visible from the transit. The effect upon a direction is, of course, the same as the effect of not setting up exactly over the point. For short sights

greater care should be taken than for long sights, and usually the plumb line is employed in place of the range pole.

5. *Imperfect Focusing (Parallax)*.—The error due to imperfect focusing is always present to a greater or less degree, but with reasonable care it may be reduced to a negligible quantity. The manner of detecting parallax was described in Art. 104*a*.

212a. All of the personal errors are accidental in character and hence cannot be eliminated. They form a large part of the resultant error in transit work. Of the personal errors, those due to inaccuracies in reading and setting the vernier and to not sighting on the exact point are likely to be the ones of greater magnitude.

213. Natural Errors.—Sources of natural errors are (1) settlement of the tripod, (2) unequal atmospheric refraction, (3) unequal expansion of parts of the telescope due to temperature changes, and (4) wind producing vibration of the transit or making it difficult to plumb accurately.

In general, the resulting angular errors are not of sufficient magnitude to affect appreciably the measurements of ordinary precision. Large errors, however, are likely to arise from the first source when the transit is set up on boggy or thawing ground. Attention is directed to the fact that settlement is usually accompanied by an angular movement about the vertical axis as well as linear movements both vertically and horizontally. When horizontal angles are being measured, the angular displacement of the circle between backsight and corresponding foresight, rather than the movement of the transit laterally from the point over which it is set, is usually what produces the large error. Errors due to adverse atmospheric conditions may nearly always be rendered negligible by choosing appropriate times for observing.

For measurements of high precision the methods of observing are such that instrumental and personal errors are kept within very small limits and natural errors become of relatively great importance. In general, natural errors are accidental in character, but under certain conditions systematic errors may arise from natural causes. On surveys of very high precision, special attempt is made to establish a procedure which will as nearly as possible eliminate natural systematic errors. Thus the instrument may be set up on a masonry pier and protected from sun and wind; also certain readings may be made at night when temperature and atmospheric conditions are nearly constant.

214. Precision of Angular Measurements.—The angular precision to be expected in transit work depends upon so many factors that it would be absurd to attempt to lay down an exact procedure to be followed to ensure a required accuracy. It is clear from the

preceding articles that, with proper methods, the important systematic errors may be practically eliminated and that the resultant error is largely accidental in character. No matter how precisely the transit may be adjusted nor how carefully it may be set up, there yet remain the errors of sighting and of reading the angle, and these are of major importance in nearly all surveying. The angular precision with which the line of sight may be directed obviously depends upon the length of sight and the character of the target or other object used to mark the point sighted, as well as upon the quality of the instrument and the skill of the observer. The precision with which an angle may be read depends upon the character of the graduations of the circle.

The following values represent, in a general way, the *maximum* errors likely to occur in measuring horizontal angles under the average conditions of practice, instruments being in fair condition and in fair adjustment except as otherwise stated.

Case 1.—Short sights, point indicated by range pole obscured near ground. Range pole plumbed by eye. Single observation of angles. Error 02′ to 04′.

Case 2.—Long sights, but otherwise as stated for case 1. Error 01′ to 02′.

Case 3.—Unobscured but steeply inclined sights; no special attention given to making horizontal axis truly horizontal; single measurement of angle. Error 01′ to 02′.

Case 4.—Unobscured sights on well-defined points; sights not steeply inclined. Single observation of angle, vernier reading to minutes. Error 30″ to 01′.

Case 5.—As for case 4, but transit in excellent condition and in good adjustment. Angles estimated to ½ min. Error 20″ to 30″.

Case 6.—As for case 4 but angle doubled, the telescope being reversed between sights. Error 15″ to 30″.

Case 7.—Unobscured sights on well-defined points. Sights not steeply inclined. Verniers reading to 30″. Single observation of angle represented by mean of readings of both verniers. Transit in excellent condition and in good adjustment. Error 15″ to 30″.

Case 8.—As for case 7, but verniers reading to 10″. Also instrument set up with great care. Error 10″ to 15″.

Case 9.—Unobstructed sights on well-defined points. Instrument set up with great care. Sights not steeply inclined. Transit in good condition. Vernier reading to 30″. Angles repeated six times with telescope direct and six times with it reversed. Error 02″ to 04″.

Case 10.—As above, but transit reading to 10″. Observations taken at favorable times. Error 01″ to 02″.

The *average* angular errors will of course be materially less than the values given for the preceding cases. Also, since the errors are largely

accidental in character, the resultant error in the sum of a series of measured angles may be expected to vary as the square root of the number of angles involved.

215. Numerical Problems.

1. Thirty spaces on a transit vernier are equal to 29 spaces on the graduated circle and one space on the circle is 15′. What is the least count of the vernier?

2. Sixty spaces on a transit vernier are equal to 59 on the circle and the circle is divided into 15′ spaces. What is the least count of the vernier?

3. Referring to Fig. 204*a*, suppose that the distance AC is 317.2 ft. and the angle ACB is 67°13′. What is the distance AB?

4. Referring to Fig. 204*b*, if the length of the line AC is measured and found to be 517.2 ft. and the length of AD is found to be 315.5 ft., what is the distance AB?

5. Referring to Fig. 205*b*, a straight line AX is run at random from A in the general direction of AB, point B not being visible from A. A swing offset is measured from B to the line AX and found to be 63.4 ft. The transit is set up at E, and EY (perpendicular to AX) is erected. The swing offset from B to EY is 6.1 ft. Also the distance AE is 2638.9 ft. Calculate the angle α which must be laid off from the random line in order to establish points on the straight line AB, and determine the length AB. What must be the precision of α in order that the line established shall fall within 0.1 ft. of the point B?

6. What error would be introduced in the computed value of the angle α of problem 5 if the swing offset distance from B to EY had been neglected and AE had been assumed to be the base of a triangle of which AB is the hypotenuse?

7. Having given the data of problem 5, it is proposed to establish points on the line AB by perpendicular offsets from C and D. What must these offsets be if AC = 937.6 ft. and AD = 1932.0 ft.?

8. A transit for which the graduated circle is numbered from 0° to 360° in each direction is used to measure an angle by repetition. The angle is multiplied five times in both clockwise and counter-clockwise directions and the following readings are observed:

	Measured clockwise	Measured counter-clockwise
Initial reading	63°17′	318°26′
One measurement	189°49′	84°58′
Five measurements	335°58′	231°09′

Calculate the most probable value of the angle.

9. Before beginning the construction of a 300 by 500-ft. armory, lines marking its sides were laid off on the ground. This was accomplished by first establishing one of the 500-ft. sides and then erecting 300-ft. trial

perpendiculars at both ends. The 90° angles were laid off as accurately as possible with a 01′ transit. The angles laid off were then repeated six times with the telescope direct and six times with it reversed. The actual angles laid off were found to be 89°59′50″ and 90°00′25″. What are the values of the offsets from the ends of the 300-ft. trial perpendiculars to establish the true corners?

10. Two points *A* and *B*, 5,280 ft. apart, are to be connected by a straight line. A random line run from *A* in the general direction of *B* is found by calculation to deviate 03′18″ from the true line. On the random line at a distance 1,250.6 ft. from *A* an intermediate point *C* is established. What must be the offset from *C* to locate a corresponding point *D* on the true line?

11. What error would be introduced in the measurement of a horizontal angle if, through nonadjustment, the horizontal axis were inclined 03′ with the horizontal and sights were taken to points at the same elevation as the transit? If the horizontal axis were inclined 3° and sights were taken to points at the same elevation as the transit?

12. Same as problem 11, but sights inclined at positive angles of 30°.

13. Same as problem 11 but one sight inclined at +30° and the other at −30°.

14. In prolonging a straight line the transit is set at *B*, a backsight is taken to *A*, and the telescope is plunged to set *C* 1,000 ft. in advance of *B*. *A* and *B* are at the same elevation but the vertical angle from *B* to *C* is +15°00′. If the vertical axis were inclined 01′ with the true vertical in a vertical plane making 90° with the direction of the line, what would be the linear offset error in the located position of *C*?

15. Same as problem 14, but *A*, *B*, and *C* all in the same plane.

16. Same as problem 15, but vertical angles of +15°00′ for both backsight and foresight.

17. A line *AB* is prolonged by setting up the transit at succeeding points *B*, *C*, *D*, etc., backsighting on preceding points, *A*, *B*, *C*, etc., and plunging the telescope. If the line of sight made an angle of 15″ with the normal to the horizontal axis and the procedure were such that each backsight was taken with the telescope in the direct position, what would be the angular error in the segment *FG*? What would be the offset error in the position of the point *G* if the segments *AB*, *BC*, etc., were uniformly 500 ft. long?

18. In measuring a horizontal angle the error of setting up the transit is 0.03 ft., the direction of displacement being such as to produce a maximum angular error. What error is introduced in a 60° angle if the length of sights is 50 ft.? If the length of sights is 1,000 ft.?

19. Same as problem 18, but 120° angle.

20. If the ratios of linear precision to be maintained on the various parts of a survey were $\frac{1}{1,000}$, $\frac{1}{5,000}$, $\frac{1}{20,000}$, and $\frac{1}{40,000}$, about how closely should the corresponding horizontal angles be observed in order that a consistent relation should exist between precision of angles and precision of distances?

21. It is desired to determine by calculation the length of a side of a right triangle, the angle opposite and the hypotenuse being measured. If the angle is 20°, with what precision should it be measured in order that the ratio of precision in the calculated length be $\frac{1}{20,000}$?

22. Same as problem 21, but a measured angle of 70°.

23. It is desired to determine by calculation the length of a side of a right triangle, the angle opposite and the side adjacent thereto being measured. If the angle is 20°, with what precision should it be measured in order that the ratio of precision be $\frac{1}{20,000}$? If the angle is 70°?

Measuring Angles About a Point

Sta.	Point	Angle	Bearing	Angle by Bear	Discrepancy
O	1		N57½°W		
		48°03'		48¼°	12'
	2		N9¼°W		
		54°26'		54¾°	19'
	3		N45½°E		
		86°32'		86½°	02'
	4		S48°E		
		31°07'		30¾°	02'
	5		S19¼°E		
		77°10'		77¼°	05'
	6		S60°W		
		62°44'		62½°	14'
	1		N57½°W		
	Sum.	360°02'			

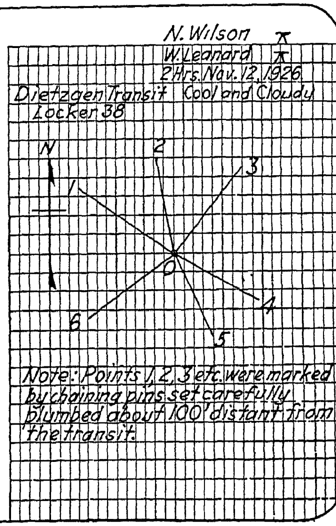


Fig. 216*a*.

216. Field Problems.

1. Measurement of Angles with Transit

Object.—To measure several angles about a point, and to check the values of the angles by the use of magnetic bearings.

Procedure.—(1) Set up and level the transit at any point *O*. (2) Set six chaining pins, 1, 2, 3, etc., at about equal distances from the transit, forming six angles at the station *O*. (3) According to the procedure of Art. 196 measure each of the angles, using the *A* vernier only and always resetting the vernier on each backsight. (4) The sum of the measured angles should not differ from 360° by more than ±03′. (5) If this difference is exceeded, the angles should be remeasured until the sum falls within the stated limits. (6) Release the compass needle, sight on each point, and, according to the method of Art. 184, read and record the magnetic bearing to each pin. (7) Calculate the angles by bearings and

compare with the transit angles. The discrepancy between any transit angle and the same angle by bearings should not exceed 30′ (for sample notes, see Fig. 216*a*).

Hints and Precautions.—The pins should be set as nearly vertical as possible with reasonable care. They may be plumbed by the vertical cross-hair of the transit. If each pin is run through a piece of paper, piercing it in several places, the paper will form an excellent background for sighting the pin.

2. Prolongation of a Line by Double-sighting with Transit

Object.—To produce a straight line with precision, setting stakes at intervals of approximately 300 ft. (see Art. 199).

Procedure.—(1) Set two points about 300 ft. apart in such position as to afford an open view for 1,000 ft. or more in advance. (2) Set the instrument on the forward point. Backsight with the telescope reversed. (3) Plunge the telescope and set a stake on the line a hundred paces in advance. Mark a point on the stake exactly on line. (4) Take a second backsight upon the rear stake in the same manner as before, with the telescope direct. Plunge the telescope again, and mark a point on the advance stake. (5) If this point is not coincident with the first point set, a point midway between them is on the line. (6) Set the transit over this point, and advance by the same process, backsighting on the next point in the rear. Continue in this way for the desired distance. (7) Check the work by setting the instrument over the first point, sighting carefully on the next point, and then noting the linear error of the points set by double-sighting, without moving either horizontal motion.

3. Intersection of Lines with Transit

Object.—To bisect the three angles of a triangle and to mark the point of concurrence of the bisectors.

Procedure.—(1) Drive three stakes, A, B, C, at the vertices of a roughly equilateral triangle, with sides 300 or 400 ft. in length. Tack each stake. (2) Set up the transit and measure the angle at A. Lay off one half of the measured amount, thus establishing the bisector of angle A. On the bisecting line of sight and on an estimated bisector of angle B drive a stake o, and drive a tack halfway. Set two more tacked stakes, m and n, on the bisecting line of sight about 10 ft. from and on opposite sides of o. (3) Set the transit over B and locate the position of the bisector as at A. Drive a stake on this line and under a cord stretched from m to o or n to o, as the conditions require. Tack the exact point of intersection, p. (4) Set the transit on C, measure the angle and bisect as at A and B. (5) Measure the discrepancy between this bisector and the point of intersection of the first two bisectors at p to hundredths of feet. Also measure the angular discrepancy; it should not exceed 02′. (For sample notes, see Fig. 216*b*.)

4. Measurement of Angle When Transit Cannot Be Set at Vertex

Object.—To measure the angle between two walls of a building (see Art. 202).

Procedure.—As outlined in Art. 202; the points *a*, *b*, *c*, etc., to be tacks in stakes driven in appropriate locations.

5. Prolongation of Line Past Obstacle (Offset Space Limited)

Object.—To prolong a line, determined by two fixed points, past an obstacle when the conditions are such as to limit the lengths of the offsets.

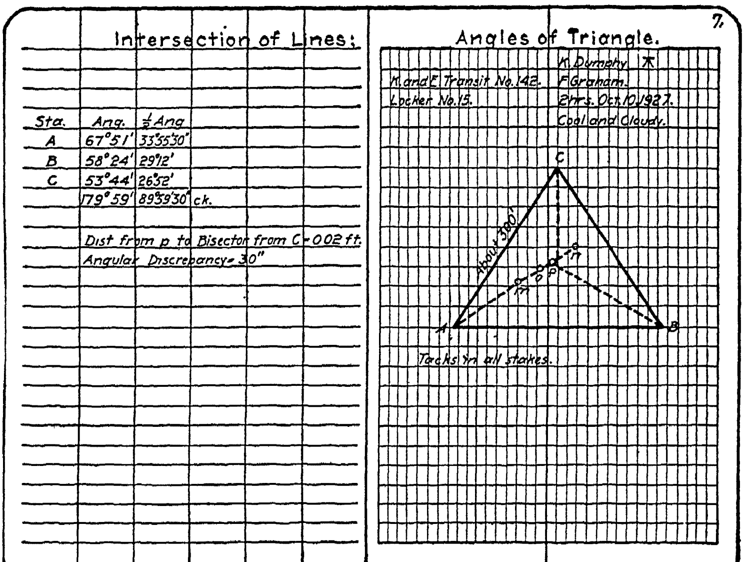


Fig. 216*b*.

Procedure.—As outlined in Art. 203. The lengths of offsets should be measured very carefully. If the instrument is in good adjustment, the points *E* and *F* may be set by a single reversal, or if a clear sight can be obtained the transit may be set up at *C* and the points *E* and *F* may be established without reversal.

If the obstacle is imaginary, check the accuracy of the work by setting the transit at *A* and locating *G* and *H* by the direct prolongation of *AB*.

6. Determination of Inaccessible Distance

Object.—To obtain the distance between two points on opposite sides of a river (see Art. 204).

Procedure.—As outlined in Art. 204. The transit points should be tacked stakes. If the river is imaginary, after the distance has been computed by each method, check it by direct measurement.

7. Running a Straight Line between Two Points Not Intervisible

Object.—To establish points along a straight line joining two given points not intervisible (see Art. 205, cases 2 and 3).

Procedure.—As outlined in Art. 205, cases 2 and 3. Under case 2 take two points on opposite sides of a hill. Check the located position of C (Fig. 205*a*) by occupying that station and by the method of double-sighting prolonging the line AC to B. Note the error at B.

Under case 3, determine offsets from intermediate points on the line AX (Fig. 205*b*) to the line AB and establish the corresponding points on AB by tape measurements. Then lay off the angle α from AX and establish a second set of points on the line AB by the method of double-sighting. Note the discrepancies.

8. Angles by Repetition

Object.—To obtain a more accurate determination of the angles between various stations about a point than would be possible by a single measurement (see Art. 206).

Procedure.—(1) Set up the transit very carefully over the point. (2) Set the A vernier at zero, read the B vernier, and record the readings. (3) With the telescope in its direct position (bubble down), measure one of the angles in a clockwise direction, and record both vernier readings to the smallest reading of the vernier. (4) Leaving the upper motion clamped, again set on the first point and again measure the angle in a clockwise direction (thus doubling the angle). (5) Continue until six "repetitions" (readings) have been secured. Record both vernier readings and the total angle turned. (6) In a like manner, setting the B vernier at zero, measure the explement of the angle in a *counter-clockwise* direction with the bubble *up*, but read the horizontal circle as though the *angle itself* had been measured clockwise. (7) Go through the same process for all other angles about the point. (8) Compute the value of each of the angles for each direction turned and compare with the single measurement. (9) Find the mean of each of these sets of single angles. For a transit reading to single minutes the *total error should not exceed* $10'' \sqrt{\text{number of angles}}$. (10) Adjust the angles so that their sum will equal 360° by distributing the error equally among the mean values. Keep notes in form shown by Fig. 216*c*.

Hints and Precautions.—(1) Level the transit very carefully before each repetition, but do not disturb the leveling screws while a measurement is being made. (2) The mean of each set of single angles should furnish a value free from instrumental errors. The station adjustment is an attempt to properly distribute the accidental errors to meet the known condition, that there are 360° about a point. (3) Do not become

confused when calculating the total angle turned. Observe how the horizontal limb is graduated, and do not omit a full turn. (4) The instrument should be handled very carefully. When turning on the lower motion, the hands should be in contact with the *lower* plate (not the alidade) and when making an exact setting on a point, the last movement of the tangent-screw should be *clockwise* or against the opposing spring. (5) After each repetition the instrument should be turned on its lower motion in a direction *opposite* to that of the measurement. (6) The single measure-

Angles About △ 47						By Repetition.			8.
									B. Hobbs, ⊼
Sta.	Direction	Bubble	Ver. A.	Ver. B.	Ver. Mean.	Single	Mean	Adj.	G. Whit, Notes.
From-To						Ang.		Ang.	2hrs. August 10, 1928.
50	↻	Down	0°00'00"	180°00'30"	0°00'15"				Very Hot.
			89°23'	269°23'					Berger Transit No. 8.
53			536°21'00"	716°21'00"	536°21'00"	89°23'27"			One Minute Vernier
									Estimated to 30."
50	↺	Up	180°00'00"	0°00'00"	0°00'00"				Six Repetitions.
53			716°21'30"	536°22'00"	536°21'45"	89°23'37"	89°23'32"	89°23'37"	
53	↻	Down	0°00'00"	180°00'00"	0°00'00"				
			51°44'	231°44'					
48			310°21'30"	490°21'00"	310°21'15"	51°43'33"			
53	↺	Up	179°59'30"	0°00'00"	359°59'45"				
48			490°21'30"	310°21'30"	310°21'30"	51°43'37"	51°43'05"	51°43'40"	
48	↻	Down	0°00'00"	180°00'00"	0°00'00"				
			218°53'	398°53'					
50			1313°16'00"	1493°15'30"	1313°15'45"	218°52'37"			
48	↺	Up	180°00'30"	0°00'00"	0°00'15"				
50			1493°16'30"	1313°16'00"	1313°16'15"	218°52'40"	218°52'39"	218°52'43"	
							359°59'46"	360°00'00"	ck.

FIG. 216c.

ment is taken as a check on the number of repetitions. It should agree closely but not exactly with the mean value.

9. Laying Off an Angle by Repetition

Object.—To lay off a given horizontal angle more accurately than is possible with a single setting of the vernier (see Art. 207).

Procedure.—(1) Drive and tack two stakes about 500 ft. apart. (2) Set the transit carefully over one end of the line. Sight at the point at the other end, and lay off the angle. (3) Set a stake on the line of sight about 500 ft. from the instrument (distance by pacing), and carefully set a tack. (4) By repetition measure the angle laid off, as in the previous problem, making six repetitions in each direction. (5) Find the difference between the angle laid off and the required angle, and by

trigonometry compute the linear distance that the tack must be moved perpendicular to the line of sight. (6) Set the tack accordingly.

10. Adjustments of the Transit

Object.—To make the field adjustments of the engineer's transit (see Arts. 208, 209).

Procedure.—As outlined in Art. 209.

CHAPTER XIII

SURVEYS WITH TRANSIT AND TAPE

217. General.—This chapter treats of the general methods utilized on the large variety of surveys of ordinary precision for which the transit is employed for the measurement of horizontal angles and the tape is used for the measurement of distances. The practices here described are, with minor modifications, common to land, city, topographic, and hydrographic surveys, and to location surveys for highways, railroads, and the like. Probably 90 per cent of all surveying work (not including leveling) is, with modifications, carried on as herein described.

218. Transit Party.—The transit party is usually composed of three men, the *transitman,* the *head chainman* and the *rear chainman.* The transitman directs the work of the party, operates and cares for the transit, and keeps all notes in the field notebook. The head chainman performs the duties of that position as described in Arts. 83 and 84, gives line as directed by the transitman, and is responsible for the accuracy and speed of the chaining operations. Where stakes are set he attends to their proper marking. The rear chainman carries out the duties of that position as described in Arts. 83 and 84, gives backsights as directed by the transitman, often carries and drives stakes, and assists in removing obstructions to the vision of the transitman. In wooded country, axemen (up to five or six in number) are employed in clearing the transit line of trees and brush. The head chainman keeps in close communication with the axemen and assists in their direction. Where sights are long, a rear flagman is often stationed at the transit point preceding that at which the transit is set, his duty being to give backsights to the transitman. Where a large number of observations are to be taken, the notes are kept by a recorder, who may also act as the chief of party.

219. Equipment of Transit Party.—The equipment of the transit party usually consists of a transit, 100-ft. steel tape, two range poles, stake bag, stakes, tacks, axe or hammer, two or three plumb bobs, field notebook, chaining pins, and marking crayon. A wool or silk hood is provided for the transit as a protection against rain. Often large nails are conveniently used as markers, and frequently a cold chisel is used in marking points on stone or other hard objects.

220. Transit Stations.—Any point of reference over which the transit is set up in the process of making a survey is called a *transit station*. The object marking the point may be either temporary or permanent in character. On most surveys the transit station is a stake driven flush with the ground with a tack in its top marking the exact point to which both angular and linear measurements are referred. Such a stake is generally termed a *hub*. On city streets the transit station may be a driven nail or a cross cut in the pavement or curb. In land surveying the stations are often iron pipes, stones, or other more or less permanent monuments set at the corners. In mountainous country the station marks are often cut in the natural rock.

Where the station is a hub, its location is usually indicated by a stake (called a *guard stake*) extending some distance above the ground and driven sloping so that its top is over the hub. This guard stake carries the number of the transit station over which it stands.

The transit stations of a survey are usually identified either by number or by letter, the identification mark being placed either upon the station marker or upon some other object nearby, and also of course being appropriately indicated in the field notes. Usually the number is marked with lumber crayon or keel and reads down the stake. It is common practice to drive the witness stake so that the number is face downward, thus protecting it from the weather. The hubs are often made square, say 2 by 2 in., and the witness stakes are usually flat, perhaps ¾ by 3 in.

221. Transit Lines.—Lines connecting transit stations are termed *transit lines*. If a system of lines run with the transit forms a traverse (as described in Art. 180) it is called a *transit traverse*, to distinguish it from traverses run with other instruments. The transit lines forming a triangulation system are called *lines of triangulation*, and the points at which the transit is set up are called *triangulation stations*.

Both in the field notes and as a part of the identification mark left in the field, the station number, if for a traverse, is preceded by the symbol ⊙, and if for a triangulation station, by the symbol △.

In most cases the transit stations are identified by consecutive numbers as the survey progresses. Sometimes triangulation stations are given names suggested by the locality where each is established; this is particularly true for the more precise triangulation systems covering large areas.

For many continuous traverses where lengths are measured with the tape, distances are referred to the point of beginning of the survey and stakes marked with the distance from the initial point are commonly set every 100 ft. These 100-ft. points are called *full*

stations. Points falling between full stations are called *plus* stations. The distance to any plus station is indicated as the number of *hundreds of feet* from the initial point to the preceding full station plus the distance in feet from the preceding full station to the point in question. Thus, a full station 1,200 ft. from the point of beginning would be numbered 12 + 00, or simply 12, and a plus station at say 1,927.2 ft. from the point of beginning would be numbered 19 + 27.2. The stations intermediate between transit points are usually marked by flat stakes driven vertically with the number on the side towards the zero end of the traverse, the number reading down the stake. Where conditions will not permit the driving of stakes, there may be employed nails, chisel marks, painted marks, etc.

Transit points are guarded as described above, the number for a given point being its chainage from the zero point, as for any other station on the traverse. Thus the guard stake for a transit point 1,216.3 ft. from the initial station would be marked as ⊙12 + 16.3.

222. Transit Surveys.—The field work involved in making a survey with the transit may ordinarily be divided into (1) the establishment of transit stations and transit lines by angular and linear measurements, and (2) the location of objects and points with respect to the transit lines. The transit lines may be said to form the skeleton of the survey and are sometimes spoken of as the *control* or *horizontal control;* the measurements made to determine the location of objects or to establish the position of points furnish the details with which the transit lines are clothed. For some surveys the amount of detail secured from the transit lines is little; for example, for surveys to establish the boundaries of land the transit stations are usually at corners of the property, and if the boundaries are straight, few if any measurements to details are required. For some other surveys the location of features away from the transit lines forms the greater portion of the work. For example, in certain topographic surveys, for every transit station there may be 50 points to which measurements must be taken to secure adequate information for the construction of the map. On some surveys the collection of details may take place as the work of laying out the transit lines proceeds; on others the system of transit lines is first established, and after it has been checked, the details are obtained. The latter procedure is most likely to be employed where the survey covers a considerable territory and where the methods and instruments used in running the transit lines are not those used in the collection of details.

Surveys have for their object either (1) the location of certain features of the landscape, or (2) the establishment of points and lines

of predetermined length and direction which are to be employed as a guide to the future enterprises of man. Often a single survey may accomplish both objects. For a given character of work the methods employed are fundamentally the same.

The simplest survey to be made with the transit and tape employs a single transit station over which the instrument is set. Angles and distances to surrounding points are observed, and thus objects within the range of vision of the transit are located. This is often called the method of *radiation.*

One nearly as simple consists of two transit stations connected by a single transit line, called the base line. From each station, angles with respect to the transit line are observed to objects which it is desired to locate. Thus any point is defined by the two angles taken from the transit stations and by the length of the base line. This is generally called the method of *intersection,* and is a variety of triangulation.

On surveys of any considerable extent the method of *traversing* is generally employed to establish most of the transit lines, and the two preceding methods together with some others later to be described are used in locating details. Transit traverses in one form or another probably make up more than 95 per cent of the systems of control on surveys of ordinary precision.

The *triangulation* system as a means of providing control on surveys of ordinary precision is not generally used except in hydrographic surveying and in topographic surveying in rough, open country. Where conditions are favorable, it is a very economical method in so far as time spent in the field is concerned. Generally, additional control stations are established by traverses run between triangulation points. The method of triangulation is employed extensively on surveys of precision covering wide areas, particularly in connection with topographic and hydrographic work. The opportunities where the triangulation system may be employed to good advantage over traversing are more numerous than is generally realized. Owing to the fact that practice covers such a large range of conditions, both as regards precision and as regards instruments and methods, the subject of triangulation is reserved for a later chapter.

223. Method of Radiation.—This method by itself is applicable only to surveys covering small areas. The transit is set up at any convenient station from which can be seen all points which it is desired to locate. The distance from the transit station to each of the points is measured, and the horizontal angle is observed. The angles between successive points may be measured, or the true,

magnetic, or assumed bearing or azimuth of each of the lines joining the points with the transit station may be observed.

Figure 223 illustrates the notes for the survey of a field, angles being measured between successive points.

Where it is necessary to locate only the points, as, for example, where the survey is being made for a map, the method is excellent. But trigonometric computations are necessary if the lengths and directions of land lines are to be determined.

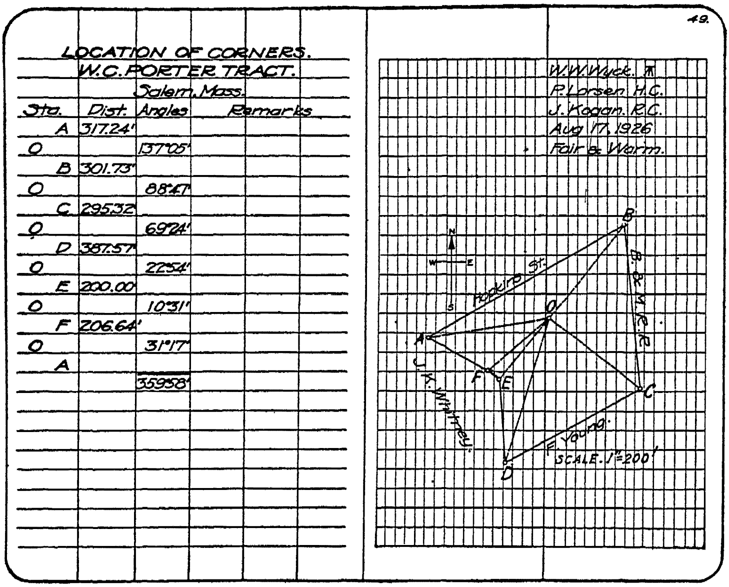


FIG. 223.—Field notes.

For example, consulting the sketch of Fig. 223, the field notes give for each of the triangles into which the figure is divided, two sides and the included angle at *O*. To determine the length of any unknown side (as *EF*) and the value of any unknown angle (as *OFE*) it is necessary to use the sine[1] of the angle at *O* (as sin *EOF*). The disadvantage of the method lies in the additional office work required to compute the lengths and directions of the boundaries and also in the weakness of the calculated values when the measured angle is small (see Art. 21).

Thus, if the angles in Fig. 223 were measured with an error of 30″, the error in the computed length *EF* would correspond to the low ratio of

[1] If solved by right-angle triangles, the sine is used directly; if solved by oblique triangles, the sine is used indirectly.

precision of $\frac{1}{1,260}$, while the ratio of precision of the computed length CD would be practically $\frac{1}{20,000}$.

Inasmuch as there is likely to be at least one small measured angle in each figure, the method while often practicable from the standpoint of economy of time in the field, is not commonly used for property-line surveys. It is generally employed for the location of details on more extensive surveys, the stations occupied being traverse or triangulation points.

224. Method of Intersection.—In Fig. 224 let the points A, B, C, etc., represent objects which it is desired to locate and let OP be some convenient line from both ends of which the unknown points are visible. The length of the base line OP is measured with the tape. The transit is set up at O, and angles to the unknown points are observed. These may be expressed as azimuths, as bearings, or as angles between successive points. A similar series of observations is made with the transit at P. In this manner each of the unknown points A, B, C, etc., becomes the vertex in a triangle of which the base line OP is the side of measured length, and in which the angles adjacent thereto are observed values; the positions of the unknown points are thus defined.

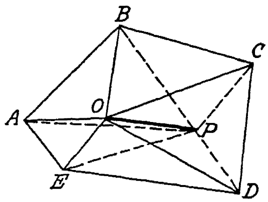


Fig. 224.

The method is not usually employed alone but is used in conjunction with other methods, particularly traversing. Where well-defined details at a considerable distance are visible from two or more transit stations, they may often be located by this method much more expeditiously than by angle and distance. Also when some landmark is visible from a number of stations of a traverse, angles to the mark taken from the several stations provide a means of checking the accuracy of the angular and linear measurements of the traverse.

In hydrographic surveying it is a very useful method of locating soundings, observations upon a boat signal being made simultaneously by two transits located at triangulation or traverse stations on shore.

Particular attention should be given to securing well-shaped triangles; that is, the angles should neither be too large nor too small. Usually on important work the attempt is made to secure angles between 30° and 150°. In Fig. 224 the triangle AOP is weak, the angles at A and P being too small and the one at O being too large; hence the uncertainty of the position of point A would be large. The triangle BOP is well shaped,

and the computed position of B would be much more certain for a given precision of angular measurement than would that of A.

Thus, if the angles were measured with an error of 30″, and the values were $OAP = 5°$ and $OBP = 45°$, the ratios of precision as governed by angular errors would be about $\frac{1}{400}$ for the computed distance AO and about $\frac{1}{7,000}$ for the computed value of BO.

The method is not, in general, well suited to surveys made for the purpose of determining the lengths and directions of boundaries, because of the large amount of computing necessary and because of the uncertainties attached to the computed values when triangles are weak. For example, in order to determine the length and direction of EA it would be necessary to solve triangle AOP, EOP, and AOE, and the weakness of the triangle AOP would, of course, be reflected in the computed direction and length of EA.

225. Traversing.—The process of running a transit traverse is much the same for one variety of traverse as for another. If the traverse is for the purpose of locating features already existing in the field, the positions of transit stations are chosen so as to facilitate the work of locating these features. If the traverse is for the purpose of establishing points and lines in accordance with predetermined measurements, it becomes simply a matter of laying off these measurements in the field. For example, the location survey for a railroad would consist of a traverse along the center line of track.

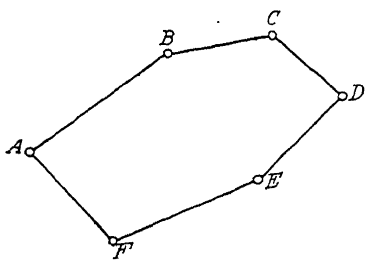


FIG. 225.

Following is a general description of the work of running a closed traverse, the transit stations being established in advantageous locations as the survey progresses, and distances being measured between successive transit stations: In Fig. 225 let A and B be selected locations for transit stations marking the first line of a traverse. Hubs defining the points are driven and properly identified by guard stakes. The transit is set up at B, the horizontal vernier is set to a given angular value, a backsight is taken on a range pole at A, and the lower motion is clamped. The line AB is then chained as described in Chap. VI, except that usually the head chainman is lined in by the transitman rather than by the rear chainman. The distance AB is recorded. The position of C is selected, and the transit point is established. The transit is turned on the upper motion until a foresight is secured on C. The upper motion is

clamped, and the angular value is read and recorded. The distance *BC* is chained and recorded in a manner similar to that for *AB*. The transit is moved forward to *C*, a backsight is taken to *B*, the point *D* is chosen, a foresight is taken to *D*, and the angle is read. The line *CD* is chained. And so the process is repeated for points *E*, *F*, etc., until the traverse is finally brought to a closure on the initial point *A*.

Usually, as a means of checking the angular measurements against large errors, *magnetic* bearings are observed on both backsight and foresight from each station and are compared with *calculated* bearings determined from the measured angles as described in Chap. XI. Also as a check, after the traverse is brought to a closure, the initial station is occupied by the transit and the angle between the first and last lines of the traverse is measured; in this way the angular error of closure is determined. Where the traverse is not long, a sketch of the traverse, together with any other desired features of the survey, is usually shown on the right-hand page of the notebook, and the measurements are tabulated on the left-hand page, the numerical notes being kept down the page in order of the observations. Even though measurements to details are not included, it is usually customary to show on the sketch the approximate location of important objects.

A continuous or open traverse may be run in exactly the same manner, but the angular error of closure cannot, of course, be determined as described above. However, the continuous traverse may begin and close on previously established lines, the directions and positions of which are known. Where stakes are set every 100 ft. the chainage is referred to the point of beginning; and if the traverse is of considerable length, the notes are usually taken *up* the page, and for sketches the traverse line is considered as being on the center line of the right-hand page. Also in the notebook a line is given to each station and plus. This manner of keeping notes facilitates sketching, since objects represented will, when the transitman has his back to the zero end of the line, appear in the same relative position with respect to the transit line in the notebook as they appear with respect to the transit line on the ground.

Usually the location of a point is indicated to the transitman by the range pole held in a position estimated to be vertical. For short sights, where unobstructed vision is possible, a pencil or chaining pin makes a finer target. Where the view near the ground is obstructed, the plumb line may be used for short sights; when the range pole is used it needs to be held with care.

Where stakes are set every 100 ft., those marking intermediate points between transit stations are not usually tacked except where they mark

the final location of some very definite line, as for example, the center line of railroad track. On rough surveys distance is measured by the rear chainman holding his end of the tape to the center of base of the last stake driven and the head chainman thrusting the end of the range pole in the ground at the other end. The rear chainman before leaving the point calls out the number of the station as marked on the stake, and the head chainman replies with the number of the station at the range pole, at the same time marking the stake which is to be set at the new station. He then removes the range pole from the ground, at the same time putting the stake in its place, and the rear chainman, when he comes forward, drives the stake and checks its number. Often the head chainman carries a small notebook in which he records the number of each station as soon as the stake is marked and set. On more accurate surveys, linear measurements are carried forward by means of chaining pins as described in Chap. VI, each stake being driven by the rear chainman as he pulls the pin. Where tacks are set in intermediate stakes the chainage is carried forward from tack to tack.

Usually when the transit is to occupy a station for any considerable length of time, there is more or less likelihood of its being disturbed. To detect any movement of the lower motion, the transitman often observes the angle to some prominent feature of the landscape just after having taken a backsight to the preceding transit point, and occasionally thereafter he again sights at the reference mark and notes the angle. If no change is observed, it is proof that there has been no accidental rotation of the horizontal circle. The transitman should invariably apply this check as the last operation before leaving any station.

The procedure of traversing with the transit as just described is, of course, modified to conform with the purpose of the survey and with the conditions under which the work is prosecuted. Where the traverse is through wooded country, it is impossible to establish transit stations in advance of the instrument until the line has been cleared. Consequently, a foresight is taken in the general direction of the proposed station and the clearing proceeds until a favorable location for the advance transit station is encountered. If 100-ft. stations are to be established, the stakes are driven on a fixed line as fast as clearing will allow. When the line is extended to the appropriate locality, the transitman lines in the hub in the same manner as the intermediate stakes.

Often one or more transit stations are necessary between points at which angles are turned. Usually the line is prolonged beyond such stations by plunging the telescope rather than by turning 180° on the upper motion. Where a number of stations lie between two adjacent angle points it is good practice to backsight at one station with the telescope in the direct position and at the next station with the telescope in the reversed position. When there is any doubt as to the accuracy of

adjustments or when the precision demands it, the line should be prolonged by the method of double-sighting (see Art. 199). In the notes the stations thus established are indicated in the same manner as are the transit points at which angles are turned.

The common methods of running transit traverses are: (1) by deflection angles, (2) by azimuths, (3) by interior angles, and (4) by azimuths from back line. The magnetic compass with which the transit is equipped may be employed for running magnetic traverses, and in this respect it is more useful than is generally appreciated, particularly for rough preliminary or reconnaissance surveys.

Formerly it was common practice to run bearing traverses by means of the horizontal circle graduated in quadrants as illustrated in Fig. 192*c*, but the azimuth method has such marked advantages over the bearing method that the latter is not often used except with the compass.

226. Deflection-angle Traverse.—This method of running traverses is probably more commonly employed than any other, especially on continuous traverses where only a few details are located as the traverse is run. It is used, practically to the exclusion of other methods, for the location surveys for roads, railroads, canals, pipe lines, etc. It is employed to a less extent in land surveying and in establishing control traverses for topographic and hydrographic surveys.

Successive transit stations are occupied and at each station a backsight is taken with the *A* vernier set at zero and the telescope in the reversed position. The telescope is then plunged to the direct position, the foresight is taken by turning the instrument on its upper motion, and the deflection angle is observed. The angle is recorded as right *R* or left *L*, according to whether the upper motion is turned clockwise or counter-clockwise.

Figure 226*a* illustrates the field notes for a closed traverse run by the deflection method. It will be seen that they are kept down the page. Magnetic bearings have been observed forward and back from each station. For any closed traverse the summation of the deflection angles, considering those turned to the right as being of opposite sign to those turned to the left, should equal 360°. The actual sum indicated in the notes shows that there existed an angular error of closure of 01′. Deflection angles computed from the observed magnetic bearings are shown in next to the last column of the left-hand page. These values will be free from local attraction and should agree closely with the corresponding observed deflection angles. In the last column are the bearings as calculated from the observed deflection angles, assuming the calculated and observed

bearings of BC to be the same. Usually one or the other of the last two columns is omitted. The values given in both are used as a rough check as the work of running the traverse progresses. The calculated bearings are often shown on land plats and are useful in computing areas and coordinates.

Figure 226*b* shows the notes for a portion of a continuous or open deflection-angle traverse where stakes are set every 100 ft. For the sketches the center line of the right-hand page represents the traverse. The notes are kept up the page, and observations are checked against

Deflection Angle
Siggs, N.B.

Sta At–To	Dist. —	Def. Ang.	Mag Bear.	Ang by Bear	Cal Bear.
A E		57°54′R	N 28°W	58° R	
B	507.65		S 30° W		S 29°37′W
B A		113°38′L	N 29½°E	113½° L	
C	784.68		S 84°E		S 84°00′E
C B		98°15′L	N 84°W	98¼° L	
D	994.60		N 2¼°W		N 2°15′W
D C		88°19′L	S 2°E	88¼° L	
E	739.72		S 89½°W		S 89°26′W
E D		117°43′L	N 89¼°E		
A	526.00		S 28¼°E	117½° L	S 28°17′E
		360°01′			

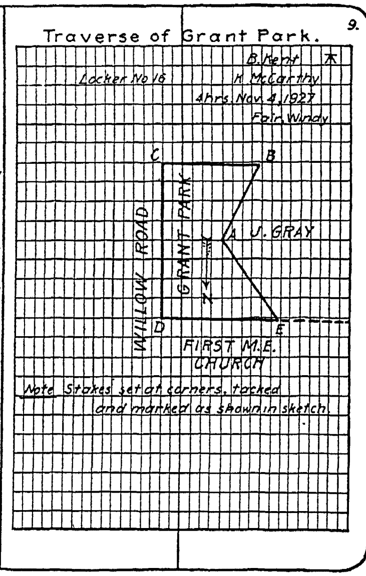


FIG. 226*a*.—Notes for deflection-angle traverse.

large errors by the close agreement between calculated bearings and observed magnetic bearings. The notes are typical of the form used on railroad, highway, and other route surveys.

The method just described is open to the objection that plunging the telescope from the reversed to the direct position introduces a constant error in each observed angle if the line of sight is not in perfect adjustment. If there is a large number of lines in the traverse, the total angular error introduced in this manner may become considerable, even though the error at each individual station is less than the least reading of the vernier. For this reason, it is better practice to set the A vernier at 180°, take the backsight, and then turn the instrument on its upper motion (instead of plunging) to the foresight. If the practice of plunging the telescope is

followed, one backsight should be taken with the telescope direct and the next backsight with the telescope reversed.

The precision of measurements is increased somewhat, and at the same time each observation is checked, by doubling the angle, a common practice on important surveys. Both the single and the doubled values are usually recorded. The deflection angle is considered as being one half of the doubled value. To eliminate instrumental errors, the first backsight from a given station is taken with the telescope direct and the second backsight is taken with the telescope reversed. The telescope is usually plunged between backsight and corresponding foresight. The procedure just outlined checks the angles within the least count of the

Preliminary Def. Ang. Traverse. Survey, I.N.Ry. C. to St.L. 10.

Sta.	Def.Ang.	Mag.B.	Cal.B.	Remarks.	
12					
+82				Water	Line
+17				"	"
11					
10					
9					
⊙8+36.3	23°14'L	N35°W	N34°47'W		
8					
+99.8					
+68				℄ Due E	to St.L.
+33.1					
7					
6+00				Ctr. of	Brook.
5					
4					
3					
2					
+23.6					
1					
⊙0+00		N11°W	N11°33'W	on C.L.	C.P.Ry.
				523'N	M'l Post-

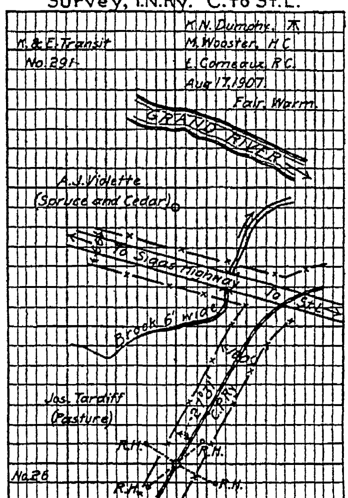


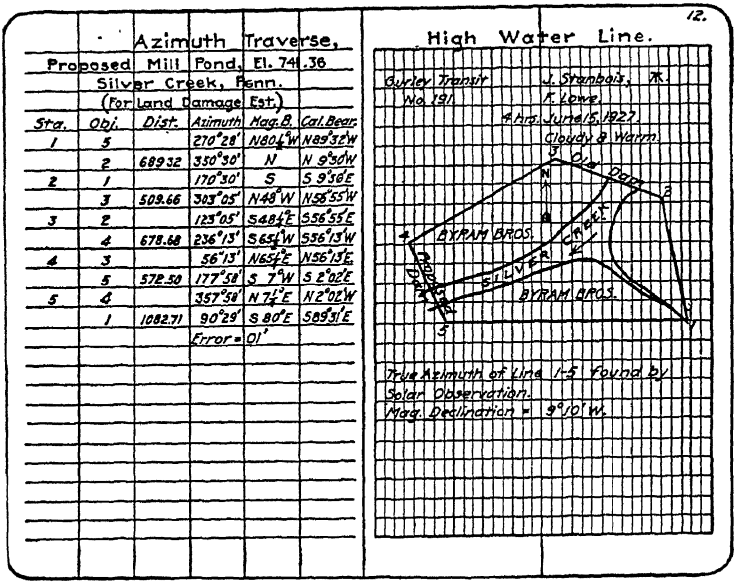
FIG. 226*b*.—Notes for continuous deflection-angle traverse.

vernier and hence furnishes a much closer verification than does the method of checking by magnetic bearings. Ordinarily if the angles are doubled, magnetic bearings are not observed.

The primary traverses of the U. S. Geological Survey are established by a procedure somewhat modified from that just described. At each station a backsight is taken with the verniers set at the last reading taken from the preceding station, the telescope is plunged, a foresight is taken, and both verniers are read. With the upper motion clamped the transit is rotated about the vertical axis, and a second backsight is taken. The telescope is again plunged, a foresight is taken, and both verniers are read. From the observed readings two angles are calculated, and the maximum allowable difference is 60″. The deflection angle is

taken as the mean of the two values thus determined. From the deflection angles, the azimuths of the traverse lines are calculated.

227. Azimuth Traverse.—The azimuth method possesses an advantage over the other common methods of traversing with the transit in that the simple statement of an angular value fixes the direction of the line to which it refers. The method is extensively used on topographic and other surveys where a large number of details are located by angular and linear measurements from transit points. The simple relation existing between azimuths and bearings



Azimuth Traverse,
Proposed Mill Pond, El. 741.36
Silver Creek, Penn.
(For Land Damage Est.)

Sta.	Obj.	Dist.	Azimuth	Mag. B.	Cal. Bear.
1	5		270°28′	N80¼°W	N89°32′W
	2	689.32	350°30′	N	N 9°30′W
2	1		170°30′	S	S 9°30′E
	3	509.66	303°05′	N48°W	N56°55′W
3	2		123°05′	S48¼°E	S56°55′E
	4	678.68	236°13′	S65¼°W	S56°13′W
4	3		56°13′	N65¼°E	N56°13′E
	5	572.50	177°58′	S 7°W	S 2°02′E
5	4		357°58′	N 7¼°E	N 2°02′W
	1	1082.71	90°29′	S 80°E	S89°31′E
			Error =	01′	

Fig. 227.—Notes for azimuth traverse.

makes it possible to calculate one from the other at a glance. Also the angular error of closure of a traverse is at once evident by the difference between the initial and final observations taken along the first line.

The azimuth of the initial line of the traverse may be referred either to a true or an assumed meridian. Successive stations are occupied beginning with the line of known azimuth. At each station the transit is *oriented* by setting the *A* vernier to read the back azimuth of the preceding line and then backsighting to the preceding transit point. The instrument is then turned on the upper motion, and a foresight on the following transit point is secured. The reading indicated by the *A* vernier gives the azimuth of the forward line.

Figure 227 illustrates the notes for a closed traverse for which the azimuths were observed as just described. Azimuths set off on the circle and used as backsight readings are called *back azimuths*, and those observed when the foresights have been taken are called *forward azimuths*.

As indicated in the notes, each line has both a forward and a back azimuth whose values differ by 180°. The traverse is started from the line 1–5 whose azimuth is 270°28′ reckoned from true north. The forward azimuth of line 1-2 is found to be 350°30′. When the transit is set up at station 2, the back azimuth is calculated by subtracting 180° from the forward azimuth (350°30′ − 180° = 170°30′), and this value is set on the vernier before the backsight to station 1 is taken. Magnetic bearings are observed, and a check against large errors is secured by noting that the calculated bearings vary from corresponding magnetic bearings by about 9°10′, the amount of the magnetic declination.

227a. The procedure just described is often modified by plunging the telescope between each backsight and the corresponding foresight, and leaving the vernier setting unchanged between a foresight and the following backsight. In other words, if AB is some transit line in the traverse and a foresight reading has been taken from A to B with telescope direct, the transit is brought forward to B without disturbing the vernier setting, and a backsight on A is taken with the telescope reversed. It is then plunged to the direct position (which orients the transit), and the foresight to C is obtained by turning on the upper motion. The reading of the A vernier gives the azimuth of the line BC. The advantage of plunging the telescope over changing the vernier reading by 180° lies in the increased speed with which the azimuths may be measured and in the smaller probable error of determining azimuth of a line due to accidental errors of reading the vernier. That is, so far as errors of reading are concerned, no error is introduced beyond the first setting, the reason for this being that only one initial setting of the vernier is made, that being for the first backsight taken. The disadvantages are that a mistake may be made in reading and recording an angle without its becoming evident at closure of the traverse, and that unless the line of sight is in perfect adjustment a constant angular error is introduced at each set-up. The vernier should always be read before a backsight is taken to make sure that no slip between plates has occurred while the transit was being brought forward.

227b. A third method of running an azimuth traverse possesses certain marked advantages over either of those described. The procedure is practically the same as that described in Art. 227, except that instead of the vernier reading being increased 180° for backsight, the other vernier is employed. It is evident that if there has been no slip between plates as the transit is carried forward from one station to the next, the vernier opposite to that registering the forward azimuth of the line will indicate the back azimuth. For convenience in reading this

vernier the telescope is usually plunged prior to each backsight, but it should be noted that the telescope is not plunged between backsight and following foresight. If the upper motion is left clamped after each foresight until the succeeding backsight has been taken in the manner just described, and both foresight and backsight from any point are taken with the telescope in the same position (direct or reversed) and with the same vernier, neither accidental error due to imperfect vernier settings nor constant error due to the line of sight's not being perpendicular to the horizontal axis will be introduced. To guard against reading the wrong vernier or sighting with the telescope in the wrong position, the station numbers in the notes may be alternately marked *A* and *B*, or it may be noted that at odd-numbered stations azimuths are read by means of the *A* vernier with the telescope normal, and at even-numbered stations, by means of the *B* vernier with telescope reversed.

228. Interior-angle Traverse.—This method of traversing is used principally in land surveying. So far as the field operations are concerned it is not materially different from the deflection method. At any station the vernier is set at zero and a backsight to the preceding transit point is taken. The instrument is then turned on the upper motion until the advance station is sighted, and the interior angle is read. The notes may be kept in the form of a sketch on which the observed angles and distances are shown, or the numerical values may be tabulated in form similar to that of Fig. 226*a*. Angles may be checked by means of the geometrical relation that in any polygon having n sides the sum of the interior angles is $(n - 2)\ 180°$.

229. Traverse by Azimuth from Back Line.—This method is employed less commonly than any of those previously described. It is used mostly on continuous traverses, particularly where numerous details are to be located from the traverse stations by angular and linear measurements. For such work the chances of confusion are considerably less than when the deflection-angle method is employed. At any station a backsight is taken to the preceding point with the *A* vernier set at zero. The instrument is turned on the upper motion and a foresight to the following station is taken. The clockwise angle indicated by the *A* vernier is the azimuth of the forward line referred to the back line. Notes are kept in much the same form as for deflection angles (see Figs. 226*a*, 226*b*). Traverse angles are checked either by magnetic bearings or by doubling.

230. Methods of Checking Traverses.—At the expense of some repetition, the common methods of checking traverses will here be discussed. The errors involved in traversing are of two kinds, angular and linear.

For a closed traverse certain fixed geometrical relations exist so that it is possible to determine readily the angular error of closure,

i.e., the amount that the measured sum of the angles differs from the true sum. Thus, for the deflection-angle traverse the algebraic sum of the deflections should be 360°; for the interior-angle traverse the sum of the interior angles should equal $(n - 2)$ 180°; and for the azimuth traverse the azimuth of the first line as observed at closure should equal the known or assumed azimuth of the same line as employed at the beginning of the survey.

The actual total error in measured distances cannot be determined. But the coordinates of the first point as calculated by successive angles and distances around the traverse should equal the coordinates of this same point as used at the beginning of computations, and this condition renders it possible to calculate the *linear error of closure*, due

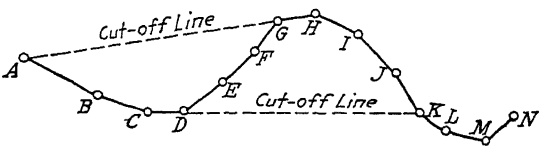


FIG. 230.

to errors both in angles and in distances, or the difference between the initial and final computed positions of the initial point of the traverse. The process of computing the linear error of closure is given in Chap. XV.

For an open or continuous traverse no such means of checking the measurements as a whole are available. But while linear errors cannot be detected, angular errors may be closely determined by astronomical observations taken at intervals as the traverse progresses. Where conditions allow, cut-off lines run between certain intermediate points are sometimes established, these lines making it possible to check parts of the traverse. Thus in Fig. 230 the line *ABC . . . N* represents a continuous traverse. If the direction of the cut-off line *AG* is observed, at both *A* and *G*, the angular measurements of the traverse from *A* to *G* may be checked, and further if the distance *AG* is measured the linear error of closure may be computed. Similar measurements for the cut-off line *DK* would enable the checking of the traverse up to *K*. Wherever conditions make running a cut-off line impracticable, there will be a portion of the main traverse line which cannot be checked by this method. The method really amounts to running a succession of closed traverses. Conditions favorable to establishing a cut-off line are met only occasionally, and then it is often not convenient to measure the length of the line.

By observing the angle to some distant landmark from each of several stations, several determinations of the rectangular coordinates of the mark may be obtained. If these values agree closely, it is good evidence that all angular and linear measurements which are involved are free from serious errors. If there is disagreement between computed coordinates, there is no way of always telling with certainty just where the error lies but usually its location may be fixed approximately.

On closed traverses it is nearly always customary to calculate the angular error of closure in the field. When the linear error is determined, as is the case on all important traverses, the calculations are made in the office. Where graphical methods are of sufficient precision, a short traverse may be checked expeditiously by plotting with protractor and scale, or by other methods later to be described; but for long, many-sided traverses the accumulative errors of plotting are likely to be large and there is no way of determining whether the linear error of closure is due to inaccuracies in the field work or in the drafting.

While the methods of checking continuous traverses already described are employed where opportunity offers, ordinarily it is impossible thus to verify more than a small portion of the total number of observations taken, and the errors in the traverse as a whole cannot be determined. Not infrequently the continuous traverse begins and ends at points, the positions of which have been accurately determined by previous field operations (for example, the triangulation stations of the U. S. Coast and Geodetic Survey). In such cases the error of closure of traverse on the known point may be determined by comparing its established or accepted coordinates with those computed from the traverse observations.

Where conditions are such as to render the magnetic needle a dependable device, magnetic bearings offer an excellent means of checking observed angles against large errors or mistakes, and on traverses of ordinary precision this check is usually applied by reading the compass needle on the transit and then comparing the observed bearing with the bearing calculated from observed transit angles.

On important traverses, particularly those of extensive surveys or of surveys that do not contain closed figures, the angular values are usually verified by doubling the angles, and often the linear measurements are checked by chaining forward and back over each line.

231. Precision of Transit Traverse.—This is affected by angular as well as by linear errors of measurement. The precision of linear measurements with the tape was discussed in Art. 92, p. 97; the precision of angular measurements with the transit was discussed

in Art. 214. It will be remembered that under ordinary conditions the angular errors are largely accidental in character, and that hence the probable error in the direction of any line in a traverse with respect to any other may be expected in general to vary as the square root of the number of set-ups between the two lines. On the other hand the important linear errors on traverses of ordinary precision are likely to be systematic in character. The precision of the position of any transit point therefore is generally influenced much more by the systematic linear errors than by the accidental errors, and except on surveys of high accuracy, the precision is usually found to vary approximately as the length of the traverse lines. In stating limits of error in transit work the ratio of linear precision $\left(\text{as } \frac{1}{5{,}000}\right)$ is used.

Unless a traverse forms a closed figure, or begins and ends on points previously established by measurements known to be practically correct, its precision is indeterminate, but if the proper procedure is employed (see Arts. 92 and 214), and if the several angles and distances making up the traverse are checked by a second measurement or by other methods previously discussed, there is reasonable assurance that the accuracy will not be below a fixed standard. The required precision of a traverse, whether open or closed, depends upon the character, purpose, and extent of the survey, and for any given case is a value to be fixed after due consideration of the factors involved.

On important or extensive surveys it is common practice for those in charge to issue definite instructions covering in detail the procedure to be followed by the several members of the surveying staff and specifying the maximum allowable discrepancies between check measurements or otherwise signifying the precision which it is desired to maintain. Where a traverse forms a closed figure, the angular error of closure is ordinarily determined through fixed geometrical relations already mentioned, and the linear error of closure is determined by trigonometric computations which will be described in a later chapter. Often limits are placed upon the angular as well as the linear error of closure.

Limitations as regards the skill of surveying personnel, quality of instruments, and field conditions are so numerous as to make any statement of the accuracy to be attained in traversing with the transit of only very general value. The following specifications have been prepared with this in mind, and the values therein given are not by any means considered fixed. The specifications give *maximum* values, and if executed by well-trained men, with instruments in good adjustment, and under average field conditions, the error of closure should, in general, be not more than half the specified amount. It should be noted that even

in rough chaining, the effects of the systematic errors can be greatly reduced by easily calculated corrections applied to the measured values (Arts. 86 to 92). This fact is frequently overlooked, even by experienced surveyors. In these specifications it is assumed that a standardized tape is used.

231a. Specifications for Traversing. *Class* 1.—Total error of closure not to exceed $\frac{1}{1{,}000}$. Transit angles to be read to the nearest minute. Sights to be taken on a range pole plumbed by eye. Allowable angular error of closure, $1'30''\sqrt{n}$ in which n is the number of observations. Distances to be measured with a 100-ft. steel tape. Disregard slopes under 3 per cent. On slopes over 3 per cent, distances to be measured, either on the slope and corrections applied, or with the tape held level using an estimated standard pull. Pins or stakes to be set within 0.1 ft. of the end of the tape. Precision sufficient for many preliminary surveys, horizontal control for topographic surveys plotted to intermediate scale, and for land surveys where the value of the land is low.

Class 2.—Total error of closure not to exceed $\frac{1}{3{,}000}$. Transit angles read carefully to the nearest minute. Sights taken on range pole carefully plumbed. Allowable angular error of closure to be $1'\sqrt{n}$. A correction to be applied to the linear measurements if the temperature differs more than 15°F. from standard. Disregard slopes under 2 per cent. Distances on slopes over 2 per cent to be measured either on the slope and corrections applied, or the tape to be held level and the standard pull carefully estimated. Pins or stakes to be set within 0.05 ft. of the end of the tape. Precision sufficient for most land surveys, for the location of railroads, highways, etc. By far the greater number of transit traverses fall in this class.

Class 3.—Total error not to exceed $\frac{1}{5{,}000}$. Transit angles read twice with the instrument reversed between readings. Sights taken on a plumb line or on a range pole carefully plumbed. Allowable angular error of closure, $30''\sqrt{n}$. Temperature to be determined within 10°F. and corrections applied to the linear measurements. All tape measurements on slopes to be corrected. If the tape is held level, the pull to vary not more than 5 lb. from the standard, and corrections for sag to be applied. Pins to be set within 0.05 ft. of the end of the tape. Precision sufficient for much of the work of city surveying, for surveys of important boundaries, and for the control for extensive topographic surveys.

Class 4.—Total error of closure not to exceed $\frac{1}{10{,}000}$. Transit angles read twice with the instrument reversed between readings, each reading being taken as the mean of both *A* and *B* vernier readings. Verniers reading to 30″. Instrument in excellent adjustment. Allowable angular error of closure $15''\sqrt{n}$. Temperature of tape to be determined within 5°F. and corrections applied. Slopes to be determined within 2 per cent and corrections applied. When the tape is held level, the tension to vary

not more than 3 lb. from standard and corrections for sag to be applied. Pins to be set within 0.03 ft. of the end of the tape. Precision sufficient for accurate city surveying and for other specially important surveys.

232. Referencing Transit Stations.—In many instances transit stations are in locations where they are likely to be disturbed or to be destroyed before the usefulness of the transit lines as lines of reference has expired. Many of the hubs marking the location of highways, railroads, and other works of man are bound to be uprooted or covered during the progress of construction, and they must be replaced, often more than once, before construction is completed. Such hubs marking the transit stations are usually tied by angular or linear measurements to temporary wooden hubs called *reference hubs*, or to other objects that are not likely to be disturbed. A

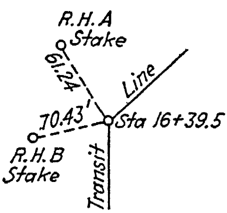


FIG. 232*a*.

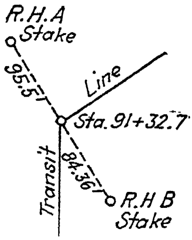


FIG. 232*b*.

transit station is said to be *referenced* when it is so tied to nearby objects that it can be readily replaced. The manner of referencing a transit station is indicated in the notes by an appropriate sketch showing not only the angular and linear reference measurements but also the character of the reference marks. Particularly on land surveys, the corners should be tied to nearby objects which can be readily found, which are not likely to be moved or obliterated, and which are of a more or less permanent character.

Often in land surveying a corner is said to be *witnessed* when angular and linear measurements are taken to nearby objects of the character just mentioned, and each of the objects is called a *witness mark*. This designation is unfortunate and leads to the confusion of ties or reference marks for a corner which has been established, with witness corners which are markers set on each of the four land lines leading to a corner when that corner falls in a place where it would be either impossible or impracticable to establish or to maintain a monument. The importance of properly referencing corners established on land surveys can hardly be overemphasized.

Figures 232*a* to 232*d* illustrate several methods of referencing a transit station. The station shown in Fig. 232*a* is tied by linear

measurements to two reference hubs. While the original measurements can be made very expeditiously the field work connected with replacement of the station is not so simple, and if either of the reference hubs is disturbed the station cannot be relocated. The tie lines should intersect at a favorable angle (preferably about 90°) otherwise the station cannot be relocated with certainty.

Figure 232*b* shows a station referenced by setting the two reference hubs on a line passing through the station and by taking linear measurements from these hubs to the station. Replacement is accomplished by setting up the transit at R.H.*A* and sighting to R.H.*B* (or *vice versa*) and then laying off along the line thus established the

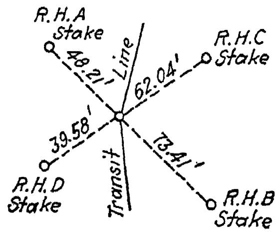


Fig. 232*c*.

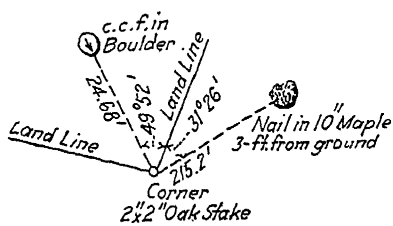


Fig. 232*d*.

given distance from reference hub to station; only one of the linear measurements is necessary, but the other is desirable as a check. If either of the reference hubs is destroyed the station cannot be relocated.

Figure 232*c* represents a third method, often employed when there is any likelihood of the reference hubs being accidentally lost. The station is at the intersection of the line *AB* and the line *CD*. With the measurements as shown, any two reference hubs may be destroyed and still the station may be relocated. Sometimes it is more convenient to place all the reference points to one side of the transit line and to locate the station by angular measurement only.

Figure 232*d* is typical of the manner in which corners are referenced in land surveying.

233. Details from Transit Lines.—On nearly all transit surveys, a part of the work consists in the location of certain details with respect to the transit lines. The nature and number of details to which measurements are taken depends upon the purpose of the survey and upon the character of the country through which the survey runs. On the one hand a survey for the purpose of establishing or relocating boundaries of land would include the location of only a few important objects close to the transit lines. On the other hand a

complete topographic survey might include the location of all natural and artificial features of the terrain.

The accuracy with which details are located likewise depends upon the purpose of the survey. In retracing property lines not infrequently the actual lines are obstructed by hedges or buildings, so that it is necessary to run transit lines approximately parallel with the property lines, and to locate the corners by measurements from transit lines; such measurements would need to be taken with a precision as great as that for the transit line. The survey for a map ought to be so conducted that all objects may be correctly shown within the scale of the map. Thus if the scale of the map were 1 in. = 1,000 ft. there would be no particular advantage in taking measurements closer than the nearest 10 ft. from a transit line to, say, a building; but if the scale were 1 in. = 10 ft. measurements ought to be taken to tenths of feet. Generally, if details are located solely for map-making purposes, the required precision of measurements to details is less than the required precision of measurements for the transit lines; this is particularly true for extensive surveys.

Angular measurements are usually made with the transit. In most cases angles are read to minutes, but where angles are to be used only in mapping operations and their correct numerical value is of no particular consequence, usually nothing is to be gained by reading closer than 05′. For the ordinary transit, angles may be estimated to the nearest 05′ without the aid of the vernier, and where a number of observations are to be made, estimating the angles in this way results in a material saving in time. When details are located with respect to stations intermediate between transit stations, angular measurements are frequently made with some hand instrument such as the Brunton pocket transit or the sextant.

Linear measurements to details are made with the 100-ft. steel tape, with the metallic tape, with the stadia, or sometimes by pacing. Where distances are long and where a high precision is required, the steel tape is employed. Where a considerable number of short distances are to be measured, the metallic tape may be used more expeditiously than the steel tape and such measurements are sufficiently accurate for most purposes. Where the survey is for the purpose of securing data for a map, distances to details are often obtained by the stadia method (see Chap. XIV), which is sufficiently accurate except for very large scales. Distances to details of indefinite outline (for example the bank of a stream or the edge of a wood) are sometimes determined by pacing.

On most surveys the details are located as the work of establishing the transit lines progresses; thus as a traverse is being run, at any transit station after the foresight to the following station has been observed, the transit is left in position until angles to details have been read. These

observations with the transit are called *side shots*, to distinguish them from the traverse angles.

233a. Locating Details.—Following are descriptions of the common methods of locating details with the transit and tape. In general, no matter how many points of an object have been located, its dimensions should be determined by direct measurement.

1. *By Angle and Distance from Transit Station.*—As illustrated by Fig. 233*a*, any given point on the object is located by an angle and a distance. At a given transit station distances are measured to such details as may be conveniently located from that station, and usually

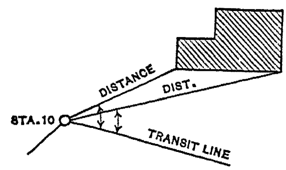


FIG. 233*a*.

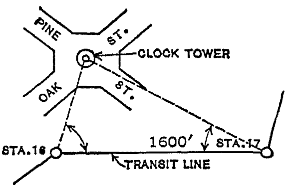


FIG. 233*b*.

at the same time the transit is set up at the station and angles are observed between the transit line and the given details. The method is widely used, particularly where the country through which the survey passes is open and where the details are not remote from transit stations. Generally the notes show by sketches the character and general position of the objects thus located. Where details are not numerous the angles and distances are shown in the proper location on the sketch, but if a considerable number of observations are to be made from a single transit station, each observed point is given a number on the sketch, and angles and distances are tabulated opposite corresponding numbers on the left-hand page of the notebook. If angular values are not placed on the sketch, care should be taken that the notes make evident the manner in which the angles were observed and the transit line to which the angles are referred. Where details are numerous, azimuth angles, rather than deflection angles, are observed for the reason that the chances of later confusion are much less. The azimuths may be referred either to a fixed meridian, or, if the transit line is a traverse, may be referred to the back line from each transit station.

2. *By Angles from Two Transit Stations.*—This method is particularly useful in locating distant or inaccessible objects which can be seen from two or more transit stations. As shown by Fig. 233*b*, the position of any point is located with respect to the transit line if angles to the point are taken from at least two stations. The par-

ticular advantage of the method is that no linear measurements, other than those made in running the transit lines, are required; hence the field work is reduced. The disadvantage is that the distance from transit station to point sighted can be determined only by computation (except that a rough value can be scaled from a map). Also the location of the point becomes indefinite as the angle at the point approaches 0° and 180°. The angular values may be tabulated or may be shown on a sketch.

3. *By Distances from Two Stations.*—On transit traverses for which stakes are set every 100 ft. this method of locating details sometimes expedites the work, particularly when the details are close to the traverse line, yet distant from the nearest transit station. A given object is located by linear ties from two traverse stations. Thus, in Fig. 233*c* the stone bound at the fence corner is located by distances from stations 4 + 00 and 5 + 00 neither of which is a transit station.

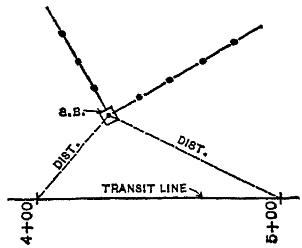


FIG. 233*c*.

4. *By Perpendicular Offsets from the Transit Line.*—This method is adapted to the location of irregular or curved boundaries, streams, drives, and other features of character similar to those mentioned which closely parallel the transit lines. As indicated by Fig. 233*d*,

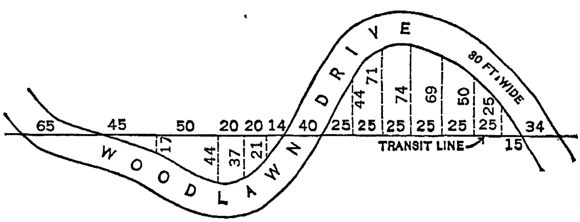


FIG. 233*d*.

a point is located by measuring the distance along the transit line to the foot of the perpendicular offset through the point and then measuring the length of the perpendicular offset. Features such as those mentioned are located sometimes by offsets at regular intervals, but more often at critical points which will make the offsets come at irregular intervals. Where stakes are placed every 100 ft. along the transit line, the station and plus of the foot of each perpendicular is secured, rather than the distance between offsets as shown.

Unless the measurements are to be of more than ordinary precision the directions of the perpendiculars are usually estimated by eye,

except where offsets are long; in this case their directions are established with the tape, magnetic compass, sextant, or optical square. Where extreme precision is required the offsets are laid off with the transit.

In the notes, the nature of the features is shown by sketches. Sometimes the numerical values of offsets are written on the sketch (as in Fig. 233*d*), but where a large number of measurements are taken it is more convenient to tabulate these values on the left-hand page of the notebook, the stations and plusses of the offsets being shown in the first column and corresponding values of the offsets with their directions (right or left) from the transit line in the second column.

5. *By Angle from One Station and Distance from Another.*—This method is not employed as frequently as the others just described, but occasionally it will be found useful. Angles are measured with the transit set up at a transit station; distances are measured from stations intermediate between transit stations. Figure 233*e* illustrates a situation where the method would prove advantageous, the lake making impossible the direct measurement of the distance from the transit station at 16 + 90.5 to the corner of the building, but there being no obstacle to the measurement of the distance from station 15 + 00 to the corner.

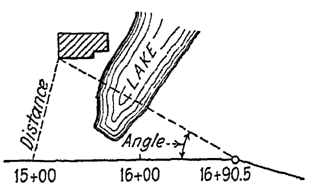


Fig. 233*e*.

6. *By Range Ties and Swing Offsets.*—In addition to the methods previously described, if the features to be located are buildings, the work of location may be facilitated by ranging in (by eye) the sides of the building and finding the points of intersection of lines thus defined with the transit line. The station and plus of a point of intersection together with the distance along the range line from the point of intersection to a corner of the building is called a *range tie*.

A *swing offset* is the perpendicular distance from a transit line to a point and is determined by swinging the tape about the point as a center and taping the least distance from the point to the transit line.

Figure 233*f* shows how a group of buildings near a traverse line may be located by tape measurements only. The building at the right is completely located by two range ties, one intersecting the traverse line at 12 + 18 and the other intersecting the traverse line at 13 + 26. The location of the building is checked (assuming that the lengths of sides are measured) by the check tie to station 14 + 00. The

position of the middle building of the group is established by ties to station 12 + 00 and station 13 + 00, three ties being sufficient to locate the building and the fourth tie being sufficient to check the location. The position of the building on the left is fixed by the range tie intersecting the traverse line at 11 + 11 and by the swing offset. The location is checked by the range line tying the three buildings together.

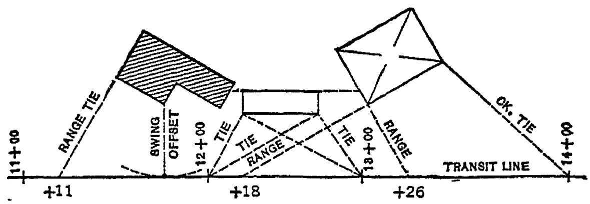


FIG. 233*f*.

REFERENCES

1. CUNNINGHAM, P. D., "The Accurate Measurement of Angles with the Engineer's Transit," *Eng. News*, Vol. 45, No. 22, p. 402, May 30, 1901.
2. GOULD, E. S., "Hints upon Transit Surveys," *Eng. News*, Vol. 43, No. 1, p. 2, Jan. 4, 1900.
3. HOLDEN, E. H., "Primary Traverse Work in the Bronx, New York City," *Eng. News*, Vol. 63, No. 20, p. 586, May 19, 1910.
4. STEWART, J. T., "Field Methods of Triangulation in the Plains Country in Montana," *Eng. News*, Vol. 55, No. 15, p. 407, Apr. 12, 1906.
5. "Tunnel Alignment in Three Aqueduct Tunnels," *Eng. News*, Vol. 67, p. 1183, June 20, 1912.
6. "Topographic Instructions of the United States Geological Survey," Latest Edition. Government Printing Office, Washington, D. C.
7. "General Instructions for the Field Work of the United States Coast and Geodetic Survey." Special Publication No. 26. Government Printing Office, Washington, D. C.
8. See references under "Topographic Surveying."

CHAPTER XIV

STADIA SURVEYING

234. Purpose.—The stadia method of measuring distances is adapted to a wide variety of conditions, and is extensively employed on topographic and hydrographic surveys and other surveys conducted for the purpose of securing data for the plotting of maps (see Art. 245). It is far more rapid than chaining, and under certain conditions is as accurate. It is a useful means of checking more precise measurements.

In this chapter the principles of the stadia are explained, and stadia surveys are described.

235. The Stadia Method.—The equipment for stadia measurements consists of a telescope with two horizontal hairs, called *stadia hairs*, and a graduated rod, called a *stadia rod*. The telescopes of most transits, all plane-table alidades, and many levels are furnished with stadia hairs in addition to the regular cross-hairs, one stadia hair being above and the other being an equal distance below the horizontal cross-hair. The rod is usually graduated in decimals of a foot, but may be graduated in decimals of a meter or a yard.

The process of taking a stadia measurement consists in observing through the telescope the apparent interval between the two stadia hairs, the rod being held in a vertical position. The interval thus determined, called the *stadia interval* or *stadia reading*, is a direct function of the distance from the instrument to the rod, as will shortly be demonstrated. Thus, the distance from the instrument to any given point is determined by observing the stadia interval on the rod held at the point.

236. Stadia Hairs.—Stadia hairs are usually mounted on the same ring and in the same plane as the horizontal and vertical cross-hairs. Under these conditions the stadia hairs are not adjustable with respect to one another and hence the distance between hairs remains unchanged. Both stadia hairs and cross-hairs are simultaneously visible and in focus. The advantage of fixed stadia hairs is that the interval between them cannot be accidentally altered. The disadvantage is that the hairs may be so placed as to produce an inconvenient stadia interval factor (ratio of distance to interval), but considering the precision with which the hairs are usually placed

by the manufacturer, the interval factor is so nearly 100 that frequently it may be so considered without appreciable error.

To eliminate the possibility of confusing the stadia hairs with the horizontal cross-hair in ordinary transit or level work, the stadia hairs are sometimes mounted in a plane a short distance in the rear of the plane of the cross-hairs. Under these conditions the stadia hairs and cross-hairs are not simultaneously visible, and it is necessary to change the focus of the eyepiece to render visible the stadia hairs when the ordinary cross-hairs have been in use. Stadia hairs mounted as just described are called *disappearing stadia hairs.*

Formerly instruments were manufactured with adjustable stadia hairs, so that the interval between hairs could be regulated to make the interval factor any desired quantity. In general, the adjustable feature is not regarded with favor, owing to the fact that the adjustment is likely to be accidentally disturbed.

237. Stadia Rods.—Any leveling rod of the self-reading type may be used as a stadia rod, but the common leveling rod graduated in hundredths of feet as illustrated by Fig. 109*a*, p. 135, is suitable only for short sights, say less than 400 ft. For longer sights a rod with larger and heavier graduations is necessary. The graduations shown by Fig. 109*b* are suitable for distances up to 700 ft. Figures 109*c* to 109*e* illustrate patterns which combine fine graduations by means of which stadia intervals may be accurately read at short distances and heavy graduations by means of which stadia intervals may be readily observed at long distances. These rods are equally adapted for use in stadia surveying and in leveling. For very long distances the graduations of Fig. 109*e* are superior to those of Figs. 109*b* and 109*c*. For stadia work alone the finer graduations are usually omitted and numbers indicating feet and tenths are often not shown, as illustrated by Fig. 237.

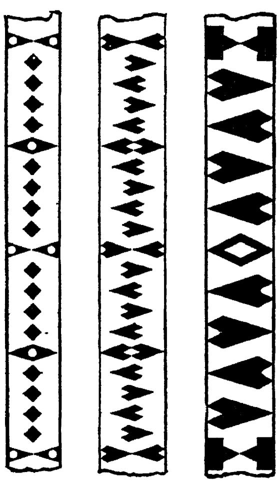


FIG. 237.—Stadia rods.

For general stadia work where the length of sight may be 1,500 ft. or greater, the rods are usually 3 or 4 in. wide and 10 to 15 ft. long, and the finest division is 0.05 ft. Stadia rods are usually made in one piece with graduations painted on the rod. For ease of transportation they are sometimes hinged or made in sections with a sleeve on

one end of each section. While rods of the patterns shown in Figs. 109 and 237 are procurable from manufacturers, most stadia rods are "home made" to the specifications of the individual surveyor. It is therefore not surprising that the number of patterns is large.

The wood for a rod should be well seasoned and should be from a light, tough, straight-grained species such as white spruce. The paint should be one which will withstand weather yet one which does not have a high gloss. The lacquer paints which can be applied with a brush, yet which dry quickly, are suitable. If the rod is varnished, it may be rubbed to a dull finish with powdered pumice or a similar abrasive.

So-called flexible stadia rods, consisting of graduated oilcloth ribbons, are on sale by instrument manufacturers. When such a ribbon is tacked to a board it makes a very satisfactory stadia rod. When removed from the board and rolled up, it occupies little space and is easily transported.

238. Observation of Stadia Interval.—On transit or plane-table surveys the stadia interval is determined by setting the lower hair on a foot mark and reading the position of the upper hair. The stadia interval is then mentally calculated with much less chance of error than would be the case if the lower hair were allowed to take a random position on the rod. When the vertical angle is taken to a given mark on the rod, the corresponding stadia interval is observed with the lower hair on the foot mark which renders a minimum displacement of the horizontal cross-hair from the mark to which the vertical angle is referred.

Thus, if a vertical angle were taken with the line of sight cutting the rod at 4.9 ft. and for this position of the horizontal cross-hair the lower stadia hair fell at 2.3 ft., the telescope would be rotated about the horizontal axis until the lower hair was at 2.0 ft., when the horizontal cross-hair would fall at 4.6 ft.

239. Principle of the Stadia.—Figure 239 illustrates the principle upon which the stadia method is based. In the figure the line of sight of the telescope is horizontal and the stadia rod is vertical. The stadia hairs are indicated by the points a and b, the distance between the hairs being i. The apparent positions of the hairs on the rod are A and B, and the interval apparently intercepted by the stadia hairs on the rod is s.

In optics it is shown that a ray of light passing through the optical center of a lens remains undeviated in direction, and further, that rays which are parallel on one side of the lens are all brought to a focus at a fixed point on the optical axis. This point is called the *principal focus*, and its distance from the optical center is called the *focal length* of the lens.

In the figure imagine that aa' and bb' are parallel rays emanating from the stadia hairs a and b. Then F is the principal focus through which these rays pass after emerging from the objective, f is the focal length of the objective, and the emerging rays take the positions $a'FA$ and $b'FB$. Also imagine that aOA and bOB are rays emanating respectively from a and b which pass undeviated through the optical center O.

Since all rays from a given point are brought to a focus on the opposite side of the objective, it follows that if rays from a passing through O and F are brought to a focus at A, the reverse is true,

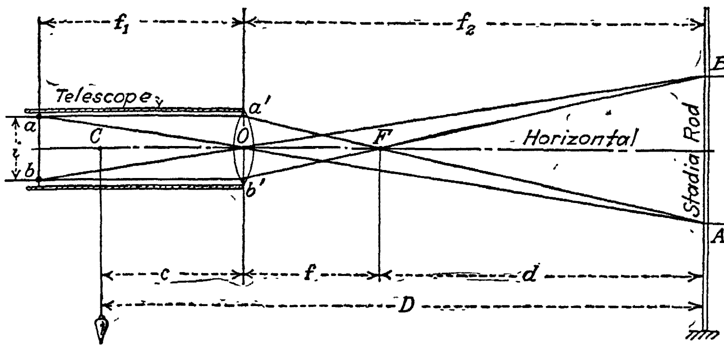


FIG. 239.

namely, that rays from A which pass through F and O are brought to a focus at a. These two points a and A or any other corresponding points as b and B, are called *conjugate foci* and their distances from the optical center of the objective measured along the optical axis, designated as f_1 and f_2 in the figure, are called the *conjugate focal distances*.

Since $ab = a'b'$, by similar triangles

$$\frac{f}{i} = \frac{d}{s}$$

Hence the horizontal distance from the principal focus F to the rod is $d = \frac{f}{i}s = Ks$, in which $K = \frac{f}{i}$ is a coefficient called the *stadia interval factor*, which for a particular instrument is a constant so long as optical conditions remain unchanged. Thus, for a horizontal sight it is clear that, as measured from the principal focus (the principal focus is at a fixed distance in front of the objective), the horizontal distance to the rod is obtained by multiplying the stadia interval factor by the stadia reading.

The horizontal distance from the center of the instrument to the rod is then

$$D = Ks + (f + c) \qquad (1)$$

in which f is a constant for a given instrument and c, though a variable depending upon the position of the objective, may for all practical purposes be considered a constant. This expression is employed in transforming stadia readings to horizontal distances when sights are horizontal and the rod is held vertically.

240. Stadia Constants.—The value of $(f + c)$ is usually determined by the manufacturer and is placed on the inside of the instrument box. The focal distance f may be determined with all necessary accuracy by focusing the objective on a distant point and then measuring the distance from the cross-hair ring to the objective. Similarly a mean value of c can be determined by measuring the distance from the vertical axis to the objective when the latter is focused for an average length of sight. Usually $(f + c)$ is nearly 1 ft. and under ordinary circumstances may be considered as 1 ft. without error of consequence.

241. Interval Factor.—As before stated, the nominal value of the stadia interval factor $K = \frac{f}{i}$ is usually 100. With an instrument having fixed stadia hairs the interval factor may be determined by observation. The usual procedure is to set up the instrument in a position where a horizontal sight can be obtained, and with a tape to lay off, from a point distant $(f + c)$ in front of the instrument, distances of 100 ft., 200 ft., etc., up to perhaps 1,000 ft., stakes being set at these points. The stadia rod is then held on each of the stakes thus established, and the stadia interval is read. The interval factor for each distance is obtained by dividing the distance from the principal focus ($(f + c)$ in front of the instrument) to the stake on which the rod was held, by the corresponding stadia interval read on the rod. Owing to errors in observation and perhaps to errors from natural sources, the values of K for the several distances are not likely to agree exactly. The mean is chosen as the most probable value.

To overcome any prejudicial tendencies on the part of the instrumentman, observations may be made on the rod held at random distances from the instrument, these points being marked by stakes. The distances from instrument to these stakes may then be measured with the tape, and K may be calculated as explained above.

With adjustable stadia hairs the interval factor is made 100 by moving the hairs until their rod intercept is $\frac{1}{100}$ of the distance from the principal focus to the rod, this distance being determined with a tape. The stadia

hairs are so adjusted that the horizontal cross-hair bisects the space between them, each in its turn being moved vertically until the distance between it and the horizontal hair is the proper half-interval, as indicated by the rod intercept.

242. Inclined Sights.—The principle of the stadia was explained in Art. 239 and there was derived the expression

$$D = \frac{f}{i}s + (f + c)$$

giving the horizontal distance from the center of the instrument to the rod when the line of sight is horizontal and the rod is held vertically. In stadia surveying horizontal sights are the exception rather

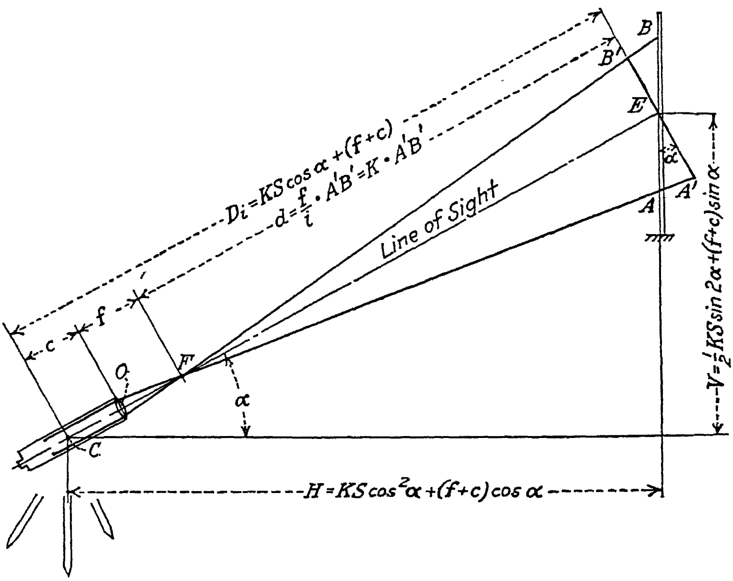


Fig. 242.

than the rule, and usually where inclined sights are taken, not only are horizontal distances determined, but vertical distances or differences in elevation are determined as well. The problem therefore resolves itself into finding the horizontal and vertical projections of an inclined line of sight, the terminals of which are the center of the instrument and a point on the rod. For convenience in field operations the rod is held vertically.

Figure 242 illustrates an inclined line of sight, AB being the stadia interval on the rod in its vertical position and $A'B'$ being the cor-

responding projection normal to the line of sight. Evidently the inclined length of the line of sight from center of instrument is

$$D_i = \frac{f}{i} \cdot A'B' + (f + c) \qquad (2)$$

For all practical purposes the angles at A' and B' may be assumed to be 90°. Then letting $AB = s$, $A'B' = s \cos \alpha$. Making this substitution in Eq. (2), and letting $K = \frac{f}{i}$, the inclined distance is

$$D_i = Ks \cos \alpha + (f + c) \qquad (3)$$

The horizontal component of this inclined distance is

$$H = Ks \cos^2 \alpha + (f + c) \cos \alpha \qquad (4)$$

which is the general expression for determining the horizontal distance from center of instrument to rod, when the line of sight is inclined.

The vertical component of the inclined distance is

$$V = Ks \cos \alpha \sin \alpha + (f + c) \sin \alpha$$

or

$$V = \tfrac{1}{2}Ks \sin 2\alpha + (f + c) \sin \alpha \qquad (5)$$

which is the general expression for the difference in elevation between the center of the instrument and the point where the line of sight cuts the rod. Equations (4) and (5) are known as the *stadia formulas for inclined sights*.

243. Stadia Reductions.—In practice horizontal distances and differences in elevation are not ordinarily computed by actually solving the two stadia formulas, but are obtained by the use of tables, diagrams, slide rules, etc., which are based upon these two equations. Table IX gives, for each 02′ of vertical angle up to 30°, the horizontal distances (from principal focus to rod) and differences in elevation for Ks = 100 ft., computed from the first member of Eqs. (4) and (5). For any other value of Ks, the tabular quantities are to be multiplied by the value of Ks in hundreds of feet. The table also gives the horizontal distances and differences in elevation for three values of $(f + c)$, indicated as c in the table. Tables varying in arrangement somewhat from that of Table IX will be found in numerous publications, some of these tables being very elaborate and being so designed that the desired quantities may be obtained directly without further computation. If Table IX or a similar table is used, the necessary multiplications may be carried out with sufficient accuracy by use of the ordinary slide rule.

Diagrams showing graphically the quantities $Ks \cos^2 \alpha$ and $\frac{1}{2}Ks \sin 2\alpha$ for all ordinary distances are published in a variety of forms, but it is a simple matter to prepare such a diagram and surveyors often prepare diagrams of their own design. The use of a stadia reduction diagram is considerably faster than is the use of tables. The relative accuracy of the two methods depends upon the scale of the diagram, but values taken from the usual stadia diagrams are sufficiently accurate for the ordinary purposes of stadia surveying.

A rapid and convenient means of calculating horizontal distances and differences in elevation is by means of a *stadia slide rule*, of which there are several patterns. The type which for ordinary use

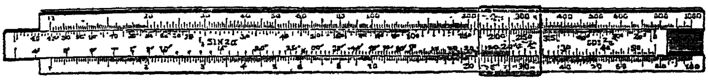


FIG. 243.—Stadia slide rule.

seems preferable to all others is constructed as is the ordinary slide rule, except that on the slide are given values of $\cos^2 \alpha$ and $\frac{1}{2} \sin 2\alpha$, these quantities being plotted to a logarithmic scale. This type is illustrated by Fig. 243. It may be obtained in lengths of either 10 or 20 in. The index of the slide (near right end) is set at the value of Ks on either the upper or the lower scale, as the case may be. The horizontal distance from principal focus to rod ($Ks \cos^2 \alpha$) is found by setting the runner at the observed vertical angle on the "$\cos^2 \alpha$" scale, and the corresponding difference in elevation is found by setting the runner to this same angle on the "$\frac{1}{2} \sin 2\alpha$" scale. The stadia slide rule is equally suitable for field or office use.

244. Permissible Approximations.—More approximate forms of the stadia formulas are sufficiently precise for most stadia work. Generally distances are calculated to feet and elevations are calculated to tenths of feet. Under these conditions, for side shots where vertical angles are less than 3°, Eq. (4) for horizontal distances may properly be reduced to the form

$$H = Ks + (f + c) \tag{6}$$

But for traverses of considerable length, owing to the systematic error introduced, this approximation should not be made for vertical angles greater than perhaps 2°.

Owing to unequal refraction and to accidental inclination of the rod, observed stadia intervals are in general slightly too large (see Art. 251). To offset the systematic errors from these sources the $(f + c)$ constant is frequently neglected on surveys of ordinary precision.

Hence in any ordinary case Eq. (4) may with sufficient accuracy be expressed in the form

$$H = Ks \cos^2 \alpha \text{ (approximate)} \tag{7}$$

Also Eq. (5) may generally be expressed in the form

$$V = \tfrac{1}{2}Ks \sin 2\alpha \text{ (approximate)} \tag{8}$$

For work of ordinary precision, the error introduced through using these approximate formulas will be negligible.

The determination of horizontal distances and differences in elevation by algebraic, graphical, or mechanical methods is considerably simplified by making Eqs. (7) and (8) the basis of calculations; hence these forms of the stadia formulas are generally employed. When K is 100 the common practice is mentally to multiply the stadia interval by 100 at the time of observation, and to record this value in the field notebook. This distance Ks is often called the *stadia distance*. Thus, if the stadia interval were 7.37 ft., the stadia distance recorded would be 737 ft. The degree of approximation using these formulas may be still further reduced by adding 1 ft. to the observed stadia distance.

245. Uses of the Stadia.—Uses of the stadia are as follows:

1. In differential leveling the backsight and foresight distances are conveniently equalized or balanced if the level is equipped with stadia hairs.

2. In profile leveling or cross-sectioning the stadia is a convenient means of finding distances from level to points where rod readings are taken.

3. In rough trigonometric or indirect leveling with the transit, the stadia method is more rapid than any other. The line of trigonometric levels is run as described in Art. 99, p. 115, except that stadia intervals are observed and differences in elevation are calculated by the stadia formula. Stadia trigonometric leveling is described in Art. 246.

4. In transit surveys of low precision where only horizontal angles and distances are required, stadia is more rapid than chaining. It may be used either in running traverse lines or in locating details from such lines. Stadia intervals are observed as each point is sighted. Horizontal angles are measured, but vertical angles are observed only when of sufficient magnitude to make the horizontal distance appreciably different from the stadia distance (say when greater than 3°) and then are estimated without reading the vernier, this being sufficiently accurate for the determination of horizontal distances. The transit-stadia method of running such surveys is described in Art. 247.

5. On transit surveys of low precision, particularly topographic surveys, where not only the relative positions of points in a horizontal plane but also the elevations of these points are desired. Both horizontal and vertical angles are measured and the stadia interval is observed as each point is sighted. The transit-stadia method of making observations when both horizontal position and elevation are desired is described in Art. 248.

6. Where the plane table is used (see Chap. XXIII), stadia observations are made with the telescopic alidade in the same manner as with the transit, but calculations of horizontal distances and differences in elevation are made in the field and are immediately plotted instead of being recorded in the form of notes.

246. Indirect Leveling by Stadia.—Where the required precision is low and the country is rolling or rough, the stadia method of indirect leveling offers a rapid means of determining differences in elevation. The instrument used is the engineer's transit, and it should preferably be provided with a sensitive control level for the vertical vernier in order that index error may be readily eliminated. With the ordinary transit having a vertical circle reading to single minutes, differences in elevation are usually calculated only to the nearest 0.1 ft. In general the average length of sight in stadia surveying is considerably greater than in differential leveling.

In running a line of levels by this method, the transit is set up in a convenient location. A backsight is taken on the rod held at the initial bench mark, by first observing the stadia interval and then by setting the horizontal cross-hair on some arbitrarily chosen rod reading and measuring the vertical angle to this mark. A turning point is then established in advance of the transit, and similar observations are taken, the vertical angle being measured with the horizontal cross-hair set on the same index mark as before. The transit is moved to a new position in advance of the turning point, and the process is repeated. The stadia distances and vertical angles are recorded, also the rod reading which is used as an index when vertical angles are measured. If it is impracticable to sight at this chosen index reading, the vertical angle is measured with the line of sight directed to some other graduation, usually a whole number of feet above or below the index, and this rod reading is given in the notes.

Figure 246 shows a suitable form of notes, the arrangement being somewhat the same as for differential level notes. Opposite a particular bench mark or turning point in the notes are given the observed values for both backsight and foresight and also the computed differences in elevation as determined by the approximate stadia formula, Eq. (8). In the notes the rod index or the rod

graduation on which sights are generally taken is shown as 6.0 ft. In the columns headed "Observations" are recorded both vertical angles and stadia distances. Thus for T.P. 147 the foresight stadia distance is 215 ft. and the foresight vertical angle is $+15°07'$.

For any set-up, the difference in elevation determined from either the backsight or foresight observation is the difference in elevation between the index mark on the rod and the center of the instrument, and the algebraic sum of the backsight and foresight differences is the total difference in elevation between the two positions of the index mark. It is clear that so long as the index mark to which vertical

STADIA LEVELS FOR INDIAN						TRAIL RECONNOISSANCE 48	
Sta.	Backsight Obs	Backsight Diff Elev	Foresight Obs	Foresight Diff Elev	Elev	Rod Index 6.0'	J.L. Black, ⊼ V.A. Alden, Rod
BM_{42}	−7°31' 518	+67.1			5972.4	Stone at John's Camp	July 24, 1927 Fair + Hot
TP_{146}	−6°04' 349	+36.7	+9°12' 641	+101.1	6140.6		
TP_{147}	−17°28' 197	+56.5	+15°07' 215	+54.1	6231.4		
TP_{148}	−5°31' 502	+48.0	+4°36'(8.0) 797	+61.6	6349.5		
TP_{149}	−12°36' 460	+98.2	+18°41' 744	+225.4	6622.9		
BM_{43}	+2°05' 820	−29.8	+3°12' 275	+15.3	6736.4	Stone at Summit at 15" Fir	
TP_{150}	+5°30' 644	−61.5	−10°42' 723	−131.7	6574.9		
TP_{151}	+23°24' 925'	−337.2	−4°46' 842	−69.8	6443.6		
TP_{152}	+4°16' 475	−35.2	0°(11.4) 215	−5.4	6101.0		
BM_{44}			−2°15' 612	−24.2	6041.6	Stone at Bottom of Canyon near Adit	

FIG. 246.—Stadia level notes.

angles are taken is unchanged, the difference in elevation determined as just described gives also the difference in elevation between the two points on which the rod is held, the actual value of the index reading being of no consequence. Thus the difference in elevation between B.M. 42 and T.P. 146 is given by the sum of the backsight difference $+67.1$ and the following foresight difference $+101.1$ and the elevation of T.P. 146 is therefore $5{,}972.4 + 67.1 + 101.1 = 6{,}140.6$ ft., since for both sights the vertical angle was taken with the line of sight directed to 6.0 ft. on the rod.

For the line of levels for which notes are shown, when a foresight was taken to T.P. 148 it was found expedient to sight at the 8.0-ft.

mark instead of the 6.0-ft. mark. This is shown in the notes by the value of 8.0 in parenthesis following the vertical angle $+4°36'$. Since the vertical angle is measured to a point 2.0 ft. above the adopted index mark and the sight is a foresight, the difference in elevation is taken as 2.0 ft. less than that given by the vertical angle and stadia distance $(63.6 - 2.0 = 61.6)$.

When the conditions are favorable it is preferable to read the rod with line of sight horizontal, as in differential leveling, for this eliminates the necessity of stadia reduction. Thus the foresight to T.P. 152 is taken with the telescope level, the rod reading being 11.4. Since this is $11.4 - 6.0 = 5.4$ ft. above the adopted index mark, the foresight difference in elevation is -5.4.

247. Transit-stadia Surveying: Elevations Not Required.—For certain reconnaissance or preliminary surveys, rough surveys for the location of boundaries, and detailed surveys for maps, where only the horizontal position of objects and lines is desired, the transit-stadia method is sufficiently accurate and considerably more rapid and economical than corresponding surveys made with transit and tape. The surveying procedure depends somewhat upon the nature and purpose of the survey but in a general way it parallels that which would be employed were distances measured with tape instead of by stadia. The field party for transit-stadia work consists of a transitman, usually a recorder, and sometimes one, but usually two, and occasionally three, rodmen. The field work consists in determining horizontal angles and directions by any of the methods described in Chaps. XI to XIII, and in observing stadia intervals together with approximate vertical angles when they are of sufficient magnitude to make the horizontal distances appreciably different from the observed stadia distances. Horizontal distances are computed by a stadia formula or are determined by one of the devices based thereupon. Horizontal distances are expressed to the nearest foot.

When the survey consists of a traverse it is customary to observe the stadia interval both forward and back from each set-up of the transit. In this way two independent stadia observations are made for each line or distance in the traverse. The closeness of agreement between the two values for each line is a check against mistakes, and the mean is taken as the most probable value.

Figure 247 is a sample page of notes for a closed traverse. The recorded value of the interval factor is 100.2. The directions of the lines are determined by azimuths and are checked by magnetic bearings. The rod intervals are given, rather than stadia distances, on account of the fact that K is not 100. Vertical angles to the nearest 10′ are observed and recorded for courses AB and DE, these angles

being of sufficient magnitude to make a horizontal correction necessary.

Stadia intervals are determined by sighting the lower stadia hair at some convenient foot mark and observing the position of the upper stadia hair. The vertical angle is then roughly observed, say to the nearest 10′, without reading the vernier. Whenever possible, sights are taken with the telescope approximately level, thus making the stadia distance equal to the horizontal distance and thereby eliminating the necessity for

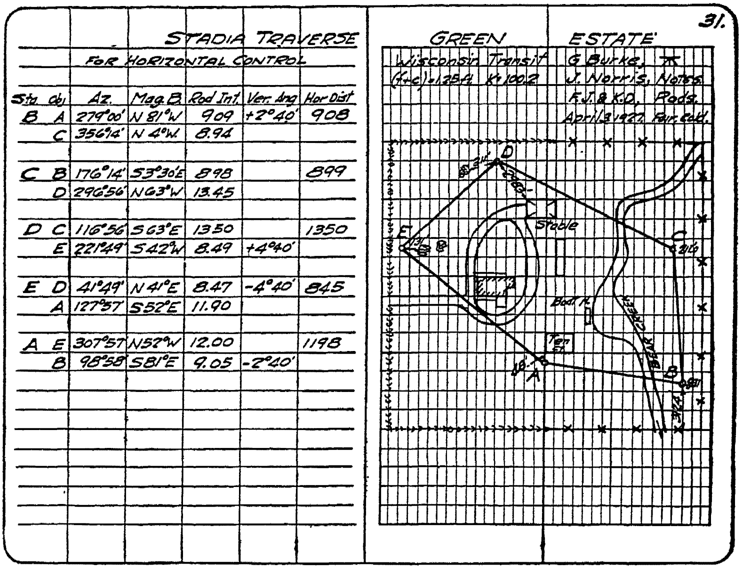


FIG. 247.—Stadia traverse notes.

reading the vertical angle. When the stadia interval is in excess of the length of the rod, the separate half-intervals are observed and their sum is taken.

Where details are to be located by stadia measurements from transit stations, the observations may be made at the same time that the transit station is established or may be made later. Figure 248*b* is a suitable form of notes when measurements to details are taken as the stadia traverse which forms the horizontal control is established, except that in practice, when elevations are not required, only those vertical angles of sufficient magnitude to make horizontal corrections necessary are recorded and differences in elevation are not computed. Figure 248*a* shows similar notes where numerous details are observed from a single transit station after it has been

established, as when the horizontal control is a triangulation system or is a taped distance.

The precision with which side shots are taken, of course, depends upon the intended use of the data. Thus, if they were to be used in plotting a map to the scale of 1 in. = 1,000 ft., distances to the nearest 10 ft. would be sufficiently accurate and horizontal corrections for angles below 6° could be neglected without affecting the precision of the map. Where details are varied in character, carefully prepared sketches, such as are illustrated in the figures just referred to, are absolutely necessary for the correct interpretation of the notes. The lack of clear sketches is the cause of more difficulty than the inexperienced can imagine.

248. Transit-stadia Surveying: Elevations Required.—This method is extensively employed on topographic and similar surveys where the elevations of points as well as their locations in a horizontal plane are desired. The field procedure of locating points consists in observing directions usually by azimuths, and distances by stadia, as described in the preceding article. In addition, differences in elevation are determined either by direct leveling when it is practicable to do so, or more usually from observed vertical angles and stadia distances. The field party consists of an observer, one or more rodmen, and usually a recorder.

In topographic surveying this method may be employed merely for the collection of details, the horizontal and vertical control being established by other means, or it may be utilized for establishing control as well as for details.

248a. In the former case the instrument is set up at a triangulation or traverse station, the elevation and position of which are known. The height of the instrument (H.I.) above the station over which it is set is measured with a rod or tape. The transit is oriented by taking a backsight to a station whose azimuth is known, this azimuth being set off on the horizontal circle. The upper motion is loosened, and sights to desired points are taken.

Figure 248*a* is a page of notes showing observations taken from station *C* of a traverse, the elevation of the station having been previously determined as 423.9. The H.I. is 4.4 ft. The transit is oriented by sighting to *B*, the azimuth of the line *CB* being set off on the horizontal circle prior to taking the sight. In the first column are given the numbers of the side shots, their positions being shown on the sketch. In the second column are the azimuths of the several points sighted; in the third column are the rod intervals. In the case illustrated by the notes the interval factor was not 100. Had it been, the intervals might have been replaced by the corresponding stadia distances, as given by the expression *Ks*. In the fourth column are the observed vertical

angles, and the following columns show respectively the calculated horizontal distances, differences in elevation, and elevations.

In measuring vertical angles it is customary, when practicable, to sight at a rod reading equal to the height of instrument above the station over which the transit is set. In this way, the difference in elevation between the center of instrument and the H.I. on the rod is the same as the difference in elevation between the station over which the transit is set and the point on which the rod is held. When the line of vision is obstructed so that the H.I. cannot be sighted, the horizontal cross-hair is set on some other graduation of the rod, usually a whole number of feet

Topographic Details, Black Estate. 32.

Wisconsin Transit (f+c) = 1.25, K = 100.2

G. Burke, ⊼. M. D. Rand, Notes. F. J. & K. D., Rods. Apr. 4, 1927. Cloudy, Cold.

Inst. at C. El. 423.9, H.I. = 4.4

Obj.	Az.	Rod Int.	Ver. Ang.	Hor. Dist.	Diff. El.	Elev.	
B	176°14′						
1	10°21′	7.23	−3°11′	723	−40.2	383.7	Water's Edge-Corner
2	3°14′	7.02	−3°17′	702	−40.2	383.7	" " -On Line
3	352°45′	5.64	−4°11′	563	−40.9	383.0	" "
4	7°18′	5.76	−4°04′	575	−40.9	383.0	" "
5	349°10′	(7.14 x	4) +3.3	714	−31.9	392.0	Line (Intervals)
6	16°55′	5.50	−2°50′	551	−27.3	396.6	"
7	315°20′	(7.86 x	5) −2.5	786	−36.8	387.1	" (")
8	349°15′	4.13	−5°46′	410	−41.4	382.5	Water's Edge
9	339°30′	5.40	−4°22′	539	−41.1	382.8	Bank Brook 6′ wide
10	0°05′	3.71	−4°12′	371	−27.2	396.7	" " 15′ "
11	344°40′	4.85	−4°54′	484	−41.4	382.5	" " 15′ "
12	25°00′	2.86		288	+ 1.2	425.1	Direct Levels.
13	307°45′	4.88	−4°56′	487	−42.0	381.9	Water's Edge.
14	319°10′	4.02	−5°56′	400	−41.6	382.3	" "
15	309°45′	5.80	−3°00′	581	−30.7	393.2	
16	318°25′	3.27	−4°36′	327	−26.3	397.6	
B	176°15′	ck. "	"				
17	340°00′	6.34	−3°08′	635	−34.7	389.2	
18	278°35′	2.51	−5°43′	250	−25.0	398.9	
19	276°20′	3.07	−7°56′	303	−42.3	381.6	Water's Edge.
20	277°40′	4.24	−5°40′	422	−41.9	382.0	" "

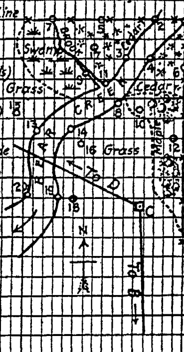


Fig. 248*a*.—Stadia notes.

above or below the H.I., and this difference in rod readings is recorded in the notes. When the corresponding difference in elevation is calculated, proper allowance is made for the difference between the H.I. and the recorded rod reading. Thus, in the case illustrated by the notes, had the vertical angle of $-3°17'$ for object 2 been taken to a rod reading of 6.4 ft. instead of the H.I. = 4.4, the difference in elevation would have been $-40.2 - 2.0 = -42.2$ ft.

When the detail to be observed is at nearly the same elevation as the point over which the transit is set, there is a marked advantage in determining difference in elevation by direct leveling. The notes of Fig. 248*a* show that object 12 was observed in this manner. The observed rod reading 3.2 might have been shown, but instead, the difference in elevation has been calculated by mentally subtracting 3.2 from 4.4 giving 1.2, which is recorded in the notes.

Shots 5 and 7 of the stadia notes of Fig. 248*a* illustrate the stepping method (Art. 249).

Generally where measurements are made solely for plotting a map, horizontal angles are estimated to 05′ without reading the vernier. Vertical angles are usually measured to minutes, and differences in elevation are calculated to tenths of feet. Where elevations to the nearest foot are sufficiently accurate, the vertical angles, except for long shots, may properly be read by estimation without use of the vernier.

The usual procedure of observing is to sight on the rod, the lower stadia hair being set on a foot mark such that the horizontal cross-

PRELIMINARY (STADIA) SURVEY OF I. N.RY., BRIGHTON TO CAMBY

Obj.	Az.	Mag.B.	Stad.Dist.	Vert.Ang.	Hor.Dist.	Diff.Elev.	Elev.	Nov. 27, 1935 J.C.Clark, ⊼
	Inst. at Sta. P49; H.I. = 4.7						785.1	Cold T.N.Tillman, Notes
P48	169°34′	S10°30′E	637	−2°27′	636	−27.2	757.9	W.W.&H.H., Rods
P50	38°21′	N38°15′E	681	+1°14′	681	+14.6	799.7	On slope.
491	151°10′		366	−7°21′	360	−45.6	739.5	West bank Green River.
492	126°35′		418	−5°59′	413	−43.3	741.8	East " " "
493	78°05′		385	−5°36′	381	−37.4	747.7	West " " "
494	81°20′		387	−5°40′	383	−38.0	747.1	East " " "
495	298°55′		214	+6°34′	211	+24.3	809.4	Top slope.
	Inst. at Sta. P50; H.I. = 4.9						799.7	
P49	218°21′	S38°30′W	683	−1°13′	683	−14.5	785.2	
501	294°40′		415	+4°38′	412	+33.4	833.1	Top slope.
502	16°00′		308	0° on 2.1	308	+2.8	802.5	On slope.
503	137°35′		374	−6°36′	369	−42.8	756.9	West bank Green River.
504	136°10′		486	−5°52′	481	−49.4	750.3	East " " "
505	5°45′		322	+7°36′	316	+41.8	841.5	Top slope.
P51	59°38′	N59°30′E	529	0° on 10.1	529	−5.2	794.5	On slope.
506	94°25′		487	−3°36′	485	−30.5	769.2	West bank Green River.

FIG. 248*b*.—Stadia notes.

hair falls somewhere near the H.I., and to read the stadia interval. The horizontal cross-hair is then set on the H.I., the horizontal angle is read without clamping the upper motion, the instrument is rotated about the vertical axis until the vertical circle is in a convenient position for reading, and finally the vertical angle is observed.

When numerous observations are to be taken from a single station, sights to some object the azimuth of which is known are taken at intervals in order to make sure that there is no undetected movement of the lower motion of the transit. The notes of Fig. 248*a* show a check measurement of this kind taken to station *B* after observations to object 16 had been completed.

248b. Where the required precision is not high, the stadia traverse with elevations of traverse stations determined by vertical angle and

stadia distance is a very rapid means of establishing horizontal and vertical control. The procedure is the same as that already described in Art. 247, except that in addition, vertical angles are observed both for the backsight and for the foresight from each station, the telescope being sighted at a rod reading equal to the height of instrument above the transit station over which the transit is set up.

Figure 248*b* is a page of notes for a stadia traverse for which side shots are taken as the work of running the traverse progresses. Directions of the traverse lines are determined by azimuth and roughly checked by observed magnetic bearings, and stadia distances are recorded rather than the rod interval, the interval factor being practically 100. In calculating the length of a traverse line and the difference in elevation between its terminal points, the mean of the vertical angles and the mean of the stadia distances observed from each end are employed.

249. Stepping Method.—Where the slope of the ground is so small as to make the horizontal distance practically equal to the inclined

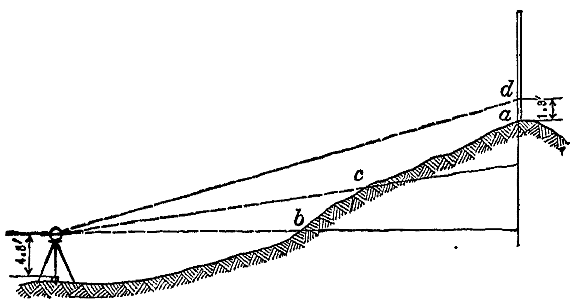


Fig. 249.—Stepping method.

distance, instead of reading the vertical angle and computing the difference in elevation as described in the preceding articles, the practice of determining the difference in elevation directly by the so-called *interval* or *stepping* method is sometimes followed. To illustrate the method, suppose that in Fig. 249 the difference in elevation between the point over which the transit is set up and the point *a* on which the rod is held is desired. The transit is sighted at the rod, and the stadia interval is observed as, say, 4.55 ft. The line of sight is then rotated in a vertical plane until the telescope is level, and the position of the horizontal line of sight on some clearly defined object of the landscape is noted at *b*. The telescope is raised until the lower stadia hair cuts *b*, and the position of the upper stadia hair on the landscape is noted at *c*. The telescope is again rotated about the horizontal axis until the lower hair cuts *c*, when the upper hair is seen to intersect the rod at *d*, which is at a rod reading of say 1.3 ft. Then if the height of instrument (H.I.) is, say, 4.5 ft., the difference in elevation between the instrument station and *a*, assum-

ing that no error is introduced by reason of the line of sight being inclined when the stadia interval was observed, is $4.5 - 1.3 + 2 \times 4.55 = 12.3$ ft.

While the illustration is for a positive vertical angle, it is clear that a similar procedure might be followed for negative angles, except that the upper hair would be set on successive points and the position of the lower hair on the landscape would be noted. It is evident that for vertical angles of sufficient magnitude to cause an appreciable difference between the observed stadia interval and that which would be observed with sight horizontal, this method would not be applicable. In practice it is generally limited to sights for which the inclination is less than 2°, or where the number of steps does not exceed three. The method is generally used for side shots in fairly flat country where direct leveling is impracticable. The precision obtainable is higher than might be expected considering the fact that points marking the successive steps are not established by the surveyors but are chosen from among the objects of the landscape upon which the stadia hairs happen to fall. If the transitman keeps the reference mark constantly in view through the telescope while revolving the telescope about the horizontal axis between steps, the error need be little, if any, more than that of reading the rod at the corresponding distance, even though the object sighted may not be readily identified when once the eye has left it. Thus a point of reference might be the stem of a leaf, a pebble in a clot of earth, the tip of a weed, a seam in a rock, a flower, or even the edge of a cloud.

Shots 5 and 7 of the stadia notes of Fig. 248*a* further illustrate the stepping method.

250. Beaman Stadia Arc.—This is a specially graduated vertical arc which is attached to the vertical circle of the transit as illustrated in Fig. 250. The stadia arc has no vernier, but settings are read by an index mark, the plate on which the mark is engraved being attached to the vertical vernier and extending over the vertical circle to the edge of the stadia arc.

The outer row of graduations, labeled "Vert," is used in determining differences in elevation, the reading of the stadia arc being 50 when that of the vertical circle is 0°, and the numbers being so arranged that for positive vertical angles the stadia-arc readings are greater than 50 while for negative vertical angles the stadia-arc readings are less than 50. To determine the difference in elevation between the center of the instrument and a point on which the rod is held, the telescope is rotated about the horizontal axis until the rod is sighted and the stadia distance is observed. It is then further rotated until the nearest graduation on the "Vert" scale is made to coincide with the index mark, and the position of the horizontal cross-hair on the rod is read. The difference in elevation between center of instrument and point last sighted on the rod is given by the observed stadia interval multiplied by the setting of the index of

the stadia arc less 50. This difference combined with the height of instrument and rod reading, gives the difference in elevation between the point over which the transit is set up and the point at which the rod is held. Following is an illustrative example:

Example: The observed stadia interval with the rod held on a given point is 4.11 ft. and with the index to the Beaman stadia arc at 41 (as shown in figure) the horizontal cross-hair falls at 3.3 ft. on the rod. The height of instrument (vertical distance from transit station to center

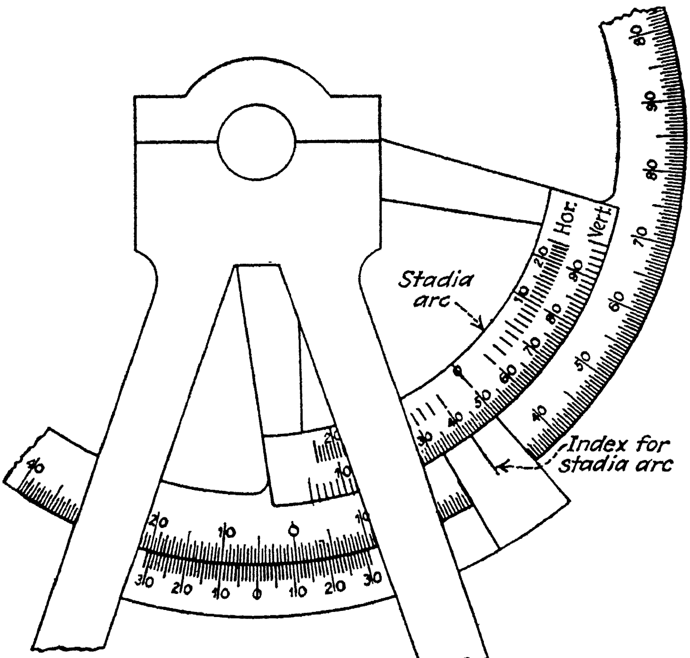


Fig. 250.—Beaman stadia arc.

of instrument) is 4.5 ft. The elevation of the point on which the rod is held is to be found, the elevation of the transit station being 100.0.

The difference in elevation between center of instrument and point sighted on rod is

$$(41 - 50)4.11 = -9 \times 4.11 = -36.99, \text{ or say } -37.0 \text{ ft.}$$

The difference in elevation between transit station and point on which rod is held is

$$4.5 - 37.0 - 3.3 = -35.8 \text{ ft.}$$

The required elevation is $100.0 - 35.8 = 64.2$ ft.

The inner row of graduations of the arc, labeled "Hor," gives corrections in feet per hundred feet to observed stadia distances to obtain the corresponding horizontal distances. Thus, with setting as shown in the figure, the horizontal correction is about 0.9 ft. per hundred. Hence if the stadia distance is, say, 412 ft. the correction to the nearest foot is −4 ft. and the corresponding horizontal distance is 412 − 4 = 408 ft.

The use of the Beaman stadia arc makes unnecessary the reading of vertical angles and reduces the determination of difference in elevation to the very simple process of multiplying the stadia interval by a whole number which, if the number be small, may be done mentally. On the other hand, it requires more time to set the arc than it does to read the vertical angle, hence from the standpoint of rapidity of operation in the field, the vertical-angle method has the advantage. The general opinion seems to be that under average conditions the stadia arc is not as rapid nor as convenient a means of determining difference in elevation as is the stadia slide rule, vertical angles being observed as described in the preceding article.

The Beaman stadia arc is merely a mechanical device for quickly laying off angles for which the function $\frac{1}{2} \sin 2\alpha$ bears a simple relation to the difference in elevation. Consulting Table IX it will be seen that the differences in elevation are, for example, respectively 1, 2, 10, and 20 ft. per 100 ft., for vertical angles of 0°34′, 1°09′, 5°46′, and 11°47′. The Beaman arc merely facilitates the setting of these and other angles for which $\frac{1}{2} \sin 2\alpha$ is a simple multiple of 0.01, the graduations on the "Vert" scale being so spaced that when the index is one division on either side of 50, the vertical vernier reads 0°34′; or when the index is ten divisions on either side of 50, the vertical vernier reads 5°46′, etc.

251. Errors in Stadia Surveying.—Many of the errors of stadia surveying are obviously not peculiar to this work alone, but are common to all similar operations of surveying. The errors of direct leveling, most of which also have their effect upon the determination of difference in elevation by methods described in this chapter, are discussed in Art. 126, p. 166. Sources of error in the measurement of horizontal angles with the transit are discussed in Arts. 210 to 213. Sources of error in calculated distances, both horizontal and vertical, based upon observed stadia intervals are as follows:

1. *Stadia Interval Factor Not That Assumed.*—This produces a systematic error in all calculated distances, the error being proportional to the error in the stadia interval factor. The case is parallel with that of the tape which is too long or too short. If the stadia hairs are fixed, the interval factor is not likely to change appreciably, but may change slightly with variations in natural conditions.

When the value of the interval factor is closely determined by observations, as described in Art. 241, and the stadia measurements are taken under conditions paralleling those existing when the interval factor was determined, the error from this source may be reduced to a negligible quantity.

2. *Rod Not Standard Length.*—If the spaces on the rod are uniformly too long or too short, a systematic error, proportional to the stadia interval, is produced in each computed distance. Errors from this source may be kept within comparatively narrow limits if the rod is standardized and corrections for erroneous length are applied to observed stadia intervals. Except for stadia surveys of more than ordinary precision, errors from this source are usually of no consequence.

3. *Incorrect Stadia Interval.*—This refers to the inability of the instrumentman to observe exactly the rod interval between the stadia hairs. Under usual conditions the error is accidental in character; following the theory of probability, in a series of connected observations (as a traverse) it may be expected to vary as the square root of the number of sights. Errors from this source are the ones principally affecting the precision of computed distances. They may be kept to a minimum by proper focusing to eliminate parallax, by taking observations at a time of day when atmospheric conditions are favorable, and by care in observing. Where a high precision is required, stadia measurements may be taken by sighting on a rod with two targets, one fixed, upon which the upper stadia hair is set, and the other movable, it being brought to the position of the lower stadia hair.

4. *Rod Not Plumb.*—This produces an error in the vertical angle, since in measuring vertical angles the horizontal cross-hair is set on a rod reading equal to the instrument H.I., but this error is of small consequence. It also produces an appreciable error in the observed stadia interval and hence in computed distances, this error for a given distance and for a given inclination of the rod being greater for large vertical angles than for small angles. It may be eliminated by using a rod equipped with a plumbing level. Where the rod is not so equipped, the error from this source is likely to be large for steeply inclined sights.

5. *Unequal Refraction.*—It has been determined by experiment that unequal refraction of light rays in layers of air close to the earth's surface introduces systematic positive errors in stadia measurements. While errors from this source are of no consequence in ordinary stadia surveying, they may be of sufficient magnitude to be of importance on the more precise surveys. The periods most favorable for equal refraction are at times when it is cloudy or

during the early morning or late afternoon when the sun is shining. On precise stadia surveys where it is necessary to work under a variety of atmospheric conditions, it is proper to determine the stadia interval factor for each condition and to apply the proper factor to all observations taken under a given condition.

251a. While the sources just listed produce errors in both horizontal distances and differences in elevation, errors in vertical angles also have their effect upon these computed values. From a study of the stadia formulas it is evident that errors in vertical angles, within the usual range of values, are relatively unimportant in their effect upon computed *horizontal* distances. Thus the ratio of precision corresponding to a 01′ error in angle is about $\frac{1}{20,000}$ when the angle is 5° and about $\frac{1}{6,000}$ when the angle is 15°. This makes it clear that so far as precision of horizontal distances is concerned, the governing quantity is likely to be the observed stadia interval rather than the observed vertical angle. On the other hand, the errors in vertical angles are relatively important in their effect upon the precision with which *differences in elevation* are determined. For example, an error of 01′ in any vertical angle within the usual range produces an error in elevation of nearly 0.1 ft. in a distance of 300 ft.

With the ordinary transit, vertical angles are read to single minutes, with an error of perhaps ½′, by means of the vernier, or to 05′, with an error of perhaps 3′, by estimation without the vernier; for an average length of sight of say 500 ft., the error in stadia distance need not exceed 2 ft. if care is taken in observing, though it might under some conditions amount to as much as 5 ft. For most stadia work vertical angles are less than 5°, but in rough country vertical angles may be 15° or more. For the 500-ft. distance the error in difference in elevation corresponding to the angular error of ½′ is less than 0.1 ft. and that corresponding to the angular error of 3′ is about 0.4 ft., regardless of the magnitude of the vertical angle within the ordinary range of values. For a vertical angle of 5°, an error of 2 ft. in the stadia distance produces an error of less than 0.2 ft. in the computed difference in elevation, while an error of 5 ft. in the observed distance produces an error in difference in elevation of more than 0.4 ft. For a vertical angle of 15°, the corresponding errors in difference in elevation are 0.5 ft. and 1.3 ft.

By comparing the above figures it may be concluded that under normal conditions, where vertical angles are small the effect of observational errors in vertical angles may be expected to be somewhere near the same as that due to observational errors in stadia intervals; but where vertical angles are large the errors in observed stadia intervals are likely to be the ones which most affect the precision of computed differences in elevation. Thus it is clear that to maintain a given precision in computed

values of difference in elevation, stadia intervals must be observed with much greater refinement where vertical angles are large than where they are small.

252. Precision of Stadia Surveying.—For surveys of ordinary precision made with the transit and tape, where only horizontal angles and distances are measured, it has been shown that the principal errors are the systematic errors of chaining and for this reason the precision of such surveys is likely to vary directly with the distance.

In transit-stadia surveying, say for a traverse, where horizontal and vertical angles and stadia intervals are observed, the precision with which the relative positions of points, in plan and elevation, are determined is dependent upon errors from each of these three sources. It has been shown that the principal errors of both horizontal and vertical angular measurements are likely to be accidental in character. The principal error in stadia measurements may be either systematic or accidental, depending upon conditions. If the stadia rod is standardized and proper corrections are applied for erroneous length, and if the interval factor is accurately determined and the rod is carefully plumbed, the principal error is that of observing the stadia interval. Under such circumstances, the errors which mainly control the computed position of points, both in plan and elevation, are largely accidental in character, and hence it is to be expected that these errors will in the long run tend to vary as the square root of the distance. This marks one of the important advantages of surveying with the transit and stadia over surveying with the transit and tape, and it explains why the precision obtained on extensive transit-stadia surveys often compares favorably with the precision obtained on similar surveys for which measurements are made with the transit and tape.

Unless care is taken to eliminate the systematic errors mentioned, the resultant error in plan is likely to vary more nearly as the distance, as is the case with tape measurements. Under these circumstances the advantage just mentioned is lost and the transit-stadia survey, regardless of its extent, would in general be considerably less precise than the survey made with transit and tape. If observed vertical angles are small, large systematic errors in stadia observations or reductions will have comparatively small effect upon computed differences in elevation, and the resultant error may be expected to vary more nearly as the square root of the distance; on the other hand, if vertical angles are large, systematic stadia errors are likely to be the chief contributors to the resultant errors in elevation, and the error may be expected to vary directly with the distance.

The factors which influence the precision of stadia surveying are numerous. The quality of the instrument, the accuracy of graduation of the rod, the character of the country, the skill of the observer, the care with which the rod is held, the length of sight, and the condition of the weather all affect the results. Following are estimates which it is believed are fairly representative of several classes of stadia work, these estimates being based upon the results secured on surveys run under a variety of conditions. In accepting these estimates it should be borne in mind that the conditions surrounding no two surveys are alike and that a definite statement of the precision that can be obtained with a given course of procedure is impossible.

1. For side shots where a single observation is taken with sights steeply inclined and with no particular care taken to insure the rod's being plumb, horizontal distances may have a precision lower than $1/100$ and individual differences in elevation may be in error 2 ft. or more per 1,000 ft. of horizontal distance.

2. Under the same conditions as in (1) but with small vertical angles and reasonable care used in approximately plumbing the rod and with lengths of sights not less than 200 ft. nor more than 1,500 ft., the precision of horizontal distances should be not lower than $1/500$; if vertical angles are observed to minutes, differences in elevation should not be in error more than 0.3 ft. per 1,000 ft. of horizontal distance and if vertical angles are estimated to 05′, the error in difference in elevation need not exceed 1 ft. per 1,000 ft. of horizontal distance.

3. For a rapid stadia traverse of considerable length run through rough country with numerous long sights, angles being measured to minutes but without special precaution to eliminate systematic errors, the error of closure may be as low as 25 ft. per mile in plan and 3 ft. per mile in elevation.

4. For conditions as in (3) but for country fairly level so that all vertical angles are small, the error of closure ought not to exceed 15 ft. per mile in plan and 0.5 ft. times the square root of the distance in miles in elevation.

5. For rough country with vertical angles up to 15°, angles to minutes, rod standardized, rod plumbed with level, sights limited to 1,500 ft. and taken forward and back from each transit station, and interval factor accurately determined, the error may be less than 15 ft. times the square root of the distance in miles in plan and 1 ft. times the square root of the distance in miles in elevation.

6. For the same conditions as in (5), but for level country so that all vertical angles are small, the error may be as small as 6 ft. times the square root of the distance in miles in plan and 0.3 ft. times the square root of the distance in miles in elevation.

7. For conditions as in (5) but stadia intervals determined by use of a target rod with two targets and observations made during cloudy days,

the error of closure in plan should not exceed 4 ft. times the square root of distance in miles.

253. Problems.

1. To determine the stadia interval factor, a transit is set up the distance $(f + c)$ in the rear of the zero end of a level base line 800 ft. long, the base line being marked by stakes set every 100 ft. A rod is then held at successive stations along the base line. The stadia interval and each half-interval observed at each position of the rod are tabulated below.

Calculate the lower, upper, and full interval factor for each distance, and find the mean values.

Distance $- (f + c)$, feet	Lower interval, feet	Upper interval, feet	Full interval, feet
100	0.49	0.50	0.99
200	0.98	0.99	1.97
300	1.47	1.48	2.95
400	1.97	1.98	3.96
500	2.46	2.47	4.94
600	2.95	2.97	5.92
700	3.45	3.47	6.91
800	3.94	3.96	7.89

2. A transit for which the stadia interval factor is 98.5 and $(f + c)$ is 1 ft. is used in making the following observations.

Observation	Stadia interval	Vertical angle
a	12.15	$+3°18'$
b	10.92	$+11°31'$

By means of Eqs. (4) and (5), Art. 242, compute the horizontal distances and differences in elevation. Check the computed values by means of Table IX.

3. The following observations are taken with a transit for which the interval factor is 100.0 and $(f + c)$ is 1 ft.

Observation	Stadia interval	Vertical angle
a	10.00	$+0°30'$
b	10.00	$+10°00'$
c	10.00	$+25°00'$

By means of Eqs. (4) and (5), Art. 242, compute the horizontal distances and differences in elevation. By means of the approximate Eqs. (7) and (8), Art. 244, determine the same quantities and note the errors introduced by the approximations.

4. If the observations of problem 3 were taken with a 12-ft. rod which was unknowingly 0.5 ft. out of plumb with top leaning towards the transit, what would be the amount and sign of the error introduced in each computed horizontal distance and difference in elevation?

5. If the observations of problem 3 were taken with the top of the rod leaning 0.5 ft. away from the transit, what would be the amount and sign of the error introduced in each computed horizontal distance and difference in elevation?

6. What conclusions may be drawn from the results of problems 4 and 5?

7. For the observations of problem 3, if each vertical angle contains an error of 01′ what error will be introduced in each computed horizontal distance and difference in elevation?

8. For the observations of problem 3, if each stadia interval is in error 0.1 ft. what error will be introduced in each computed horizontal distance and difference in elevation?

9. What conclusions may be drawn from the results of problems 7 and 8?

10. Following are the notes for a line of stadia levels:

Station	Backsight		Foresight	
	Vertical angle	Stadia interval	Vertical angle	Stadia interval
B.M.$_1$	$-4°36'$	3.37	....	....
T.P.$_1$	$-1°39'$	2.95	$+2°17'$	2.74
T.P.$_2$	$-3°48'$	4.11	$+5°04'$	1.47
T.P.$_3$	$-5°39'$	2.07	$+0°34'$	3.47
B.M.$_2$	$+0°32'$	4.76	$+0°55'$	4.16
T.P.$_4$	$+1°18'$	1.42	$-7°42'$	2.10
T.P.$_5$	$+2°19'$	3.42	$-10°08'$	3.12
T.P.$_6$	$+3°43'$	1.88	$-4°24'$	3.61
B.M.$_3$	....	....	$-2°00'$	1.41

The elevation of B.M.$_1$ is 641.12. The interval factor is 100.0 and $(f + c) = 1.2$ ft. By use of Table IX calculate the elevations of all B.M.'s and T.P.'s.

11. Following are vertical angles and stadia measurements for a transit-stadia traverse. The elevation of station A is 301.7 ft. The stadia interval factor is 100, and $(f + c) = 1$ ft. Calculate the horizontal lengths of the several courses and the elevations of the several transit stations.

Station	Object	Stadia interval	Vertical angle
B	*A*	4.84	$-3°12'$
	C	5.95	$+4°41'$
C	*B*	5.92	$-4°40'$
	D	2.46	$-1°05'$
D	*C*	2.47	$+1°05'$
	E	8.45	$-0°49'$
E	*D*	8.50	$+0°48'$
	F	4.37	$+8°13'$
F	*E*	4.34	$-8°14'$
	G	12.45	$+2°22'$
G	*F*	12.40	$-2°21'$
	H	7.18	$+0°31'$

12. Following are the vertical angles and stadia intervals taken in connection with the location of points from two transit stations. The stadia interval factor is 100 and $(f + c) = 1.0$ ft. Calculate the horizontal distances and elevations.

Object	Azimuth	Stadia interval	Vertical angle	
Instrument at D		Elevation 417.5	H.I. = 4.6	
C	144°31′	3.47	−1°32′	
41	73°25′	5.80	−1°14′	
42	157°15′	6.30	−1°26′	
43	207°05′	7.04	−0°58′	
44	332°10′	−(8.25 × 3)	+2.5	Intervals
45	299°00′	7.56	−0°44′	On 9.2 ft.
46	290°20′	−(7.25 × 2)	−1.4	Intervals
47	265°10′	3.72	−5°36′	
48	249°35′	1.41	−7°03′	
49	221°40′	3.04	−3°32′	
50	203°30′	0.90	−3.1	Direct levels
51	191°25′	1.95	−7.4	Direct levels
52	170°20′	−(3.16 × 3)	+1.8	Intervals
E	163°18′	4.91	+2°17′	
Instrument at E		H.I. = 5.0		
D	343°18′	4.89	−2°18′	
53	74°25′	2.08	−3°37′	
54	102°15′	1.27	+5°37′	
55	157°05′	2.95	+2°46′	
56	190°25′	3.11	+2°44′	
57	182°20′	3.74	+3°58′	
58	217°35′	4.63	+4°37′	
59	239°40′	4.45	+2°08′	
60	244°10′	3.06	+0°36′	
61	291°30′	2.45	+1.6	Direct levels
62	302°00′	2.01	−3°44′	
63	342°25′	0.60	−7°32′	
64	15°10′	1.96	−4°02′	
65	29°50′	2.68	−2°03′	
66	41°15′	3.12	+0°41′	
67	50°00′	3.98	+0°49′	
68	62°55′	3.54	+1°36′	
F	63°39′	4.13	+1°51′	

CHAPTER XV

METHODS OF PLOTTING

254. General.—The methods of plotting described in this chapter are those employed in mapping areas of limited extent where the earth's surface is assumed to be plane and all meridians are assumed to be parallel. These methods are applicable to surveys for railroads, highways, irrigation and drainage systems, and the like, as well as to most topographic and hydrographic surveys, except those of wide extent, and to rural and urban land surveys.

255. Process of Making a Map.—The mechanical processes of the preparation of maps were described in some detail in Chap. IV. Regardless of their purpose or kind, maps are usually so plotted that features are shown in the same relative position as they exist on the ground, at a given scale. Hence the data of a survey furnish the information that is necessary to plot the map, and the operations of plotting are in a sense the reverse of the operations of surveying. Some maps, notably those made for the purpose of delineating the boundaries of real property, show numerically the dimensions of the main features, and their worth is unimpaired if they are not drawn to true scale. The value of most maps, however, depends upon the accuracy with which details are shown, and the general aim is to plot the more definite features so that they will be located within proper limits of error.

Just as the first step in the field work for a survey of any considerable extent consists in establishing points of horizontal control so that their relative positions will be known with a required degree of precision, so the first step in mapping is to plot these points of horizontal control so that their relative positions will be correctly shown, irrespective of the intervening distance. And just as the lines joining points of horizontal control on the ground form the skeleton of the survey, so do the corresponding lines when represented upon paper form the skeleton upon which the details of the map are hung. In general, therefore, the process of mapping involves: (1) the plotting, by more precise methods, of points of horizontal control which are generally transit stations and which may be either traverse or triangulation points, or both; and (2) the plotting, by less precise methods, of features which go to make up the map, generally called

the map details, measurements to these details being given in the form of angles and distances from the lines and points in the horizontal control system.

Most maps are plotted wholly in the office from data taken in the field, but where conditions are favorable and the objects to be shown are numerous, maps are often plotted much more expeditiously in the field as the survey progresses. As a general rule, the points of primary horizontal control are plotted in the office, but often when details are mapped in the field, points of secondary horizontal control are fixed on the ground only as it becomes necessary to establish such points to expedite the location of details.

256. Methods of Plotting Horizontal Control.—Horizontal control may be plotted: (1) by use of the protractor, (2) by the tangent method, (3) by the chord method, or (4) by the coordinate method. In any case, distances are measured with the engineer's scale, and points are pricked with a needle. Traverse and triangulation stations are indicated by appropriate symbols (see Fig. 269*b*), and control lines are carefully drawn with a hard pencil having a fine point. Points are numbered or lettered to conform to the system employed in the field work.

257. Protractor Method.—Where the control system is not extensive and the map is small, a fairly large protractor provides a sufficiently accurate means of plotting angular values (see Art. 61*a*, p. 62). Obviously the precision with which angles may be laid off is likely to vary directly with the diameter of the protractor. The largest protractor commonly manufactured has a diameter of 14 in. It can be shown that even for this maximum size and with most careful plotting the probable error for a single angle is not less than 05′, and that for the 6-in. protractor (a size in common use) the accidental error of plotting may amount to 15′. It is apparent that the angular precision likely to be obtained with any protractor is low.

Where a traverse is to be plotted, the common practice is to fix by estimation the position of the first line (as AB, Fig. 257*a*) and then by placing the protractor at the forward point (as B) to lay off the deflection angle to the succeeding line, and having determined its direction by a light line of indefinite length, to lay off along this line the given distance (as BC) to the succeeding point (as to C); and thus to repeat the process for each traverse point. An objection to this procedure lies in the fact that any error in the direction of one line affects to a like degree the directions of all succeeding lines, and the linear error in the position of succeeding points increases with the distance. Thus in Fig. 257*a* if CBC' represents the error made in laying off the angle at B, then the corresponding linear errors at the

succeeding points are CC', DD', and EE' and the magnitude of these errors varies directly with the distance from the point at which the angular error occurs. For example, if in Fig. 257*a* the angular error at B were 10′, and the distances B to C and B to E were respectively 10 in. and 30 in., then the linear error in the position of C due to the given angular error alone would be 0.03 in. and that in the position of E would be 0.1 in. When it is recalled that distances may be scaled within 0.01 in., it becomes obvious why this method will not yield results sufficiently accurate for extensive maps with many points of control.

257a. When the directions of lines of a traverse are given either in the form of azimuths or bearings, a meridian line may be drawn through each station and the direction of the succeeding line may be

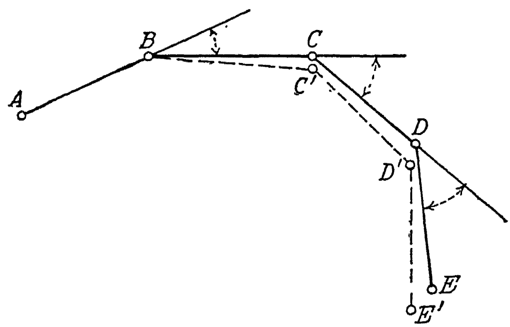


FIG. 257*a*.

laid off with respect to the meridian. A better plan, however, is to erect a meridian near the middle of the sheet and, using any desired point on the meridian as an origin, to lay off the directions of all the lines. As illustrated by Fig. 257*b*, the line designated as $\frac{a}{b}$ is laid off at an angle with the meridian equal to the bearing or azimuth of AB on the ground; and so on, for other lines in the traverse. The traverse is then plotted by transferring the directions thus determined to appropriate positions on the sheet and laying off the lengths of the traverse lines. Thus the direction $\frac{a}{b}$ (Fig. 257*b*) is transferred to the position AB (Fig. 257*c*) and the length of AB is laid off; the direction $\frac{b}{c}$ is transferred to a line of indefinite length passing through B and the distance BC is scaled along this line from B to establish C; and this process is continued for the succeeding points in the traverse.

Since the directions of all lines are referred to a common meridian, any angular error made in plotting a given line in no wise affects

the plotted directions of succeeding lines in the traverse, but causes a linear error in the position of succeeding traverse points and this error is the same for one point as for any other. Thus, if CBC' (Fig. 257c) be the error introduced in the direction of BC, then the linear error in the position of C is CC'. Since $C'D'$ is given by the direction $\frac{c}{d}$ it will, if it is assumed to be without error, be parallel to its true position; hence $DD' = CC'$ and by similar reasoning $EE' = CC'$. Where there are a considerable number of lines in the traverse, since the angular precision of one line has no influence upon the angular precision of other lines, the method just described is likely to produce

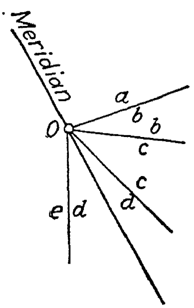


Fig. 257b.

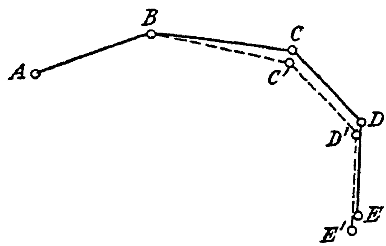


Fig. 257c.

better results than that described in Art. 257, in spite of the fact that the accidental errors of plotting the individual lines are likely to be larger when the directions are transferred as described in this article. Where traverses are established by deflection angles, the bearings or azimuths may be calculated as described in Chap. XI, and the calculated values may then be plotted from the central meridian as just explained.

The instances where the protractor may properly be employed for plotting triangulation stations are not numerous, but it is a sufficiently accurate method for rough surveys over small areas, and is often useful in checking continuous traverses when angular observations have been taken from several stations to a given landmark. The position of a station is determined by the intersection of lines laid off from two other stations marking the ends of a line of known length.

258. Plotting by Tangent Offsets.—This method is employed principally for plotting transit traverses for the control of maps of moderate size where the number of stations is small. In principle, it is similar to the protractor method described in Art. 257, with this difference: An angle is laid off by linear measurement which is a constant times the natural tangent of the angle. Usually deflection

angles are plotted, but there is an advantage in plotting azimuths or bearings from a centrally located meridian. In Fig. 258*a*, AB represents the plotted position of the initial line of a traverse and α is the deflection angle at B which it is desired to lay off in order to determine the direction of BC. By the tangent method the line last established, in this case AB, is prolonged some convenient distance, usually 10 in., to form a base line Bb, at the end of which a perpendicular bb' is erected and the distance bb' is laid off equal to the length of the base line Bb multiplied by the natural tangent of α. A line drawn from B through b' defines the direction of BC. Then the point C is plotted by laying off to scale the given distance BC. Thus the process is repeated for succeeding lines, except that generally when the deflection angle is greater than 45°, instead of making the base line a prolongation of the preceding line, it is established as a perpendicular to the preceding line at the point last plotted. Thus in the figure β is assumed to be such an angle and Cc is the base line perpendicular to BC. The perpendicular offset cc' is in this case the length of the base line Cc times the natural cotangent of the angle β. The line Cc' fixes the direction of CD, and the point D is then plotted by linear measurement from C.

Often when the traverse approaches the edge of the sheet the usual method of laying off angles will prove impracticable on account of the construction lines falling off the paper. The method shown in Fig. 258*a* for laying off the angle at D may frequently be employed, the base line being measured back along the line CD, and the tangent offset dd' being laid off as usual. The line $d'D$ is then prolonged beyond D, and E is located in the customary manner.

When a traverse is to be plotted by this method, deflection angles and distances are tabulated and the tangents of angles less than 45° and cotangents of angles greater than 45° are recorded. The traverse is roughly plotted to small scale using the protractor for laying off angles, and the shape of the traverse is noted. By means of the small-scale sketch a suitable position for the first line of the traverse is estimated, and the line is plotted. The work of plotting is then continued as described in the preceding paragraph.

The precision with which angles may be laid off by this method depends upon the care used in plotting and upon the length of base line employed. For accurate results all points should be pricked with a needle, all lines should be drawn with a hard, sharp pencil, and perpendiculars should be erected with great care. When this careful procedure is followed the error in laying off an angle using a 10-in. base line need not exceed 03′, which is about the error that

might be expected in using a protractor having a diameter of 20 in., if such a size were available.

258a. Plotting deflection angles by tangents is open to the same objections that were mentioned in Art. 257 concerning the method of laying off deflection angles with the protractor, namely, that any angular error in the direction of one line affects in like amount the directions of all succeeding lines. Hence for long traverses some persons prefer to calculate the bearings of the several lines and to

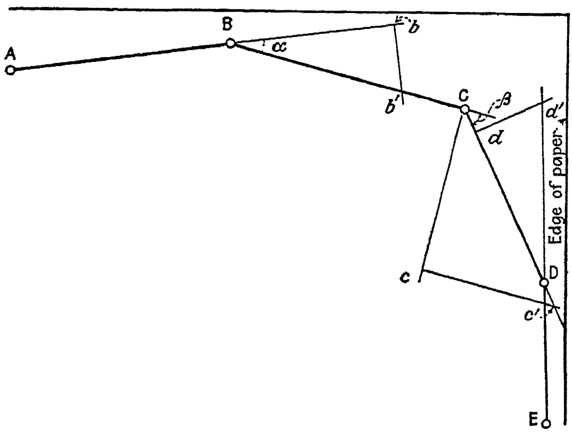


FIG. 258*a*.—Plotting by tangent offsets.

lay off the tangent distances from a meridian and base line centrally located on the sheet, as described in the following article. The process of checking is simplified, however, if deflection angles are used.

258b. When the directions of lines are given by bearings or azimuths, a meridian and base line may be established at each transit point and the following line may be plotted by methods similar to those described for deflection angles. When the traverse is many-sided, the method illustrated by Fig. 258*b* is preferable. A 20-in. square composed of four 10-in. squares with sides perpendicular to or parallel with the meridian is constructed near the middle of the sheet, as shown. The tangents of bearing angles less than 45°, or of azimuths corresponding to such bearing angles, are scaled east or west of the line YY' along the outer sides of squares forming corresponding quadrants; in a similar manner the cotangents of bearing angles greater than 45°, or azimuths corresponding to such bearing angles, are scaled north or south of the line XX'. Lines joining these points with the intersection of the X and Y axes have the same bearings as corresponding lines of the traverse.

For example, the line AB has a bearing of N37°E. Its tangent offset, 7.54 in., is therefore scaled from Y along the north side of the NE quadrant, thus locating b. The plotted position of AB must be parallel to Ob. The line BC has a bearing of S80°E, which is greater than 45°. The point c is located by scaling the cotangent distance, 1.76 in., south from the point X along the east side of the SE quadrant, and BC must be made

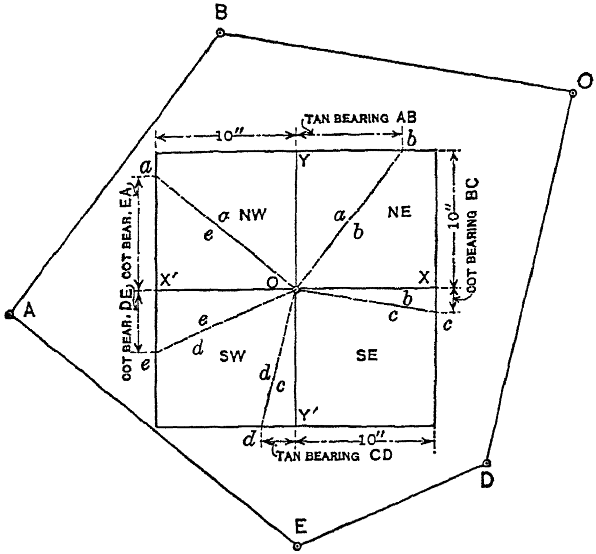


FIG. 258*b*.

parallel to Oc. In this way the directions of all lines in the traverse are laid off, the direction $\frac{a}{b}$ is then transferred to its proper position on the sheet and the distance AB is plotted to scale; the direction $\frac{b}{c}$ is plotted; and so on, around the traverse.

258c. Tangent Protractor.—A useful labor-saving device known as the *tangent protractor* consists of a scale subdivided into tangent distances (for a 10-in. base) corresponding to angles of multiples of 10′ between 0° and 45°. The values of angles, not the tangent distances, are numbered on the protractor, 45° being 10 in. (10 tan 45°) from the zero end, 30° being 5.77 in. (10 tan 30°) from the zero end, etc. The use of the protractor is identical with that of the ordinary scale, except that in plotting the tangent offsets the draftsman is guided by the angular values marked on the protractor rather than by the actual tangent offsets. To facilitate the plotting of angles between 45° and 90° by cotangent distances, a second set of angular

values is placed under the first, the cotangent scale decreasing from 90° to 45° as the tangent scale increases from 0° to 45°. The use of the protractor eliminates the necessity of determining the numerical values of tangents, but the protractor is made only for a 10-in. base and it cannot be conveniently used for any other distance.

259. Plotting by Chords.—This method is much like that of plotting by tangents described in Art. 258, except that instead of erecting a perpendicular at the end of a 10-in. base line, an arc of 10-in. radius is struck. The chord distance for the given angle is then scaled from the point of intersection between arc and base line to a point on the arc.

The chord method is generally regarded as being somewhat less accurate than the tangent method though it is not clear that there should be any appreciable difference between the two methods, provided the arcs are accurately drawn. The chord method is not in as general use as is the tangent method, probably for the reason that it requires more time to compute the chord distances than it does to compute the tangent offsets. When a table of chords is available, however, the method is considerably quicker than the tangent method.

260. Rectangular Coordinates.—This method of plotting, also known as the method of total latitudes and departures, is the only practical one for plotting extensive systems of horizontal control. It is always employed for plotting triangulation figures except those of the simplest character, and it is considerably more accurate than any of the methods so far described for plotting traverses (see Art. 265*a*). Rectangular coordinates are employed not only for plotting maps, but also frequently for calculating areas, as described in Art. 274.

The coordinate axes are a *reference meridian* (either true, magnetic, or assumed) and a line at right angles thereto called a *reference parallel*. The intersection of these lines, marking the origin, may be any point in the survey or may be a point entirely outside the survey. The azimuths or bearings of all lines are either given or are calculated from observed angles. With the direction of each line determined and its length known, the lengths of its orthographic projection upon the meridian and upon the parallel are computed. The projection upon the meridian is termed the *latitude* of the line; the projection upon the parallel is called the *departure* of the line. The origin having been chosen, the coordinates for the several control points are computed by using the latitudes and departures. The coordinate of a point measured normal to the parallel is called the *total latitude* or the *parallel distance* of the point; the coordinate measured normal to the meridian is called the *total departure* or the *meridian distance*. With the coordinate axes established on paper,

a point is plotted by laying off its total latitude and its total departure to the required scale. The succeeding articles are devoted to a more detailed discussion of the processes involved.

Before computing the latitudes and departures of the traverse lines, the angular error of closure of the traverse is determined by the known geometrical conditions, and the angles or bearings are so adjusted or corrected that the known geometrical conditions will be fulfilled (see Art. 263*a*).

261. Latitudes and Departures.—In Fig. 261*a*, *AB* represents any line the latitude and departure of which it is desired to determine,

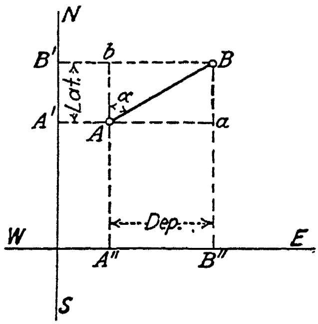


Fig. 261*a*.

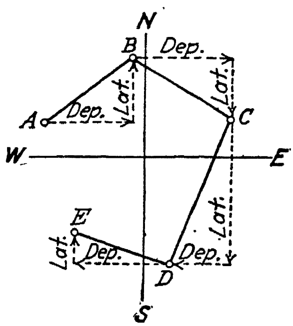


Fig. 261*b*.

and the lines NS and EW represent any meridian and any parallel. The line *AB* makes the angle α with the meridian.

Since the latitude of a line is the orthographic projection of the line upon a meridian, the latitude of *AB* is $A'B' = Ab = AB \cos \alpha$.

And since the departure of a line is its orthographic projection upon a parallel, the departure of *AB* is $A''B'' = Aa = AB \sin \alpha$.

Stated in the form of a general rule applicable to any line:

$$\text{Latitude} = \text{Length} \times \text{cosine bearing angle} \qquad (1)$$
$$\text{Departure} = \text{Length} \times \text{sine bearing angle.} \qquad (2)$$

Latitudes are designated as *North* or *positive* for all lines having a northerly bearing, and *South* or *negative* for all lines having a southerly bearing.

Departures are designated as *East* or *positive* for lines having an easterly bearing, and *West* or *negative* for lines having a westerly bearing.

Thus, referring to Fig. 261*b*, for the line *AB* the latitude is North or + and the departure is East or +. For *BC* the latitude is South or −, and the departure is East or +. For *CD* the latitude is South or −, and the departure is West or − For *DE* the latitude is North or + and the departure is West or −.

If the latitudes and departures are determined solely for purposes of map construction, the number of places to be used in computations should be such that points may be correctly plotted within the scale of the map. Thus for a scale of 1 in. = 800 ft., distances can hardly be plotted closer than to the nearest 10 ft., but to insure all coordinates being correct to the nearest 10 ft. the latitudes and departures would probably be determined to the nearest foot, and to insure the latitudes and departures being correct to the nearest foot, the intermediate computations might be carried out to tenths of feet. Computations are made with a calculating machine, or if none is available, by logarithms.

If the calculating machine is employed, the data may be kept in the following form:

Line	Bearing	Length	Cos bearing	Sin bearing	N latitude	S latitude	E departure	W departure

The bearings and lengths of the lines are first tabulated. Then the natural cosines and sines of the bearing angles are recorded. And finally for each side the latitude and departure are computed, the length being set on the machine as the multiplicand.

Where logarithms are used, the form of the following example will be found convenient.

Example: For a given course in a traverse the calculated bearing is N34°21′W and the observed length is 1,215.3 ft. The traverse is to be plotted to the scale of 1 in. = 100 ft. and it is desired that the relative position of all points be correct within the scale of the map. It is desired to compute the latitude and departure with a degree of precision consistent with the purpose for which they are to be used.

At the given scale, distances may be plotted within 2 ft. and therefore latitudes and departures should be correct to perhaps the nearest quarter foot if the traverse has many sides. Five-place logarithms will be used. The computations are:

Latitude	1,003.3 ft.
Log lat	3.00145
Log cos bearing	9.91677
Log dist	3.08468
Log sin bearing	9.75147
Log dep	2.83615
Departure	685.7 ft.

262. Error of Closure.—In any closed traverse it is obvious that the sum of the north latitudes should equal the sum of the south latitudes, and that the sum of the east departures should equal the sum

of the west departures. In other words, for any closed traverse the algebraic sum of the latitudes (ΣL) should be equal to zero and the algebraic sum of the departures (ΣD) should be equal to zero. Due to errors in field measurements of both angles and distances, the chances of these conditions being exactly realized are exceedingly remote, hence in general the unadjusted traverse would not close on paper even though the plotting were without error. The conditions just stated make it possible to determine, by means of the computed latitudes and departures, the magnitude of the error of closure. Figure 262 shows a traverse which does not close, the line $A'A = e$ being the side of error. If the sum of the latitudes and the sum of the departures of the measured sides AB, BC . . . EA' respectively be designated by the symbols ΣL and ΣD, then in order that the algebraic sum of the latitudes and the algebraic sum of the departures shall each be equal to zero, the latitude of the side of error must be $(-\Sigma L)$ and its departure must be $(-\Sigma D)$, considered algebraically. These two quantities form the base and altitude of the right-angle triangle of which the side of error is the hypotenuse, hence the error of closure

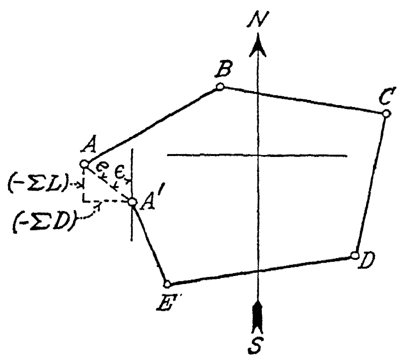


FIG. 262.

$$e = \sqrt{(\Sigma L)^2 + (\Sigma D)^2} \tag{3}$$

The direction of the side of error is in general given by the relation

$$\tan \epsilon = \frac{-\Sigma D}{-\Sigma L} \tag{4}$$

with due regard to sign, an equation which is satisfied by bearings differing by 180°. The data will make it apparent in which quadrant the bearing lies.

Example: In a given closed traverse the sum of the south latitudes exceeds the sum of the north latitudes by 9 ft. and the sum of the east departures exceeds the sum of the west departures by 12 ft. What is the linear error of closure and the bearing of the side of error?

The linear error of closure is $e = \sqrt{(9)^2 + (12)^2} = 15$ ft.

Since for the given sides the south latitudes exceed the north latitudes, the latitude of the side of error must be north, and by parallel reasoning the departure of the side of error must be west; hence the bearing angle is in the northwest quadrant.

$$\tan \epsilon = \frac{-\Sigma D}{-\Sigma L} = \frac{-12}{+9} = -1.33$$

$$\epsilon = -53°08'$$

$$\text{Bear.} = \text{N}53°08'\text{W}$$

263. Balancing the Survey.—This article deals with the practice of adjusting field observations, the principles of which have been discussed in Chap. V.

When the error of closure has been determined as described in the preceding article, corrections are usually made so that the traverse will form a mathematically closed figure, and the corrections are applied to the latitudes and departures in such manner as to make their algebraic sum equal zero. This operation is called *balancing the survey* or *balancing the traverse.* It is an operation which is performed not only prior to the calculation of coordinates for plotting, but also before computing areas. There is, of course, no possible means of determining the true magnitude of the errors in angle and distance which occur throughout the traverse, but if conditions surrounding the field measurements have been uniform it is fair to assume that errors have gradually accumulated, and corrections should be made accordingly.

There are several rules for distribution of errors, each of which will produce a mathematically closed figure and each of which is assumed to be adapted to certain conditions as regards measurements. Many surveyors, however, rely upon their own judgment, in a large measure disregarding any established rule, and arbitrarily distribute the error in accordance with their estimation of the difficulties met in the field.

Manifestly, if certain courses are over rough ground, the error of chaining these courses would be expected to be relatively large, and the correction to the observed distance should be correspondingly great; also where sights are steep and visibility is poor, larger angular errors would be expected than where conditions of observing are more nearly ideal, and hence in balancing the survey it is fair to assume that the larger changes in direction should be in the courses where conditions surrounding the observations were relatively unfavorable.

263a. Adjustment of Angular Error.—The values of the measured angles of a traverse are checked by the known geometrical conditions. The total angular error thus determined is distributed among the angles or calculated bearings of the traverse before computations of the latitudes and departures are made. This adjustment is often arbitrary, based upon a knowledge of the field conditions; but, if all angles have been measured under like conditions, the error is distributed equally to each angle in the traverse.

The total angular error is the result of both accidental and systematic errors which have affected the work. The adjustment eliminates the effects of all systematic errors to the extent that they have been constant and equal in their effect upon each angle measured.

The adjustment does not yield true values, for each measured angle is subject to accidental errors whose sign and magnitude are unknown. All that can be said about the adjustment as regards the accidental errors, is that it meets the known geometrical conditions and to that extent the adjusted values are the most probable values that can be assigned.

263b. Rules for Balancing a Survey.—Two rules are commonly used to balance a traverse, namely the *Compass Rule* and the *Transit Rule.*

The Compass Rule states that the correction to be applied to the $\left\{\begin{array}{l}\text{latitude}\\ \text{departure}\end{array}\right\}$ of any course is to the total error in $\left\{\begin{array}{l}\text{latitude}\\ \text{departure}\end{array}\right\}$ as the length of the course is to the length of the traverse. It is based upon the assumptions: (1) that the errors in traversing are accidental in nature and therefore vary with the square root of the lengths of the sides, thus making the correction to each side proportional to its length (Art. 70*a*, p. 75); and (2) that the effects of the errors in angular measurements are consistent with the effects of errors in chaining. It is the rule most commonly used (see Art. 263*c*).

The Transit Rule states that the correction to be applied to the $\left\{\begin{array}{l}\text{latitude}\\ \text{departure}\end{array}\right\}$ of any course is to the total error in $\left\{\begin{array}{l}\text{latitude}\\ \text{departure}\end{array}\right\}$ as the $\left\{\begin{array}{l}\text{latitude}\\ \text{departure}\end{array}\right\}$ of that course is to the arithmetical sum of all the $\left\{\begin{array}{l}\text{latitudes}\\ \text{departures}\end{array}\right\}$ in the traverse. It is based upon the assumptions: (1) that the errors in traversing are accidental in nature; and (2) that the angular measurements are more precise than are those of chaining.

As an example of the effects of the two rules the computations are given below for a particular case.

Example: At the top of page 367 are tabulated the lengths, bearings, latitudes, and departures for a closed traverse of six sides. The survey is to be balanced by both of the rules given in this article.

The error of closure in latitude is 15.0 ft. and in departure is 32.0 ft. The sum of the lengths of the sides is 4,997.6 ft. or practically 5,000 ft.

Line	Length, feet	Bearing	Latitude		Departure	
			N	S	E	W
AB	500.0	N	500.0	....	0.0	....
BC	848.6	N45°00′E	600.0	....	600.0	....
CD	854.4	S69°27′E	....	300.0	800.0	....
DE	1,019.8	S11°19′E	....	1,000.0	200.0	....
EF	1,118.0	S79°42′W	....	200.0	....	1,100.0
FA	656.8	N54°06′W	385.0	....	....	532.0
Sum	4,997.6	....	1,485.0	1,500.0	1,600.0	1,632.0

By the Compass Rule the correction in latitude for the course *AB* is $\frac{15}{5{,}000} \times 500 = 1.5$ ft. The north latitudes are too small and hence the correction is to be added, *i.e.*, the correction is +1.5 ft. The correction to the departure of *AB* is $\frac{32}{5{,}000} \times 500 = 3.2$ ft. The east departures are too small and hence the correction is to be added. The correction to the latitude of *CD* is $\frac{-15}{5{,}000} \times 854.4 = -2.6$ ft., and the correction to the departure of *CD* is $\frac{+32}{5{,}000} \times 854.4 = +5.4$ ft.

Using the Transit Rule, the arithmetical sum of the latitudes is 2,985.0 ft. and the arithmetical sum of the departures is 3,232.0 ft. The correction to the latitude of *CD* is $\frac{-15}{2{,}985} \times 300 = -1.5$ ft., and the correction in departure is $\frac{+32}{3{,}232} \times 800 = +8.0$ ft.

Below are tabulated the corrections in feet as determined by the two rules, for the several courses in the traverse. In the last line are

Line	Correction in latitude, ft.		Correction in departure, ft.	
	Compass	Transit	Compass	Transit
AB	+ 1.5	+ 2.5	+ 3.2	0
BC	+ 2.6	+ 3.0	+ 5.4	+ 6.0
CD	− 2.6	− 1.5	+ 5.4	+ 8.0
DE	− 3.1	− 5.0	+ 6.6	+ 2.0
EF	− 3.3	− 1.0	− 7.2	−10.8
FA	+ 1.9	+ 2.0	− 4.2	− 5.2
Sum	15.0	15.0	32.0	32.0

shown the sums of the values in the several columns, which should equal the error in latitude or departure as the case may be. This is a check on the correctness of the computations, and when the corrections have been applied, the algebraic sum of the latitudes and departures must be zero, and hence the survey is balanced.

263c. The adjustment of observations subject to accidental errors lies properly within the province of the method of Least Squares, and in the light of these principles the following comments may be made:

1. It can be shown that the Compass Rule is correct for the assumptions made.

2. The Transit Rule is merely a rule of thumb which, it is found, does not apply successfully to many cases. In fact it meets the assumptions upon which it is based only to the extent that each side is parallel to one or the other of the coordinate axes.

263d. Crandall Method.—There are conditions where the accidental errors of linear measurements are likely to be greater than those in the measurement of angles, as for example, in stadia traversing, or even in careful tape measurements where some of the systematic errors are rendered accidental in nature by reason of corrections and of special methods applied to the field measurements. In such cases we may be warranted in assuming, after any small angular error of closure has been distributed among the bearings of the traverse, that the bearings are without appreciable error, and the adjustments should properly be made to the linear measurements only. For these cases Professor C. L. Crandall has applied the method of Least Squares and has arranged the solution in such form that it can be carried through with the use of a slide rule only (see Ref. 2, p. 391). As applied to the example given above, the computations are as follows:

In these formulas the following notation is used:

L = latitude of any course
D = departure of any course
l = length of any course
v = correction to be applied to any quantity
q_L = total error in latitude
q_D = total error in departure
A and B are factors connecting various quantities in the computations, and Σ is the sign of summation.

As a matter of convenience, the distances are expressed in tape lengths by dividing the lengths l by 100 throughout.

$$A = \frac{q_D\left(\Sigma\frac{LD}{100l}\right) - q_L\left(\Sigma\frac{D^2}{100l}\right)}{\left(\Sigma\frac{D^2}{100l}\right)\left(\Sigma\frac{L^2}{100l}\right) - \left(\Sigma\frac{LD}{100l}\right)^2};$$

$$B = \frac{q_L\left(\Sigma\frac{LD}{100l}\right) - q_D\left(\Sigma\frac{L^2}{100l}\right)}{\left(\Sigma\frac{D^2}{100l}\right)\left(\Sigma\frac{L^2}{100l}\right) - \left(\Sigma\frac{LD}{100l}\right)^2} \quad (5)$$

Then

$$\left.\begin{aligned} v_{l_1} &= L_1A + D_1B \\ v_{l_2} &= L_2A + D_2B \end{aligned}\right\} \qquad (6)$$

$$\left.\begin{aligned} v_{L_1} &= v_{l_1}\frac{L_1}{100l_1} = A\frac{L_1^2}{100l_1} + B\frac{L_1D_1}{100l_1} \\ v_{L_2} &= v_{l_2}\frac{L_2}{100l_2} = A\frac{L_2^2}{100l_2} + B\frac{L_2D_2}{100l_2} \end{aligned}\right\} \qquad (7)$$

$$\left.\begin{aligned} v_{D_1} &= v_{l_1}\frac{D_1}{100l_1} = A\frac{D_1L_1}{100l_1} + B\frac{D_1^2}{100l_1} \\ v_{D_2} &= v_{l_2}\frac{D_2}{100l_2} = A\frac{D_2L_2}{100l_2} + B\frac{D_2^2}{100l_2} \end{aligned}\right\} \qquad (8)$$

Line	Length	Bearing	Latitude		Departure		$\frac{L^2}{100l}$	$\frac{D^2}{100l}$	$\frac{LD}{100l}$
			N	S	E	W			
			+	−	+	−			
AB	500.0	N	500.0	....	0.0	....	5.00	0.00	+0.00
BC	848.6	N45°00′E	600.0	....	600.0	....	4.25	4.25	+4.25
CD	854.4	S69°27′E	....	300.0	800.0	....	1.05	7.50	−2.81
DE	1,019.8	S11°19′E	....	1,000.0	200.0	....	9.80	0.39	−1.96
EF	1,118.0	S79°42′W	....	200.0	....	1,100.0	0.36	10.81	+1.97
FA	656.8	N54°06′W	385.0	....	....	532.0	2.25	4.31	−3.11
		Sum	1,485.0	1,500.0	1,600.0	1,632.0	22.71	27.26	−1.66
			$q_L = -15.0$		$q_D = -32.0$				

$$A = \frac{(-32.0) \times (-1.66) - (-15.0 \times 27.26)}{(27.26 \times 22.71) - (-1.66)^2} = \frac{+53.1 + 409.0}{621.7 - 2.8} = +0.747$$

$$B = \frac{(-15.0) \times (-1.66) - (-32.0 \times 22.71)}{(27.26 \times 22.71) - (-1.66)^2} = \frac{+24.9 + 726.7}{621.7 - 2.8} = +1.214$$

$$\begin{aligned}
v_{L_1} &= (5.00 \times 0.747) + (0.00 \times 1.214) = +3.74 + 0.00 = +3.74 \\
v_{L_2} &= (4.25 \times 0.747) + (4.25 \times 1.214) = +3.17 + 5.16 = +8.33 \\
v_{L_3} &= (1.05 \times 0.747) - (2.81 \times 1.214) = +0.78 - 3.41 = -2.63 \\
v_{L_4} &= (9.80 \times 0.747) - (1.96 \times 1.214) = +7.32 - 2.38 = +4.94 \\
v_{L_5} &= (0.36 \times 0.747) + (1.97 \times 1.214) = +0.27 + 2.39 = +2.66 \\
v_{L_6} &= (2.25 \times 0.747) - (3.11 \times 1.214) = +1.68 - 3.78 = -2.10 \\
&\qquad\qquad \text{Check: } -q_L = +14.94 \\
v_{D_1} &= (0.00 \times 1.214) + (0.00 \times 0.747) = +0.00 + 0.00 = +0.00 \\
v_{D_2} &= (4.25 \times 1.214) + (4.25 \times 0.747) = +5.16 + 3.17 = +8.33 \\
v_{D_3} &= (7.50 \times 1.214) - (2.81 \times 0.747) = +9.11 - 2.10 = +7.01 \\
v_{D_4} &= (0.39 \times 1.214) - (1.96 \times 0.747) = +0.47 - 1.46 = -0.99 \\
v_{D_5} &= (10.81 \times 1.214) + (1.97 \times 0.747) = +13.13 + 1.47 = +14.60 \\
v_{D_6} &= (4.31 \times 1.214) - (3.11 \times 0.747) = +5.23 - 2.32 = +2.91 \\
&\qquad\qquad \text{Check: } -q_D = +31.86
\end{aligned}$$

These computations are, of course, more laborious than those required by the Compass or the Transit Rule, but this solution meets the desired assumptions and will distribute the error of closure in the lengths of the lines only. The labor, after all, is not excessive and a convenient check is supplied at the close, if the sum of the separate corrections equals the total error with opposite sign. Another check is afforded if the corrected

lengths of the lines are computed from the corrected latitudes and departures.

263e. Summary.—The discussion relating to methods of balancing a survey may be summarized by the statements: (1) in many cases a careful distribution of errors on the basis of a knowledge of the field conditions is the best that can be made; (2) if systematic errors are supposed to be present in the linear measurements they can be eliminated only by applying proper computed corrections to the field measurements before any rules for balancing the survey can be applied, since systematic errors are not subject to distribution by any general rule; (3) if the error of closure is subject to accidental errors affecting to an equal extent both angular and linear measurements, the Compass Rule is valid; (4) in most cases, the Transit Rule cannot properly be used; and (5) if the accidental errors in linear measurements are assumed to be much larger than those in angles, the Crandall Method is valid.

264. Calculation of Coordinates.—When it is desired to plot points of horizontal control by the method of coordinates, the latitudes and

Carry Pond Traverse Total Lats and Deps. 48

Field Notes
Book No 4 P.27
Log Computations
on P 47 this book

Comp'td by DEL
C'k'd by PMF
Nov 4, 1927

Sta	Cal Bear	Dist Ft	Latitudes N	Latitudes S	Departures E	Departures W	Adjusted Lats	Adjusted Deps.	Total Lats N	Total Lats S	Total Deps. E	Total Deps. W
A									0	0	0	0
	N 69°07'W	2622	934			2450	+936	-2454				
B									936			2454
	N41°50'E	2722	2030		1816		+2032	+1812				
C									2968			642
	S46°20'E	946		652	686		-652	+682				
D									2316		40	
	S44°52'E	950		672	670		-670	+668				
E									1646		708	
	S38°30'W	506		396		314	-396	-314				
F									1250		394	
	S17°30'W	1312		1252		392	-1250	-394				
A									0	0	0	0
		9058	2964	2972	3172	3156	±2968	±3162				
				2964	3156							
				8	16							

$E \text{ of } C = \frac{18}{9058} = \frac{1}{503}$

FIG. 264.—Calculation of coordinates.

departures of the control lines are computed and in the case of the closed traverse the survey is balanced. The origin is chosen, and the total latitudes and departures or coordinates of the several points in the survey are computed by summing the latitudes and departures of lines between that point and the origin.

The total $\left\{\begin{matrix}\text{latitude}\\ \text{departure}\end{matrix}\right\}$ of any point equals the algebraic sum of $\left\{\begin{matrix}\text{latitudes}\\ \text{departures}\end{matrix}\right\}$ of lines lying between that point and the $\left\{\begin{matrix}\text{reference}\\ \text{reference}\end{matrix}\right.$

$\left.\begin{matrix}\text{parallel}\\ \text{meridian}\end{matrix}\right\}$ passing through the origin. $\left\{\begin{matrix}\text{North latitudes}\\ \text{East departures}\end{matrix}\right\}$ are considered positive. $\left\{\begin{matrix}\text{South latitudes}\\ \text{West departures}\end{matrix}\right\}$ are considered negative. $\left\{\begin{matrix}\text{Total latitudes}\\ \text{Total departures}\end{matrix}\right\}$ are positive or negative according to whether the corresponding points lie $\left\{\begin{matrix}\text{north or south of the reference parallel}\\ \text{east or west of the reference meridian}\end{matrix}\right\}$.

Figure 264 is an example of the tabulations ordinarily made for a closed traverse. Station *A* is seen to be the origin. The adjusted latitudes and departures are used in calculating the coordinates. In a closed traverse the additions are verified if on making the complete circuit the total latitude and total departure of the initial point check these quantities, as assumed at the beginning. In the tabulations of Fig. 264 it will be noted that this check has been performed.

265. Plotting by Coordinates.—When a system of horizontal control is to be plotted by rectangular coordinates, the size of the enclosing rectangle is determined from the total latitudes and departures; this rectangle being one whose east and west sides are meridians passing through the most easterly and westerly points of the survey and whose north and south sides are parallels passing through the most northerly and southerly points of the survey. In order to insure the proper location of points on the sheet, the enclosing rectangle and principal points of control are usually first plotted to small scale. On the drawing paper the enclosing rectangle is drawn to the required scale, estimating the position of lines by means of the small-scale sketch.

Let *HJKL* (Fig. 265) be the enclosing rectangle for the traverse of which the computations are tabulated in Fig. 264. The rectangle should be drawn with great care and it should be checked by scaling the lengths of the diagonals. The positions of the reference meridian and reference parallel are determined by scaling along the sides of the enclosing rectangle. For example, in Fig. 265, the reference meridian is found by scaling *JA* and *HM* equal to 2,454 ft., the total departure of the most westerly point of the survey. It is checked by scaling *ML* and *AK* equal to 708 ft., the total departure of the most easterly point in the survey.

If the drawing is of large size and there are a considerable number of control points to be plotted, the work will be facilitated if other meridians and parallels are constructed so that the area is divided into squares the sides of which are less than the length of the scale used and which represent some whole number of hundreds or thousands of feet at the given scale. Thus in the figure, meridians are

drawn at intervals of 1,000 ft. west of the reference meridian, and parallels at intervals of 1,000 ft. north of the reference parallel. If the scale of the drawing is 100 ft. = 1 in., the actual length of the sides of the resulting squares is 10 in. Each of the lines thus drawn is numbered with its distance from the reference meridian or reference parallel.

Each point of horizontal control is located on the sheet by plotting its total latitude and departure, and its position is verified by scaling the length of the preceding course.

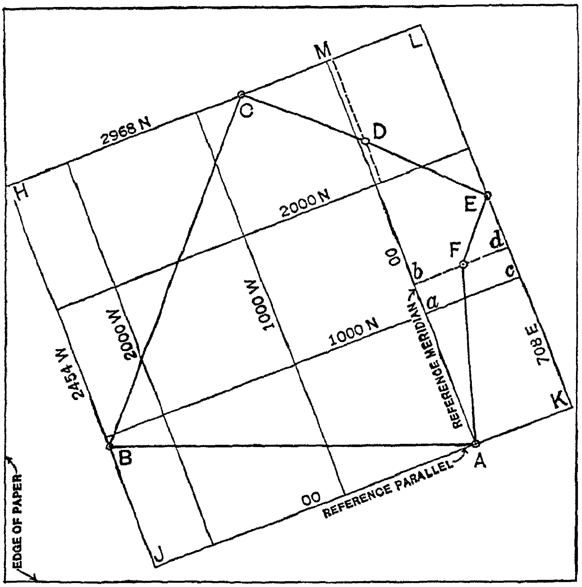


Fig. 265.—Plotting by coordinates.

For example, with the origin at A (Fig. 265) F has a total N latitude of 1,250 ft. and a total E departure of 394 ft. The point F is plotted by laying off above parallel 1,000N the distances ab and cd each equal to $1{,}250 - 1{,}000 = 250$ ft., then drawing the line bd, and finally by laying off to scale the distance $bF = 394$ ft. As a check, the distance EF as scaled from the drawing should, within the scale of the map, agree with the length of this line as computed from the adjusted latitude and departure ($EF = 506$ ft. by notes of Fig. 264).

When the angle between a given course and the meridian is small, a considerable error may be made in plotting the total departure of either end without appreciably altering the plotted length of the line. The same is true of any course making a small angle with the reference parallel when an error is made in plotting the total latitude of either end. For

this reason, where two or more of such courses are contiguous, the scaled latitude or departure (whichever is the smaller) of each course should be compared with the value from which the coordinates are calculated.

Long traverses that do not form closed figures may be more conveniently plotted without constructing the enclosing rectangle; in fact, often much of it would fall off the map. The general method is to construct the reference meridian and reference parallel as accurately as possible, and with these as a basis to draw other meridians and parallels, forming squares of convenient size, only where such lines are necessary for plotting the traverse points. For small drawings, construction lines other than those of the enclosing rectangle are usually unnecessary.

265a. The coordinate method is recognized as the most reliable method of plotting. Its single disadvantage as compared with the method of tangents or the method of chords is the increased amount of computation required preliminary to plotting. Often in the case of the closed traverse, latitudes and departures are necessarily computed for the purpose of determining the area (Art. 275), and with these values determined, the labor of computing coordinates is little more than that of computing tangent offsets or chord distances. The principal advantages of the coordinate method are: (1) that the size and shape of the drawing can be accurately determined beforehand; (2) that the accuracy of the location of any point does not depend upon the accuracy with which previous lines in the control system are plotted; (3) that the method of checking is simple and is unlike the method of plotting; and (4) that for closed traverses the field measurements are checked and the survey is balanced before plotting is begun.

266. Checking.—Checking the correctness of the plotted horizontal control is quite as essential as the verifying of field measurements. The methods of checking that may be employed are essentially the same for each of the first three methods of plotting (protractor, tangent, or chord). The process of checking the plotted location of points of horizontal control may proceed as the work of plotting progresses, but in any case it is completed before the work of plotting the details is begun. Frequently the process of checking serves not only to verify the work of plotting but also to verify the field measurements.

When the horizontal control is in the form of a traverse which has been brought to closure in the field, it should likewise close on paper, and if it does, a check is secured on the correctness of both the field work and the drafting. If for the purpose of this discussion the field work be regarded as correct, then any error of paper closure will be due to inaccuracies of laying off angles and distances. If a traverse is of considerable length it will rarely close on paper, no matter how

careful the work of plotting; if the error of closure is small, it is usually assumed to have gradually accumulated and the traverse is made to close by a gradual change in the position of the plotted lines.

Thus in Fig. 266*a* the full line $ABC'D'E'A'$ represents a traverse as first plotted, $A'A$ being a small error of closure, and the dash line $BCDEA$ shows the adjusted portion of the traverse. Where the error of closure is large, a mistake in laying off an angle or a distance has occurred, and the mistake may sometimes be quickly located by observing the simplest way of bringing the traverse to a closure. Thus, in Fig. 266*b* with the error of closure $A'A$ as shown, one might expect a mistake to have been made in the length of the side BC, and the first step in locating the error would be to scale this distance.

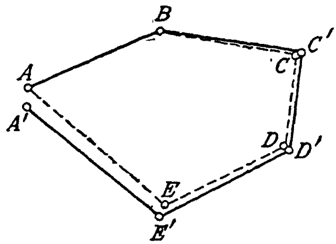


FIG. 266*a*.

Where the cause of the error of closure cannot be readily detected by measurement of the plotted angles and distances, the traverse is frequently plotted in the reverse direction, starting from the same initial point, and is continued until a point coincides with the corresponding point for the traverse as originally plotted. Thus, referring to Fig. 266*c*, $ABCD'E'A'$ represents a traverse as originally plotted and $AEDC$ represents the traverse plotted in the opposite direction from the initial point A. The two traverses come together at C, and the corrected line is therefore $ABCDEA$.

266a. When the horizontal control is a continuous or open traverse there is no absolute check on the precision of plotting as is the case with the closed figure. A satisfactory way of checking distances is as follows: Cut a strip of paper somewhat longer than the combined length of the several lines in the traverse. Let a, b, c, d, etc., be points to be marked on this strip corresponding to traverse points A, B, C, D, etc., on the drawing. Near one end and close to the edge, with a needle mark point a. Place this mark at A of the traverse and with the edge of the strip along AB, mark point b on its edge opposite B of the traverse. Move the strip to BC with b opposite the corresponding point of the traverse, and mark point c. Continue in this way around the entire traverse. Scale the total length marked off on the strip. This scaled distance should agree

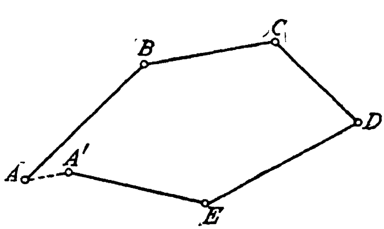


FIG. 266*b*.

very closely with the sum of the lengths according to the data used in plotting.

If deflection angles have been used in plotting, the direction of any line may be checked by calculating its azimuth or bearing and observing whether or not it makes the calculated azimuth or bearing angle with an established meridian. This check does not need to be applied to every line of the traverse, but in any case should be performed for the two end lines. If the relative directions of these two lines are found to be correct, it may be assumed that the directions of all other lines are without appreciable error, by reason of the fact that the direction of each line depends upon that of the preceding line. For traverses with numerous sides, however, this check is usually applied every five or six courses as the work of plotting progresses, and adjustment is made if appreciable error is found.

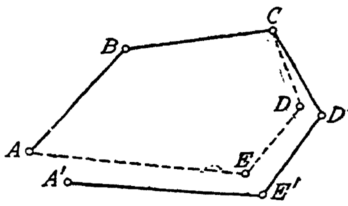


FIG. 266c.

If the traverse is plotted by using bearings or azimuths in the manner described in preceding articles, the deflection angles at the several points may be calculated and the directions of the courses may be checked by measuring the deflection angles either with the protractor or by scaling the tangent offsets or chord distances. For this case *all* courses must be checked, the reason for this being that the direction of any course may be in error without its affecting the direction of any other course.

266b. Cut-off Lines.—When during the process of running a continuous traverse, cut-off lines are established, as described in Art. 230,

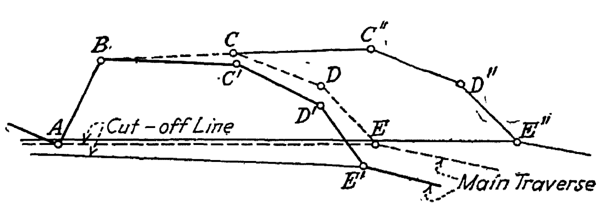


FIG. 266*d*.

both the field work and the plotting may be checked. Each cut-off line may be considered as forming the closing side of an independent traverse. Where the length of the cut-off line is measured, both angles and distances within the length intercepted by the cut-off line are checked if it closes on paper. If it fails to close, the procedure of checking is the same as for any other closed traverse, and when large errors appear to have been eliminated the portion intercepted by the cut-off line may be adjusted until the cut-off line actually closes the plotted circuit. Usually the length of the cut-off

line is not measured in the field, in which case an absolute check on the plotted length of the intercepted portion of the traverse cannot be obtained.

Figure 266*d* represents a portion of a main traverse with cut-off line of unmeasured length passing through *A* and *E*. The full line *ABC′D′E′* shows the portion as originally plotted for which the cut-off line does not pass through *A*, thus indicating an error in angle or distance or both. The line *ABCDE* shows the traverse after an adjustment in angle has been made at *B* so that the cut-off line passes through *A*. The line *ABC″D″E″* represents the same portion of the main traverse but with a gross error in the length of the second line, *BC* being the true length and *BC″* the erroneously plotted length. The cut-off line is seen to pass nearly through *A* as before, making it clear that a large error may be made in plotting a distance within the intercepted portion of the traverse and still have the cut-off line pass through the points of intercept or nearly so, provided the error is in a line that is nearly parallel to the cut-off line.

Thus the cut-off line of unmeasured length but of observed direction provides a means of checking the angles of the intercepted portion of the traverse, and also provides a means of checking against large errors in distance, except for those lines which have the same general direction as the cut-off line.

266c. Intersecting Lines.—When angles to a given station not on the traverse are observed from a number of stations of the traverse, both the field work and the plotting may be regarded as correct if the plotted positions of the lines to the given station intersect at a common point

In Fig. 266*e* the full line *ABC′D′E′* represents a portion of a continuous traverse, and the full lines intersecting at *O* and *O′* represent lines to the

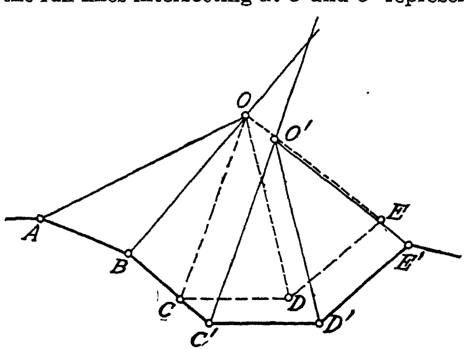


FIG. 266*e*.

station not on the traverse, the angles *OAB*, *OBC′*, *O′C′D′*, etc., being laid off in the same manner as are the angles of the traverse. The-fact

that the lines do not intersect at a common point indicates that an error exists in angle or in distance. The location of the error, and sometimes also its nature, may be determined by inspection. Thus for the case shown in the figure, OO' is nearly parallel to BC, hence it might be expected that the length of BC was in error by an amount approximately equal to OO'. If this were found to be so, then the traverse would be adjusted as shown by the line $ABCDE$ and the intersecting lines from C, D, and E (shown dotted) would pass through or approximately through the point O.

267. Omitted Measurements.—When for any reason it is impossible or impracticable to determine by field observations the length and bearing of every side of a closed traverse, the missing data may generally be computed provided not more than two quantities are omitted. When the missing quantities have been supplied, the coordinates may be computed and the traverse may be plotted (or the area may be calculated) as though all field measurements had been taken. In the process of calculating the unknown quantities it must be assumed that the observed values are without error, and hence all errors of measurement are thrown into the computed lengths or bearings. Measurements which may be supplied in this manner are:

1. Length and bearing of one side.
2. Length of one side and bearing of another.
3. Lengths of two sides for which the bearings have been observed.
4. Bearings of two sides for which the lengths have been observed.

Methods of parting land, which involve the calculation of lengths and bearings of unknown sides of a traverse, are discussed in Arts. 280–284.

267a. Length and Bearing of One Side Unknown.—The problem of determining the length and bearing of one side of a closed traverse is exactly the same as that of computing the length and bearing of the side of error in any closed traverse for which field measurements are complete. The latitudes and departures of the known sides are computed. For reasons explained in Art. 262, if the algebraic sum of the latitudes and algebraic sum of the departures of the known sides are respectively designed by ΣL and ΣD, then the length of the unknown side is

$$S = \sqrt{(\Sigma L)^2 + (\Sigma D)^2} \tag{9}$$

and the tangent of the bearing angle is

$$\tan \alpha = \frac{-\Sigma D}{-\Sigma L} \tag{10}$$

with due regard to sign.

267b. Length of One Side and Bearing of Another Side Unknown. Figure 267a represents a closed traverse for which the direction of the line $DE = d$ and the length of the line $EA = e$ are not determined by field measurements. Let an imaginary line extend from D to A, cutting off the unknown sides from the remainder of the

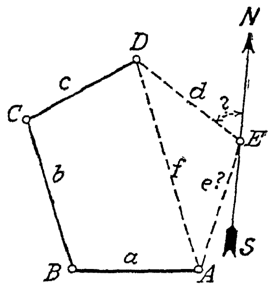


FIG. 267a.

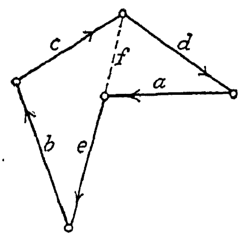


FIG. 267b.

traverse. Then $ABCDA$ forms a closed traverse for which the side $DA = f$ is unknown in both direction and length. By the method of the preceding article

$$\text{length of } f = \sqrt{(\text{Lat. } (a + b + c))^2 + (\text{Dep. } (a + b + c))^2} \quad (11)$$

and

$$\tan \text{bearing angle of } f = \frac{\text{Dep. } f}{\text{Lat. } f} \quad (12)$$

The direction of f having been computed, and the direction of e being known, the angle between the lines e and f in $\triangle DEA$ is

$$\angle DAE = \text{azimuth of } AE - \text{azimuth of } AD$$

In the triangle ADE the length of the two sides d and f and one angle DAE are known. By the relation that sines of angles are proportional to sides opposite

$$\sin DEA = \sin DAE \cdot \frac{f}{d}$$

With angle DEA known, angle ADE can, of course, be computed, and the remaining unknown length is given by the expression

$$e = f \frac{\sin ADE}{\sin DEA} = d \frac{\sin ADE}{\sin DAE}$$

Also the azimuth of DE = azimuth of $DA - \angle ADE$.

The unknown length EA and the unknown bearing of DE having thus been determined, the latitudes and departures of these lines can be computed, and the numerical calculations involved in the solution

of the triangle ADE will be verified if the algebraic sum of the $\left\{\begin{matrix}\text{latitudes}\\ \text{departures}\end{matrix}\right\}$ of the sides DE and EA is equal to the algebraic sum of the $\left\{\begin{matrix}\text{latitudes}\\ \text{departures}\end{matrix}\right\}$ of the sides AB, BC, and CD.

267c. The preceding method of solution is, in general, applicable even though two partially unknown courses are not adjacent one to the other. Obviously the latitude and departure of any line of fixed direction and length is the same for one position of the line as for any other. In other words, a line may be moved from one position to a second position parallel with the first, and its latitude and departure will remain unchanged. If this is the case, then it must also be true that the algebraic sum of the latitudes and departures of any system of lines forming a closed figure must be zero, regardless of the order in which the lines are placed. Thus the courses which are shown in the order a, b, c, d, e in Fig. 267a are given in the order a, e, b, c, d in Fig. 267b. If now it is assumed that the direction of d and the length of a (Fig. 267b) are unknown, the problem of determining these unknown quantities is seen to be identical with that explained in the preceding article for the case where the partially unknown sides were contiguous. From the latitudes and departures of e, b, and c the length and bearing of f are determined, and with the other known quantities, this length furnishes sufficient data for the solution of the triangle for which the sides are a, d, and f.

Hence, in general, for any closed traverse when two sides which are partially unknown are not adjacent one to the other, they are treated as though they were adjacent. By means of the latitudes and departures of the known sides, the length and direction of the side necessary to close the figure formed by the known sides is computed, after which the solution is identical with that described in the preceding article. To simplify the problem the data are usually plotted roughly to small scale. Following is the solution of a typical problem.

Example: Below are tabulated the measured lengths and bearings for the courses of a closed traverse a to f, together with the latitudes and departures of the known sides. The length of b and the bearing of e are not observed. The general direction of e is southwest. It is desired to compute the unknown length and direction. Quantities in parenthesis are derived from following calculations.

In Fig. 267c the lines a, c, d, and f are the courses for which the length and bearing are known. The line g is the closing side of the figure

Line	Length, feet	Bearing	Latitude N	Latitude S	Departure E	Departure W
a	500.0	N	500.0	....	0.0	....
b	unknown	N45°00′E				
	(889.9)	....	(629.3)	....	(629.3)	....
c	854.4	S69°27′E	......	300.0	800.0	....
d	1,019.8	S11°19′E	......	1,000.0	200.0	....
e	1,118.0	unknown				
		(S78°57′W)	......	(214.3)	......	(1,097.3)
f	656.8	N54°06′W	385.0	....	......	532.0
g	(625.5)	(N48°26′W)	(415.0)	....	......	(468.0)

formed by the known courses. From the tabulated quantities, the sum of the known latitudes is 1,300S + 885N = 415S, hence the latitude of g is 415N. Similarly the departure of g is 1,000 − 532 = 468W.

$$\tan \text{bearing } g = \frac{468.0}{415.0} = 1.127$$

and the bearing of g = N48°26′W

The length of $g = \sqrt{(415)^2 + (468)^2} = 625.5$ ft.

Since the bearing of b is N45°00′E,

$$\angle E = 45°00' + 48°26' = 93°26'$$

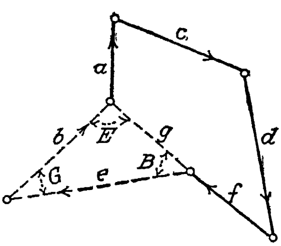


FIG. 267c.

$$\sin G = \frac{g}{e} \sin E = \frac{625.5}{1{,}118} \times 0.9982 = 0.5585$$

$$\angle G = 33°57', \ \angle B = 180°00' - 93°26' - 33°57' = 52°37'$$

$$\text{length } b = \text{length } e \cdot \frac{\sin B}{\sin E} = 1{,}118 \times \frac{0.7946}{0.9982} = 889.9 \text{ ft.}$$

$$\text{Bearing of } e = 180° - 48°26' - 52°37' = \text{S}78°57'\text{W.}$$

As a check on the calculations the latitudes and departures of the lines b and e are computed and the values are shown in parenthesis. The sum of the latitudes and the sum of the departures for courses a, b, c, d, e, and f are found to be zero, and hence the computations for determining the unknown length and bearing are correct.

267d. When the length of one side and the bearing of another are unknown, the solution described in the preceding article will in general render two values of each of the unknowns, and it is often impossible to tell which are the correct values unless the general direction of the side of unknown bearing is observed. In Fig. 267*d*,

ABCD is the known portion of a traverse and *DA* is the closing side for this portion. A line of unknown length but known direction extends from *A* in the direction of *E* and *E'*. A line of unknown direction but known length extends from *D* to intersect the line of known direction at *E* or *E'*, *DE* being equal to *DE'*. Evidently sin $\angle DEA$ = sin $\angle DE'A$, since sin $(90° + \alpha)$ = sin $(90° - \alpha)$, and further, solving the triangle for the length of the unknown side gives the value *AE* or *AE'* depending on which of the above angular values is used.

As the angle between the line of unknown length and the line of unknown bearing approaches 90°, the solution becomes weak. This is illustrated by Fig. 267*e* for which *DA* is the closing side of the traverse formed by the known lines, *DEE'* is the direction of the side of unknown length, and *AE* = *AE'* is the length of the side of unknown direction. It will be recalled (see Art. 21) that the sines of angles near 90° change slowly. Since the angle at *E* or *E'* is nearly 90°, and its value is computed through the use of its sine, a small error in computation may make a relatively large change in the angle (a difference of 0.00001 in the sine effects a change of 0°11', on either side of 90°). Also the angle at *E* is used in computing the angle at *A* and hence any error in the computed value of $\angle E$ is transmitted to the computed value of $\angle A$. Now if $\angle A$ is of average size, the change in the sine will be much more rapid (on either side of 45°, a change of 0.00001 in the sine produces a change of only $\frac{1}{20}'$ in the angle). Since the side *DE* is computed by using sin $\angle A$, the relative error in the computed value of *DE* is the same as that in sin $\angle A$ which may be many times that in sin $\angle E$.

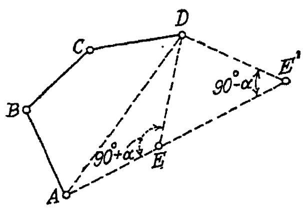


FIG. 267*d*.

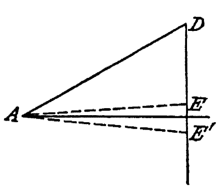


FIG. 267*e*.

Also when the angle between the partially unknown lines is nearly 90°, it is impossible to distinguish which of the two determinations is the correct one, even though the general direction of the line of unknown bearing has been observed.

The solution becomes weak when the partially unknown lines approach perpendicularity, and the solution becomes strong as the angle between these lines becomes small.

267e. Length of Two Sides Unknown.—This problem commonly occurs where angular observations are taken from two or more points in the main traverse to some landmark, the measurements being introduced as a check. It occasionally occurs on main traverse

lines where there are obstacles to the direct measurement of length but where angles are observed. The solution is so nearly identical with that for the case where the direction of one side and the length of another are unknown (Art. 267*b*) that a detailed description will not be given here.

In Fig. 267*f*, *ABCD* represents the portion of a closed traverse for which the courses are known in both direction and length, and the lines *DE* and *EA* are courses for which the directions are observed but for which the lengths are unknown. From the latitudes and departures of the known sides, the length and bearing of the closing line *DA* is computed and the angles *A*, *D*, and *E* in the triangle *ADE* are calculated from the known directions of the sides. *DE* and *EA* are determined through the relation

$$\frac{DE}{\sin A} = \frac{EA}{\sin D} = \frac{DA}{\sin E}$$

If the two lines are not adjacent to one another, the problem may be solved as though they were, as explained in Art. 267*c*. If the partially unknown lines are parallel, the problem is indeterminate. As the angle between the lines approaches zero or 180°, a small error in angle produces a relatively large error in computed length; that is, the solution becomes weak. The solution is strongest when the angle between the partially unknown lines is 90°.

267f. Direction of Two Sides Unknown.—The solution for this case is similar to that described in the preceding article. Referring

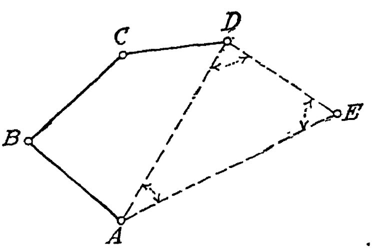


Fig. 267*f*.

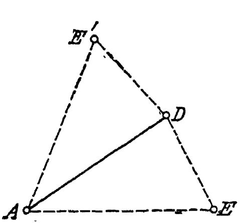


Fig. 267*g*.

to Fig. 267*f*, if *DA* is the closing side of the known portion of the traverse, its direction and length having been computed, then the lengths of the three sides of the triangle *ADE* are known, and the angles *A*, *D*, and *E* can be calculated.

In all cases the general direction of at least one of the partially unknown lines must be observed, since the values of the trigonometric functions merely determine the shape of the triangle but do not fix its position. Thus in Fig. 267*g* if *DA* is the closing line of the known portion of the traverse forming the base of the triangle of

which the courses of unknown direction but of known length are the legs, then it is evident that the vertex may fall at either E or E'.

When the partially unknown lines are parallel but not of the same length, their direction is that of the closing side of the figure formed by the known courses. When the partially unknown sides are parallel and of the same length the problem is indeterminate, since the length of the closing side of the figure formed by the known courses becomes a point.

268. Plotting Details.—In general the processes of plotting details on the map are not unlike those employed in making the field observa-

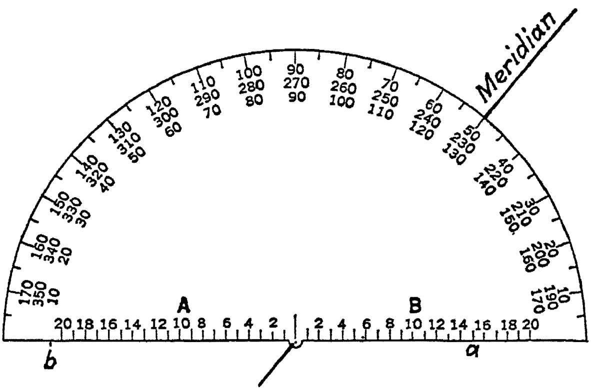


FIG. 268.—Protractor for plotting details.

tions. The work of plotting details is less refined than that of plotting the horizontal control, the aim generally being to plot objects of definite shape and size with such precision that subsequently scaled dimensions taken from the map will be correct within the allowable limit of error.

Angles are laid off with a protractor as described in Art. 61*a*, p. 62. If any considerable number of points are to be plotted from each control station, it is desirable that the protractor be of a type adapted to laying off both angle and distance in one operation.

The protractor shown in Fig. 268 may perhaps be used more rapidly than any other type. The linear scale is so designed that distances are measured outward from the center along the diametrical edge. The angle numbers are so arranged that azimuths, bearings, or deflection angles may be laid off by rotating the protractor about the station as a center until the graduation on the protractor representing the given angle coincides with the reference line. The point is then plotted at the given distance on the diametrical scale.

When the directions to details are given in azimuths or bearings measured from a given station, a reference meridian is drawn through the station and angles are laid off from this meridian. In plotting a detail of given azimuth (measured from north) the protractor is rotated until the graduation corresponding to the given azimuth coincides with a point on the meridian, and the point representing the detail is plotted at the given distance on scale *A* or *B* according as the azimuth is less or greater than 180°. Thus in the figure, the point *a* is plotted at an azimuth of 50° (measured from north) and *b* is plotted at an azimuth of 230°.

Where the position of a detail is fixed by a distance from a control station and an angle from a reference line, the point is plotted as indicated above. Where the detail is located by angles from two stations, these angles are laid off and the point is plotted at the intersection of the two lines thus defined. Where a large number of points are located by angular observations from two stations, time will be saved by using simultaneously two protractors of the type shown in Fig. 268. The point of intersection between the diametrical edges of the two protractors marks the plotted position of the detail. Where objects are located by linear measurements from two stations, they are plotted by intersecting arcs of the given radii drawn with the dividers. Where objects of somewhat indefinite form are located by perpendicular offsets from a transit line, these offsets may usually be plotted by estimating the perpendiculars with the eye. Where objects are located by other methods the process of plotting so nearly parallels that employed in the field that no further description is necessary.

The location of all important details should be checked by actual map measurements, but the position of less important objects is checked simply by inspecting the drawing. It often happens that mistakes in the field work give an object more than one location, or that a part of the field notes is confusing. In such case it is best to proceed with the plotting of other portions, for these when mapped may help in clearing up the doubtful points.

269. Conventional Signs.—Objects are represented on a map by signs or symbols, many of which have such widely accepted usage that they may be said to be conventional in character. Some of these are shown in Figs. 269*a* to 269*d*. Of course, the kind or purpose of the map defines what objects shall be represented. For many purposes, even if the necessary data were available, it would lessen the usefulness of the map if all of the objects were shown for which conventional signs are given in the figures. On the other hand, maps made for special purposes often show features not included among the conventional signs here listed. Appropriate symbols for many such features will be found among the surveying and mapping publications of the various national and state surveys. A chart of more than three hundred standard symbols adopted by the

U. S. Board of Surveys and Maps is published and for sale by the U. S. Geological Survey, Washington, D. C. (price of 1932 edition,

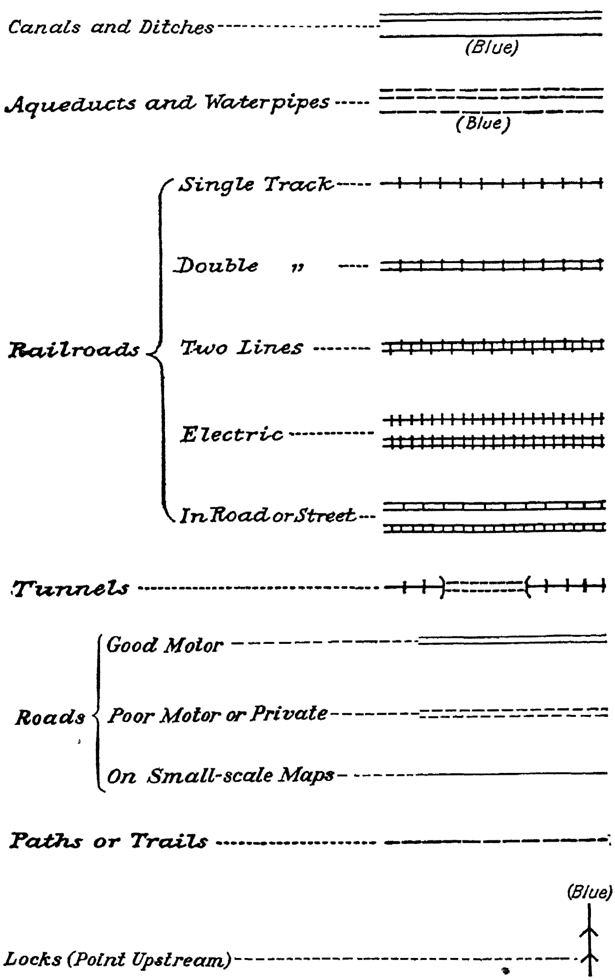


FIG. 269a.—Conventional signs; public and private works.

40 cents). The symbols are for (1) works and structures, (2) boundaries, marks, and monuments, (3) drainage, (4) relief, (5) land classification, (6) hydrography, (7) aids to navigation, (8) military use, and (9) air navigation.

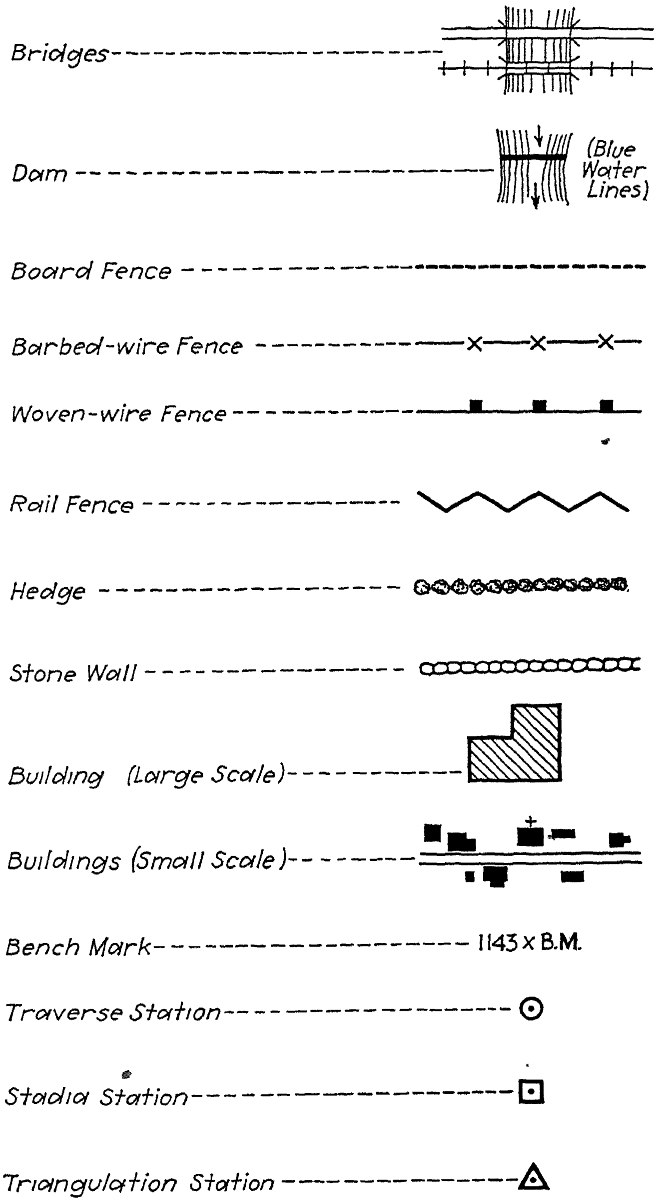


FIG. 269b.—Conventional signs; public and private works.

Where the symbols shown are not in general use, a key to their meaning should be shown on the map sheet. The size of the symbols should be proportioned somewhat to the scale of the map.

Where the map is on tracing cloth from which prints are to be made, all features are ordinarily inked in black. Where the map is

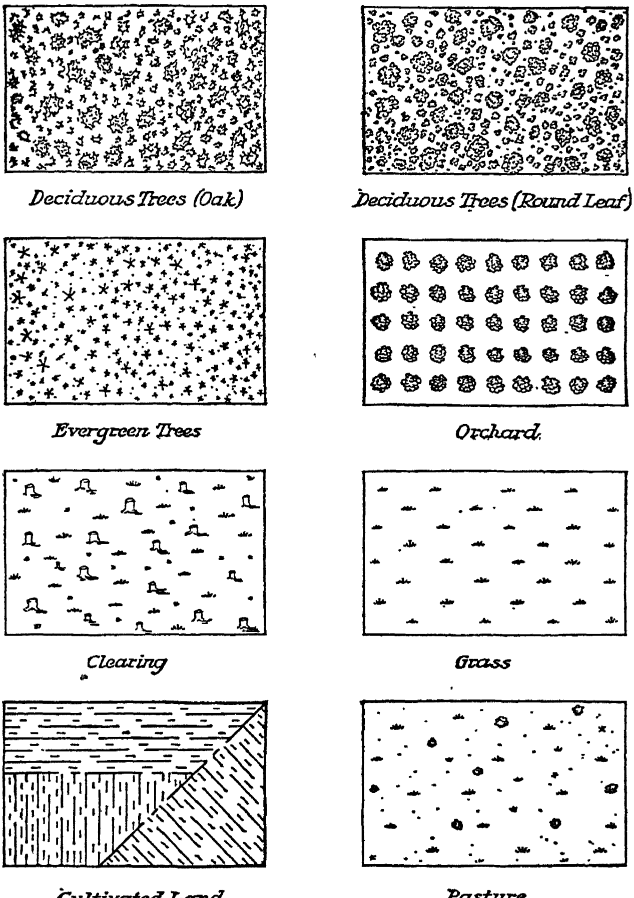


Fig. 269c.—Conventional signs; culture.

made on paper, the lakes, rivers, and other hydrographic features (Fig. 269*d*) are usually shown in blue. Often the conventional signs referring to the culture of the land are shown in green; and land forms, contour lines, etc., are shown in brown. For further suggestions regarding the plotting of topographic symbols, see Arts.

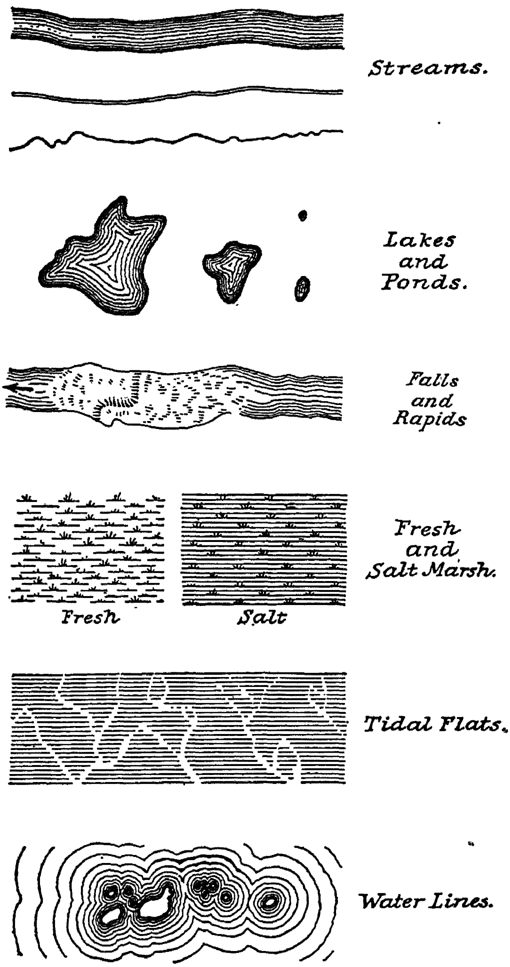


FIG. 269*d*.—Conventional signs; hydrography.

437–437*d*. When lines of horizontal control are left on the sheet, they are usually inked in black.

270. Problems.

1. With the protractor plot to the scale of 1 in. = 200 ft. the closed deflection-angle traverse for which notes are given below. Measure the error of closure and record it on the sheet. Distribute the error as suggested in Art. 266.

Station	Deflection angle	Length, feet
A	92°00′R	
		338
B	9°00′L	
		307
C	76°45′R	
		792
D	74°45′R	
		822
E	102°00′R	
		624
F	23°30′R	
		620
A		

2. Assuming the course AB in the traverse of problem 1 to be of zero azimuth, compute the azimuths of the remaining courses. Using a protractor, lay off the azimuths from a central meridian and plot the traverse at the scale of 1 in. = 200 ft., as described in Art. 257*a*. Measure the error of closure and record it on the sheet. Distribute the error of closure as suggested in Art. 266.

3. Plot the following continuous traverse at the scale of 1 in. = 400 ft. by the method of tangents, using a 10-in. base. Lay off successive lines by deflection angles as described in Art. 258. Assume the direction of the first course to be north, and compute the bearings of other courses. Check the accuracy of the plotting by methods described in Art. 266*a*.

Station	Deflection angle
118 + 75.0	
98 + 95.6	39°47′L
73 + 01.4	17°28′L
70 + 13.5	14°08′L
49 + 41.3	3°11′L
40 + 00	49°59′L
37 + 18.8	32°18′R
18 + 26.0	18°44′R
5 + 03.2	7°31′L
0 + 00	

4. Plot the following continuous azimuth traverse at the scale of 1 in. = 400 ft. by the method of tangents, establishing the directions of the several lines from a centrally located meridian as described in Art. 258*b*. From the azimuths calculate the deflection angles at the several points, and check the accuracy of the plotting as described in Art. 266*a*.

Course	Azimuth	Distance, feet
AB	142°08′	815.3
BC	181°37′	1146.0
CD	296°13′	520.8
DE	323°46′	816.5
EF	249°51′	726.4
FG	214°03′	1862.0
GH	195°45′	2795.5
HJ	191°28′	2463.7
JK	138°42′	586.4

5. Given the notes of problem 1. Assuming the direction of *AB* to be north, calculate the bearings of the several courses. Compute the latitudes and departures, using four-place logarithms. Compute the error of closure and balance the survey by the Compass Rule (Art. 263*b*).

6. Given the following notes of a survey with the transit. Calculate the latitudes and departures, using five-place logarithms. Compute the error of closure, and balance the survey by the Crandall method (Art. 263*d*).

Course	Bearing	Distance, feet
AB	N48°20′E	529.6
BC	N87°43′E	592.0
CD	S 7°59′E	563.6
DE	S82°12′W	753.4
EA	N48°12′W	428.2

7. Given the following notes for a closed traverse. (1) Calculate the latitudes and departures, using five-place logarithms, and compute the error of closure. Balance the survey by each of the three rules given in this text, and for each course find the change in length and in direction caused by the application of each of the rules.

Course	Azimuth	Distance, feet
AB	0°42′	1,221.2
BC	94°03′	541.3
CD	183°04′	795.4
DA	232°51′	646.8

(2) Add 40°00′ to each of the given azimuths, and make computations called for in (1). (3) Compare the results of (2) with those of (1), and explain reasons for variations.

8. Given the following data for a closed traverse. By the methods of Art. 267*a* compute the length and bearing of the unknown side. Make calculations with the slide rule.

Course	Bearing	Distance, feet
AB	N82°W	461
BC	unknown	unknown
CD	N68°15′E	829
DA	N80°45′E	441

9. Given the following data for a closed traverse, the length of *DE* and the azimuth of *EA* not being observed in the field. Determine the unknown quantities. The direction of *EA* is east of north, and zero azimuth is north.

Course	Azimuth	Distance, feet
AB	106°13′	1,081.3
BC	195°14′	1,589.5
CD	247°07′	1,293.7
DE	332°22′	unknown
EA	unknown	1,737.9

10. Given the following data for a closed traverse, the lengths of the courses *BC* and *DE* not having been measured in the field. Compute the unknown lengths, using five-place logarithms.

Course	Bearing	Distance, feet
AB	N9°30′W	689.32
BC	N56°55′W	unknown
CD	S56°13′W	678.68
DE	S2°02′E	unknown
EA	S89°31′E	1,082.71

11. Given the data of problem 4. Calculate the coordinates of the several points in the survey, assuming that the origin is at *A*. Plot the traverse by the coordinate method, using a scale of 1 in. = 400 ft.

References

1. Bristol, J. J., "Plotting Coordinate Surveys," *Mining and Scientific Press*, Vol. 100, p. 487, 524, April 2, 1910.
2. Crandall, C. L., "The Adjustment of a Transit Survey as Compared with That of a Compass Survey," *Proc. Am. Soc. Civ. Eng.*, Vol. 26, p. 1164, December, 1900.

3. Tuttle, G. W., Discussion of Prof. Crandall's paper on "The Adjustment of a Transit Survey," *ibid.*, Vol. 27, p. 119, February, 1901.
4. Hayes, J. L., "Mapping Practice of the Anaconda Copper Co.," *Eng. and Min. Jour.*, Vol. 116, No. 20, p. 841, November 17, 1923.
5. Rappleye, H. S., "Adjustment of Transit and Stadia Traverses," *Trans. Am. Soc. Civ. Eng.*, Vol. 95, p. 232, 1931.

CHAPTER XVI

CALCULATION OF AREAS OF LAND

271. General.—One of the primary objects of a land survey is to determine the area of the tract or tracts with which the survey is concerned. During the progress of a survey of this character a closed traverse is run, the lines of the traverse being made to coincide with property lines where possible. Where the boundaries are irregular or curved, or where they are occupied by objects which make direct measurement impossible, they are located with respect to the traverse line by appropriate angular and linear measurements. In the usual course of such a survey the lengths and bearings of all straight boundary lines are determined either directly or by computation, the irregular boundaries are located with respect to traverse lines by perpendicular offsets taken at appropriate intervals, and the radii and central angles of boundaries which form the arcs of circular curves are obtained.

The following articles explain the several common methods by means of which these data are employed in computing areas.

In ordinary land surveying, the area of a tract of land is taken as its projection upon a horizontal plane, and it is not the actual area of the surface of the land. For precise determinations of the area of large tracts, such as of a state or nation, the area is taken as the projection of the tract upon the earth's spheroidal surface at mean sea level.

In the United States the common units of area are for rural lands the acre, and for urban lands the square foot. There are 640 acres in 1 sq. mile; 1 acre = 10 sq. Gunter's chains = 160 sq. rd. = 4,047 sq. m. = 4,840 sq. yd. = 43,560 sq. ft.

272. Methods of Determining Area.—The area of a tract may be determined by the following methods:

1. By plotting the boundaries to scale as described in Chap. XV. The area of the tract may then be found by use of the planimeter as described in Arts. 157 and 158, or it may be calculated by dividing the tract into triangles and rectangles, scaling the dimensions of these figures, and calculating their areas mathematically. This method is useful in roughly determining areas or in checking those that have

been computed by more exact methods. Its advantage lies in the rapidity with which calculations can be made.

2. By mathematically computing the areas of the individual triangles into which the tract may be divided, the computations being based upon angular and linear measurements taken in the field. This method is often employed when the corners of a tract are located by triangulation from two or more stations within the boundaries. It is also used for finding the area within a traverse when it is not expedient to calculate the latitudes and departures of the sides.

3. By coordinates. Where the coordinates or total latitudes and departures of the corners of a tract have been found as described in Arts. 260 to 267, the area of the tract may be determined expeditiously by computations based upon these coordinates.

4. By double meridian distances. This method, which is a form of the method of coordinates, is almost universally employed for finding the area of lands for which the boundaries are straight lines whose lengths and bearings are known. Its advantage lies in the ease with which computations may be made and checked. The basis of computations are the latitudes and departures of the several courses forming the boundary traverse.

5. By double parallel distances. This method is in every way the equivalent of that of double meridian distances, but is less widely employed. It may be used to good advantage in independently checking areas calculated by the double-meridian-distance method.

273. Area by Triangles.—Table XXII gives the relations between the area of a triangle and its angles and lengths of sides. When the lengths of two sides and the included angle of any triangle are known, its area is given by the expression

$$\text{area} = \tfrac{1}{2}ab \sin C \tag{1}$$

When the lengths of the three sides of any triangle are given, its area is determined by the equation

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)} \tag{2}$$

in which $s = \tfrac{1}{2}(a + b + c)$.

In surveying small lots as for a city subdivision, it is common practice to omit the determination of the error of closure of each lot, and hence the calculation of latitudes and departures is unnecessary. Under such circumstances the area may be calculated by dividing the lot, usually quadrangular in shape, into triangles, as illustrated by Fig. 273*a*, for each of which two sides and the included angle have been measured. By Eq. (1), the areas of *ABD* and *BCD* are computed; the sum of these two areas is the area of the lot. The area thus found may be independently checked by computing the

areas of the two triangles ABC and CDA formed by dividing the quadrilateral by a line from A to C.

The accuracy of the field work may be investigated by computing the lengths of the diagonals. Thus BD can be determined by solving either triangle ABD or triangle BCD. The field measurements are without error if the length of BD computed by solving one of the triangles is the same as that computed by solving the other.

273a. Figure 273*b* illustrates a survey made by a single set-up of the transit at O, such as might be the case for a small lot where the property lines $ABCD$ were obstructed or where the transit could not be set up at the corners. Under these circumstances the angles about O and the distances OA, OB, OC, and OD are measured in the

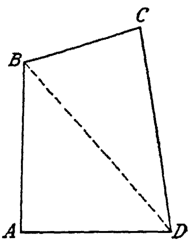


Fig. 273*a*.

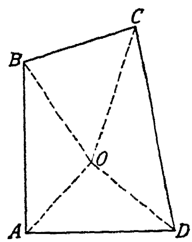


Fig. 273*b*.

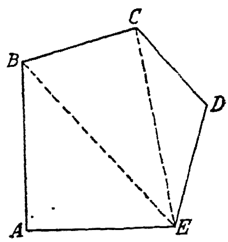


Fig. 273*c*.

field. Since in each triangle two sides and the included angle are known, the area may be determined as just described. If in addition to the above measurements, the lengths of the sides AB, BC, etc., are taken, the area of the lot may be independently checked by solving each triangle by Eq. (2).

273b. In Fig. 273*c*, the figure $ABCDE$ represents the boundary of a tract surveyed by simple triangulation, each of the angles of the three triangles into which the figure is divided being measured, but only the distance AB being determined in the field. In order to determine the length of the unknown boundaries, it is necessary to solve in succession the triangles ABE, BEC, and ECD, the lengths of all sides being determined by the relation that the length of the side of a triangle is proportional to the sine of the opposite angle. When the lengths of the sides have thus been found, the areas of the individual triangles may be computed by either of the equations given in Art. 273.

274. Area by Coordinates.—Frequently it happens that the rectangular coordinates of the points defining the corners of a tract of land with reference to some arbitrarily chosen coordinate axes are computed before the directions and lengths of the connecting lines are known. For example, the corners may be located by a system of

triangulation, neither the direction nor the length of any of the boundaries being measured directly. Or a traverse may be inside a given tract, and the corners of the property may be located by direction and distance from traverse points. In either case, the coordinates of the corners are useful in finding not only the lengths and bearings of the boundary lines, but also in computing the area of the tract. Essentially the computation is that of finding the areas of trapezoids formed by projecting the lines upon one of a pair of coordinate axes. The coordinate axes are usually a true meridian and a parallel at right angles thereto.

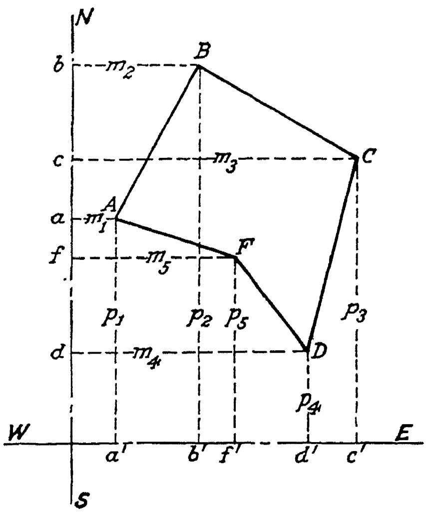


FIG. 274.

In Fig. 274, *ABCDF* represents a tract the area of which is to be determined, *SN* being the reference meridian, *WE* being the reference parallel, and the coordinates of *A*, *B*, etc., being known. For any point these coordinates are the perpendicular distance from the reference meridian, defined as the *total departure* or the *meridian distance,* and the perpendicular distance from the reference parallel, defined as the *total latitude* or the *parallel distance.* Thus, for *A* the meridian distance is $aA = m_1$ and the parallel distance is $a'A = p_1$. Meridian distances are regarded as positive or negative according to whether they lie to the east or to the west of the reference meridian; parallel distances are regarded as positive or negative according to whether they lie to the north or south of the reference parallel. In the figure all meridian and parallel distances are positive, since all points lie in the northeast quadrant.

For reasons explained in the later discussion of the D.M.D. method (Art. 275) the algebraic sum of the areas of the projection trapezoids upon the reference meridian is equal to the area of the tract. Then

$$2 \text{ area } ABCDF = (m_1 + m_2)(p_1 - p_2) + (m_2 + m_3)(p_2 - p_3) + (m_3 + m_4)(p_3 - p_4) + (m_4 + m_5)(p_4 - p_5) + (m_5 + m_1)(p_5 - p_1) \quad (3)$$

By multiplication and a rearrangement of terms in the above equation there is obtained

$$2 \text{ area } = m_2p_1 - m_1p_2 + m_3p_2 - m_2p_3 + m_4p_3 - m_3p_4 + m_5p_4 - m_4p_5 + m_1p_5 - m_5p_1 \quad (4)$$

Example 1: By the use of Eq. (4) calculate the area of the tract whose corners are defined by coordinates as shown below:

Corner	1	2	3	4	5
Meridian distance, ft.............	300	400	600	1,000	1,200
Parallel distance, ft..............	300	800	1,200	1,000	400

$$\begin{aligned} 2 \text{ area } &= (400 \times 300) - (300 \times 800) + (600 \times 800) - (400 \times 1{,}200) \\ &\quad + (1{,}000 \times 1{,}200) - (600 \times 1{,}000) + (1{,}200 \times 1{,}000) - \\ &\quad (1{,}000 \times 400) + (300 \times 400) - (1{,}200 \times 300) \\ &= 3{,}120{,}000 - 2{,}080{,}000 = 1{,}040{,}000 \text{ sq. ft.} \end{aligned}$$

$$\text{Area} = \frac{1{,}040{,}000}{2} = 520{,}000 \text{ sq. ft.}$$

274a. Equation (3) may also be written in the form

$$2 \text{ area } = p_1(m_2 - m_5) + p_2(m_3 - m_1) + p_3(m_4 - m_2) + p_4(m_5 - m_3) + p_5(m_1 - m_4) \quad (5)$$

which form leads to the following rule:

Rule.—*To determine the area of a tract of land when the coordinates of its corners are known, multiply the parallel distance of each corner by the difference between the meridian distances of the following and the preceding corners, always algebraically subtracting the preceding from the following. One half of the algebraic sum of the resulting products is the required area.*

Ordinarily the application of this rule involves somewhat less labor than does the method described in the preceding paragraph.

Example 2: Given the following data. Find the required area by applying the preceding rule:

Corner	1	2	3	4	5
Meridian distance, ft.	300	400	600	1,000	1,200
Parallel distance, ft.	300	800	1,200	1,000	400

$$\begin{aligned} 2 \text{ area} &= 300(-800) + 800(300) + 1{,}200(600) + 1{,}000(600) + 400(-700) \\ &= -240{,}000 + 240{,}000 + 720{,}000 + 600{,}000 - 280{,}000 \\ &= 1{,}040{,}000 \text{ sq. ft.} \\ \text{Area} &= \frac{1{,}040{,}000}{2} = 520{,}000 \text{ sq. ft.} \end{aligned}$$

275. Principles of Double-meridian-distance Method.—In computing area by the double-meridian-distance method, the latitudes and departures of all the courses are determined as described in Art. 261 and the survey is balanced, usually by one of the rules

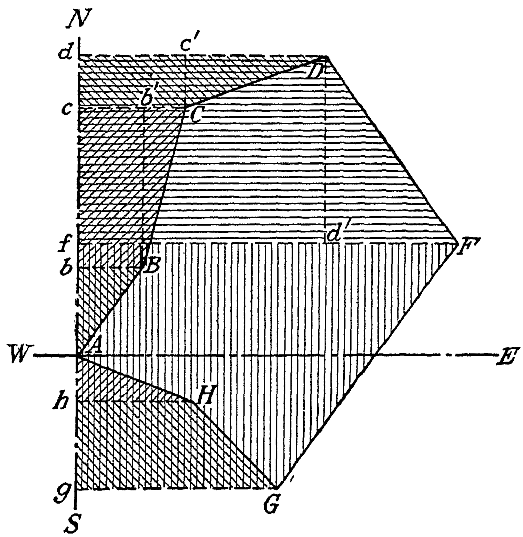


Fig. 275.

of Art. 263. A reference meridian is then assumed to pass through some corner of the tract, usually for convenience the most westerly point in the survey, and double the areas of the trapezoids or triangles formed by orthographically projecting the several traverse lines upon the meridian are computed. The algebraic sum of these double areas is double the area within the traverse.

Thus, in Fig. 275, *ABCDFGH* represents a closed traverse and *SN* represents a meridian passing through the corner *A*, which is the

most westerly point. The points B, C, D, etc., are projected upon the meridian to the points b, c, d, etc., the lines of projection Bb, Cc, Dd, etc., being perpendicular to the meridian.

The perpendicular distance from the meridian to any point in the traverse is called the total departure or the *meridian distance* of the point. Thus the meridian distance of B is Bb. Meridian distances are considered as being positive if they lie east of the reference meridian and are considered as being negative if they lie west of the reference meridian.

The sum of the meridian distances of the two extremities of a line is called the *double meridian distance* of the line. Thus the double meridian distance of BC is $Bb + Cc$. It is clear that if the meridian passes through the most westerly corner of the traverse, the double meridian distance of all lines will be positive, which is a convenience, but not a necessity, in computing.

As explained in Art. 261, the length of the orthographic projection of a line upon the meridian is the latitude of the line, latitudes being considered as positive if the direction of the line has a northerly component and negative if the direction of the line has a southerly component. Thus the latitude of BC is bc and is positive, and that of DF is df and is negative.

From the figure it is seen that each projection trapezoid or triangle for which a course in the traverse is one side, is bounded on the north and south by meridian distances and on the west by the latitude of that course. Thus the projection trapezoid for BC is $BCcb$. It is therefore seen that the double area of any triangle or trapezoid formed by projecting a given course upon the meridian is the product of the double meridian distance of the course and its latitude.

$$\text{Double Area} = \text{D.M.D.} \times \text{Latitude} \qquad (6)$$

In calculating double areas, account is taken of signs. If the meridian extends through the most westerly point, all double meridian distances are positive, hence the sign of a double area is the same as that of the corresponding latitude. Thus in the figure the double areas of AbB, $BbcC$, $CcdD$, $GghH$, and HhA are positive, the latitudes Ab, bc, cd, gh, and hA being positive; while the double areas of $FfdD$ and $GgfF$ are negative, the latitudes df and fg being negative. Those projection trapezoids and triangles for which the double areas are positive have, in the figure, been cross-hatched with sloping lines and those trapezoids for which the double areas are negative have been cross-hatched with vertical or horizontal lines. Consulting the figure, it will be seen that the projected areas outside the traverse are considered once as positive and once as negative, being cross-

hatched by both inclined and vertical or horizontal lines, hence the algebraic sum of the corresponding double areas is zero.

It is therefore clear that the algebraic sum of all double areas is equal to twice the area of the tract within the traverse. Whether this algebraic sum is a positive or negative quantity is determined solely by the order in which the lines of the traverse are considered. If the reference meridian passes through the most westerly corner, then a clockwise order of lines, as in the figure, results in a negative double area, and a counter-clockwise order results in a positive double area.

275a. When the reference meridian passes through a traverse point there is an intimate relation between the departures and double meridian distances. Thus, again referring to Fig. 275, it is seen that the D.M.D. of AB is bB, which is equal to the departure of the course both in magnitude and sign. And the D.M.D. of BC is equal to $bB + cb' + b'C$, which is equal to the D.M.D. of AB, plus the departure of AB, plus the departure of BC. Similar quantities make up the D.M.D.'s of CD and DF. For the last line HA the D.M.D. is hH, which is equal in magnitude but opposite in sign to the departure of HA.

Following are three rules for determining D.M.D.'s, which are deduced from the above relations:

1. The D.M.D. of the first course (reckoned from the point through which the reference meridian passes) is equal to the departure of that course.
2. The D.M.D. of any other course is equal to the D.M.D. of the preceding course, plus the departure of the preceding course, plus the departure of the course itself.
3. The D.M.D. of the last course is numerically equal to the departure of the course but with opposite sign.

The first two of the above rules are employed in computing values. The third rule is useful as a check on the accuracy of the computations. Assuming the departures as balanced, the D.M.D. of the last line, as found by computing the D.M.D.'s in succession around the traverse from the first line, must be numerically equal to the departure of the last line if no mistake in addition or subtraction has been made. When this condition is realized it may be concluded that the intermediate D.M.D.'s are correct, and recomputation is unnecessary. In calculating D.M.D.'s by the above rules, due regard must be given to signs.

276. Area within Closed Traverse by D.M.D. Method.—Following is a summary of the steps employed in calculating by the D.M.D. method the area within a closed traverse when the lengths and bearings of the sides are known.

1. Compute the latitudes and departures of all courses as described in Art. 261.
2. Find the error of closure in latitude and in departure as described in Art. 262.
3. Balance the latitudes and departures in accordance with one of the rules of Art. 263*b*.
4. Assume that the reference meridian passes through the most westerly point in the survey, and calculate the D.M.D.'s by the rules of the preceding article, using the corrected departures.

Area of Balsam Park, Island Pond, Vermont
By D.M.D. Method

Field Notes
Book No. 3
Page 47

Computations
Aug. 17, 1927
Computed and
Checked by J. D. M.

Line	Cal. Bear.	Dist. 66 ft. Ch.	Latitudes N	Latitudes S	Departures E	Departures W	Corrected Lats.	Corrected Deps	D M.Ds.	Double Areas +	Double Areas −
A-B	S80°29½W	34.464		5.694		33.991	− 5693	−33.990	61.812		351.89
B-C	S33 04 W	25.493		21.364		13.911	−21.361	−13.911	13 911		297.15
C-D	S33 46¾ E	33.934		28.205	18 867		−28.201	+18.867	18 867		532.06
D-E	N87 38¼ E	28.625	1.013		28.607		+ 1.013	+28.608	66.342	67.21	
E-A	N 0 27 E	54.235	54.234		0.426		+54.242	+ 0.426	95.376	5173.51	
		176.751	55.247	55.263	47.900	47.902	ΣL = 0	ΣD = 0		5240.72	1181.10
				55.247		47.900				1181.10	
				.016		.002				2 | 4059.62	
										2029.81 Sq. Ch.	
										or 202.98 Ac	

$$E. \text{ of } C. = \frac{.016}{176.751} = \frac{1}{11{,}000}$$

$$E = \sqrt{.016^2 + .002^2} = 0.016 \text{ Chains}$$

Line	A-B	B-C	C-D	D-E	E-A
Lat.	5.694	21.364	28.205	1 013	54.234
Log. Lat.	0.75542	1.32968	1.45032	0.00584	1 73427
Log. Cos.	9.21805	9.92326	9.91969	8.54899	9.99999
Log. Dist.	1.53737	1.40642	1.53063	1.45674	1.73428
Log Sin.	9.99399	9.73689	9.74509	9.99973	7.89535
Log Dep.	1.53136	1.14331	2.27572	1.45647	9 62964
Dep.	33.991	13.911	18.867	28.607	0.426
Log. Cor Lat.	0.75534	1.32962	1.45026	0.00584	1.73434
Log. D.M.D.	1.79107	1.14336	1.27570	1.82179	1.97944
Log. D.A.	2.54641	2.47298	2.72596	1.82763	3.71378
D'ble Area	351.89	297.15	532.06	67.21	5173.51

Note:
Survey Balanced
by Transit Rule

FIG. 276.—Computations for area.

5. Compute the double areas by multiplying each D.M.D. by the corresponding corrected latitude.
6. Find the algebraic sum of the double areas, and determine the area by dividing this sum by two.

These steps are illustrated in the computations shown by Fig. 276 in which the area of a tract within a transit traverse is determined. It is seen that distances are in 66-ft. chains, and that computations are made by the use of logarithms. The survey is balanced by the Transit Rule, and the corrected latitudes and departures are recorded.

The point *C* is the most westerly point in the survey, and the double meridian distances are computed by beginning with the line *CD* for

which the D.M.D. and corrected departure are of the same magnitude and sign. The D.M.D.'s for lines *DE*, *EA*, *AB*, and *BC* are calculated by the second of the rules given in Art. 275*a*. The D.M.D. of *BC*, the last line in the traverse, is seen to be numerically equal to the corrected departure of this line but with opposite sign, hence the D.M.D. computations are correct.

Below the calculations of latitudes and departures are given the logarithmic computations for double areas, each D.M.D. being multiplied by the corresponding corrected latitude.

In the last two columns of the upper tabulation are recorded the positive and negative double areas. These are algebraically added, and the result is divided by two as shown, the result being the area in square chains.

While computations by logarithms have been shown, where a computing machine is available it would ordinarily be used in preference to logarithms.

The correctness of the computations for latitudes, departures, and double areas cannot be readily checked except by recomputations of similar character. The corrections applied to latitudes and departures are checked if the algebraic sums of the corrected latitudes and departures are zero. The application of the third of the rules for D.M.D.'s as already explained, serves to verify the calculations for D.M.D.'s.

277. Double Parallel Distances.—Determining area within a closed traverse by the method of double parallel distances (D.P.D.'s) is essentially the same as the D.M.D. method described in the preceding articles. In principle the two methods are identical, and the only difference is that with double parallel distances (D.P.D.'s) the courses are projected upon a parallel, or line perpendicular to the meridian, instead of being projected upon the meridian as with the D.M.D. method.

While the D.P.D. method possesses all of the advantages of the D.M.D. method, it is used very little in practice. It is occasionally employed as an independent method of checking areas which have been computed by the D.M.D. method.

278. Areas of Tracts with Irregular or Curved Boundaries.—As before stated, it frequently happens that the boundary of a tract of land follows some irregular or curved line, such as a stream or road. In such cases it is customary to run a traverse in some convenient location near the boundary and to locate the boundary by offsets from the traverse line. Figure 278*a* may be taken as representing a typical case of this kind, *AB* being one of the lines in the traverse. For such a case the determination of area of the entire tract involves computing the area within the closed traverse, by

methods which have already been described, and adding to this the area of the irregular figure between the traverse line AB and the curved boundary. The data which are available for computing the irregular area consist of offset distances as aa', bb', cc', etc., and the corresponding distances along the traverse line as Aa, Ab, Ac, etc. Where the boundary is irregular, as from a' to f', it is necessary to take offsets at points of change, hence the offsets for a boundary of this character are, in general, taken at irregular intervals. Where a segment of the boundary is straight, as from f' to g', offsets are taken only at the ends. Where the boundary is a gradual curve, as from g' to m', offsets are ordinarily taken at regular intervals.

An irregular boundary is not usually a succession of straight lines between offsets, but often the boundary itself is not a sharply defined line and if the offsets are taken sufficiently close together, the error involved in considering the boundary as straight between offsets is small as compared with the inaccuracies of the measured offsets. When this assumption is made, as is usually the case, the areas between offsets are of trapezoidal shape, the assumed boundary taking some such form as that illustrated by the dotted lines $g'h'$, $h'k'$, $k'l'$, etc., in Fig. 278*a*. Irregular areas computed on such an assumption are said to be calculated by the *Trapezoidal Method* or by the *Trapezoidal Rule.*

When the curved boundaries are of such definite character as to make it justifiable, a somewhat higher degree of accuracy in the computed area may be obtained by assuming that the boundary is made up of segments of parabolas as first suggested by Simpson. When the area within a curved boundary is found by making this assumption, it is said to be computed by *Simpson's Method* or by *Simpson's One-third Rule.*

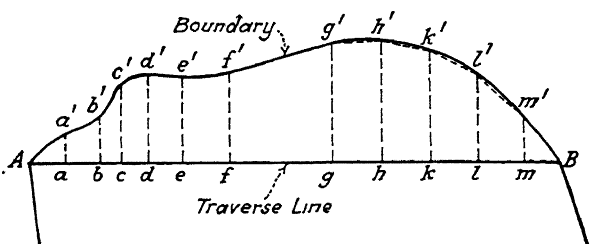


FIG. 278*a*.

278a. Trapezoidal Method: Offsets at Regular Intervals.—Let Fig. 278*b* represent a tract lying between a traverse line AB and an irregular boundary CD, offsets h_1, h_2 . . . h_n having been taken at

the regular intervals d. Summing up the areas of the trapezoids, the total area is

$$\text{Area} = \frac{h_1 + h_2}{2} \cdot d + \frac{h_2 + h_3}{2} \cdot d + \frac{h_3 + h_4}{2} \cdot d \cdots + \frac{h_{(n-1)} + h_n}{2} \cdot d$$

$$\text{Area} = d\left(\frac{h_1 + h_n}{2} + h_2 + h_3 + h_4 \cdots + h_{(n-1)}\right) \tag{7}$$

which is the expression commonly used for finding an irregular area when offsets are taken at regular intervals.

Equation (7) may be conveniently expressed in the form of the following rule:

Trapezoidal Rule.—*Add to the sum of the intermediate offsets, the average of the end offsets. The product of the quantity thus determined and the common distance between offsets is the required area.*

Example 1: By the Trapezoidal Method find the area between a traverse line and curved boundary, rectangular offsets being taken at intervals of 20 ft. and the values of the offsets in feet being $h_1 = 3.2$, $h_2 = 10.4$, $h_3 = 12.8$, $h_4 = 11.2$, $h_5 = 4.4$. Applying the above rule, there results

$$\text{Area} = 20\left(\frac{3.2 + 4.4}{2} + 10.4 + 12.8 + 11.2\right) = 764 \text{ sq. ft.}$$

278b. Simpson's Method: Offsets at Regular Intervals.—In Fig. 278c let AB be a portion of a traverse line, let DFC be a portion

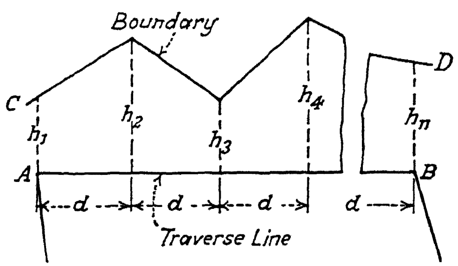


FIG. 278b.

of the curved boundary assumed to be the arc of a parabola, and let h_1, h_2, and h_3 be any three consecutive rectangular offsets from traverse line to boundary taken at the regular interval d.

The area between curve and traverse line may be considered as composed of the trapezoid $ABCD$ plus the area between the parabolic arc DFC and the corresponding chord DC. It is one of the properties of the parabola that this latter area is equal to two-thirds the area of enclosing parallelogram $CDEFG$. This being recognized, the area

between the traverse line and curved boundary within the length of $2d$ is given by the expression

$$A_{1,2} = \frac{(h_1 + h_3)}{2} 2d + \left(h_2 - \frac{h_1 + h_3}{2}\right) 2d \cdot \frac{2}{3}$$
$$= \frac{d}{3}(h_1 + 4h_2 + h_3)$$

Similarly for the next two intervals

$$A_{3,4} = \frac{d}{3}(h_3 + 4h_4 + h_5)$$

Summing up these partial areas for $(n - 1)$ intervals, n being an odd number and representing the number of offsets, the total area is

$$\text{Area} = \frac{d}{3}[h_1 + h_n + 2(h_3 + h_5 + \cdots h_{(n-2)}) + 4(h_2 + h_4 + \cdots h_{(n-1)})] \quad (8)$$

which is conveniently expressed in the following rule which is applicable to any case where the number of offsets is odd and the interval between the offsets is uniform.

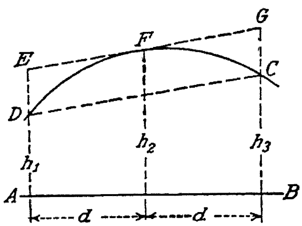


FIG. 278c.

Simpson's One-third Rule.—*Find the sum of the end offsets, plus twice the sum of the odd intermediate offsets, plus four times the sum of the even intermediate offsets. Multiply the quantity thus determined by one third of the interval between offsets, and the result is the required area.*

Example 2: By Simpson's One-third Rule find the area between a traverse line and a curved boundary, rectangular offsets being taken at intervals of 20 ft. and the values of the offsets in feet being $h_1 = 3.2$, $h_2 = 10.4$, $h_3 = 12.8$, $h_4 = 11.2$, $h_5 = 4.4$. (The statement of the problem is the same as for example 1, Art. 278*a*.)

Applying Simpson's Rule there results

$$\text{Area} = \tfrac{20}{3}[3.2 + 4.4 + 2(12.8) + 4(10.4 + 11.2)]$$
$$= 797 \text{ sq. ft.}$$

278c. It will be found that results obtained by using Simpson's Rule are greater or smaller than those obtained by use of the Trapezoidal Rule, according as the boundary curve is concave or convex towards the traverse line. Some appreciation of the variations between the two methods will be gained by studying examples 1 and 2. It will be seen that the two results differ by more than 4 per cent. Under average conditions the difference will be much less than this, but in an extreme case it may be much larger.

The result secured by the use of Simpson's Rule is in all cases the more accurate, but the rule is not so easily applied as the Trapezoidal Rule. The latter approaches the former in accuracy to the extent that the irregulár boundary has curves of contrary flexure thereby producing the compensative effects mentioned above.

278d. Trapezoidal Method: Offsets at Irregular Intervals.—The method of coordinates described in Art. 274 may be applied to this problem by assuming the origin as being on the traverse line and at the point where the first offset is taken. The coordinate axes are then the traverse line and a line at right angles thereto. The rule of Art. 274*a* may then be modified to the following:

Rule.—*Multiply the distance, along the course, of each intermediate offset from the first, by the difference between the two adjacent offsets, always subtracting the following from the preceding. Also multiply the distance of the last offset from the first by the sum of the last two offsets. The algebraic sum of these products, divided by two, is the required area.*

18

Area Between Meander Line and Bog Brook
on
Brigham Farm

Field Notes
Drawer 11,
Book T6
Pages 37-39

Computations
Aug. 16, 1927
Comp'td by J. O. S.
Checked by P. Q. D.

Line E-F				
Dist. from E, ft.	Length of Offset, ft	Difference	Products	
			−	+
0=E	0.0			
20	14.3	− 23.1	460	
65	23.1	+ 4.7		310
87	96	+ 18.5		1610
100	46	− 8.1	810	
131	177	− 13.9	1820	
148	18.5	− 19 4	2870	
160	37.1	− 6.5	1040	
200	250	+ 12.9		2580
225	242	+ 2.0		450
250	23.0	+ 58		1450
300	18 4	+ 5.0		1500
350	18 0	− 20.4	7140	
385	38 8	+ 9.3		3580
400	8.7	+ 38.8		15520
413.7=F	0.0	+ 8.7		3600
			14,140	30,600
				14,140
			2	16,460
				8,230 sq ft
			or	0.189 Ac.

Note: For area computations within traverse see page 17 of this book

FIG. 278*d*.—Computations for area using trapezoidal rule; offsets at irregular intervals.

The application of the rule to a specific problem is illustrated by the computations of Fig. 278*d*, where the area between a meander line and a stream is determined.

279. Area of Segments of Circles.—A problem of frequent occurrence in the surveying of city lots and of rural lands adjacent to the curves of highways and railways is that of finding the area when one or more of the lines of the boundary is the arc of a circle.

In Fig. 279*a*, *ABCDEQF* may be taken as a boundary of this character, for which it is convenient to run a traverse along the straight portions of the boundary, and to make the chord *EF* the closing side of the traverse, the length of the chord $EF = L$ and the middle ordinate $PQ = M$ being measured in the field.

In calculating the area, it is convenient to divide the tract into two parts: (1) that within the polygon formed by the traverse *ABCDEF*, for which the area is found by the double-meridian-distance method, and (2) that between the chord *EPF* and the arc *EQF*, which is the segment of a circle. The area of this segment is found exactly by subtracting the area of the triangle *OEPF* from the area of the circular sector *OEQF*. If I is the angle and R is the radius whose arc is *EQF*, then by Art. 373*b*, p. 568,

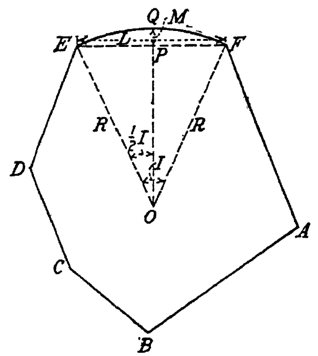


FIG. 279*a*.

$$\tan \frac{1}{4}I = \frac{2M}{EF} \tag{9}$$

and

$$R = \frac{L}{2 \sin \frac{1}{2}I} = \frac{M}{\text{vers } \frac{1}{2}I} \tag{10}$$

The area of the circular sector *OEQF* is $A_s = \frac{\pi R^2 I^\circ}{360}$, in which I° is expressed in degrees.

The area of the triangle *OEF* is

$$A_t = \frac{R^2}{2} \sin I$$

The area of the segment is, then, exactly

$$A = A_s - A_t = R^2\left(\frac{\pi I^\circ}{360} - \frac{\sin I}{2}\right) \tag{11}$$

Example 1: Find the area of a circular segment when the chord length is 275.0 ft. and the middle ordinate is 31.35 ft.

By Eq. (9)

$$\tan \tfrac{1}{4}I = \frac{2 \times 31.35}{275.0} = 0.2280$$

$$\tfrac{1}{4}I = 12°51'; I = 51°24' = 51°.40$$

By Eq. (10)

$$R = \frac{L}{2 \sin \tfrac{1}{2}I} = \frac{275.0}{2 \times 0.4337} = 317.0 \text{ ft.}$$

By Eq. (11)

$$A = (317.0)^2\left(\frac{3.142 \times 51.40}{360} - \frac{0.7815}{2}\right) = 5{,}810 \text{ sq. ft.}$$

279a. The area of a parabolic segment is

$$A_p = \tfrac{2}{3}LM \tag{12}$$

in which the letters have the same significance as before. This expression may be employed for finding the approximate areas of circular segments, the precision decreasing as the size of the central angle I increases. The following example illustrates the error involved in applying this expression to the conditions of example 1.

Example 2: By Eq. (12) find the approximate area of the circular arc of example 1, and determine the per cent of error introduced through using this approximate expression.

$$A_p = \tfrac{2}{3} \times 275.0 \times 31.35 = 5{,}750 \text{ sq. ft.}$$

This value is

$$\frac{5{,}810 - 5{,}750}{5{,}810}100 = 1.0 \text{ per cent too low.}$$

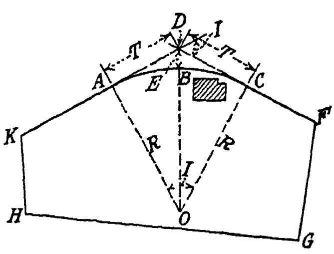


Fig. 279b.

When the central angle is small the error involved in using Eq. (12) for circular arcs is often negligible; thus, when $I = 30°$, the error is less than 0.2 per cent. But for large values of I, the error introduced is so great as to render the approximate expression of little use; thus, when $I = 90°$, the error is about 3 per cent, and when $I = 180°$ the error is about 15 per cent.

279b. When tangents to the curve are property lines, it is sometimes more convenient to establish the traverse as illustrated by Fig. 279*b*. Here KA and FC, which are tangent to the curve ABC, are run to an intersection at D and the distances AD and CD and the angle I are measured. Also E is usually measured as a check.

The work of finding the area is conveniently divided into two parts: (1) that of computing the area within the polygon $ADCFGHK$ by the double-meridian-distance method, and (2) that of calculating the external area between the arc ABC and the tangents AD and CD. The latter subtracted from the former is the required area.

The external area may be found by subtracting the area of the circular sector $OAC = A_s = \frac{\pi R^2 I^\circ}{360}$ from the area $OADC = TR$, in which T is the tangent distance $AD = CD$, and R is the radius of the curve. If R is unknown it may be found by the relation

$$R = \frac{T}{\tan \frac{1}{2}I} = \frac{E}{\text{exsec } \frac{1}{2}I} \text{ (see Art. 373}b\text{, p. 568)} \qquad (13)$$

280. Partition of Land.—The problems involved in the partition or division of lands are so numerous as to preclude the possibility of discussing each one, but four of the simpler cases frequently encountered in the subdivision of irregular tracts will be described in the succeeding articles. Methods of subdividing the U. S. public lands are given in Chap. XX.

In general, where a given tract is to be divided into two or more parts, a resurvey is run, latitudes and departures are computed, the survey is balanced, and the area of the entire tract is determined. The corrected latitudes and departures are further employed in the computations of subdivision.

The cases here to be discussed are: (*a*) to find the area cut off by a line running between two points in the boundary; (*b*) to find the area cut off by a line running in a given direction from a given point in the boundary; (*c*) to cut off a required area by a line passing through a given point in the boundary; and (*d*) to cut off a required area by a line running in a given direction.

281. Area Cut Off by a Line between Two Points.—In Fig. 281 let $ABCDEFG$ represent a tract of land to be divided into two parts by a line extending from A to D. A survey of the tract has been made, the latitudes and departures have been balanced, and the area has been computed.

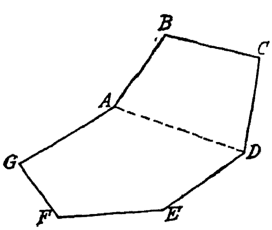


FIG. 281.

It is desired to determine the length and direction of the cut-off line AD without additional field measurements, and to calculate the area of each of the two parts into which the tract is divided.

Either of the two parts may be considered as a closed traverse with the length and bearing of one side DA unknown. Considering the

part $ABCDA$, the latitudes and departures of AB, BC, and CD are given, hence the latitude, departure, length, and bearing of DA may be determined as described in Art. 267*a*. The area of either part may now be found by the method of double meridian distances (Art. 276).

A check on the field work and computations is obtained by actually measuring the length and direction of the line DA and noting the agreement between observed and calculated values. A further check is secured by noting that the sum of the areas of the two parts, each calculated independently, is equal to the calculated area of the entire tract.

282. Area Cut Off by a Line Running in a Given Direction.—In Fig. 282, $ABCDEFG$ represents a tract of known dimensions, for which the balanced latitudes and departures are given; and DH represents a line running in a given direction which passes through the point D, dividing the tract into two parts.

It is desired to calculate from the given data the lengths DH and HA and the area of each of the two parts into which the tract is divided.

Each of the two parts may be considered as a closed traverse for which the lengths of two sides are unknown; the values of the unknown lengths may be computed as described in Art. 267*e*. Considering the part $ABCDHA$, the latitudes and departures of AB, BC, and CD are known, from which the length and bearing of DA is calculated. In the triangle ADH the lengths of the sides DH and HA are found, and their latitudes and departures are calculated. With the latitudes and departures of AB, BC, CD, DH, and HA known, the area of $ABCDHA$ is computed by the D.M.D. method.

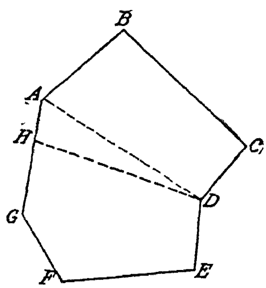


Fig. 282.

In the field the length and direction of the side DH is laid off from D, and a check on field work and calculations is obtained if the point H thus established lies on the line GA and if the calculated distance HA agrees with the observed distance.

The computations are further verified by seeing that the sums of the latitudes and departures of AB, BC, CD, DH, and HA are equal to zero. (This is on the assumption that the latitudes and departures of both DH and HA are based upon the lengths of these lines as computed from the triangle ADH.) The area computations may be checked by

observing that the sum of the areas of the two parts, each computed independently, is equal to the area of the entire tract.

283. To Cut Off a Required Area by a Line through a Given Point.—In Fig. 283*a*, *ABCDEFA* represents a tract of land of known dimensions, the corrected latitudes and departures of the sides being given. And *G* represents a point in the boundary through which a line is to pass cutting off a required area from the tract. It is

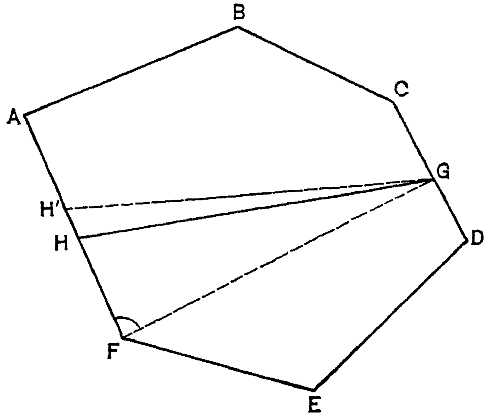


FIG. 283*a*.

assumed that the area within the tract has been computed by the D.M.D. method and that a sketch of the tract has been prepared.

To find the length and direction of the dividing line the procedure is as follows:

A line *GF* is drawn to that corner of the traverse which, from inspection of the sketch, will come nearest being on the required line of division. The latitude and departure of *CG* are calculated. Then in the traverse *ABCGFA* all sides are known except *GF*, which is unknown in both length and direction. By the methods of Art. 267*a*, the latitude, departure, length, and bearing of *GF* are determined. By the D.M.D. method the area of *FABCG*, the amount cut off by the line *FG* is computed. The difference between this area and that required is found.

In the figure it is assumed that *FABCG* has an area greater than the desired amount, *GH* being the correct position of the dividing line. Then the triangle *GFH* represents this excess area, and since the angle *F* may be calculated from known bearings, there are given in this triangle, one side *FG*, one angle *F*, and the area. The length *HF* is calculated by the relation

$$HF = \frac{2 \times \text{area } GFH}{FG \sin F}$$

The triangle is then solved for angle G and length GH.

By the known direction of GF and the angle G, the bearing of GH is calculated. The latitudes and departures of the lines FH, GH, and HA are computed.

In the field the line GH is established by laying off the length GH in the required direction, and a check is obtained on field work and computations if the position of H thus established falls on the line FA, and if the computed distance HF or HA agrees with the measured distance.

55

Computations for Balsam Park Division
to Divide into two Equal Parts
by a Line through "A'"

Field Notes
Book No. 3, Page 47.
Area Computations
Page 7 this Book

Comp't'd by O.P.J. Aug 17, 1927
Checked by H.A.C. Aug. 17, 1927

Total Area (see P.7)

Line	Cal. Bear	Dist. 66 ft. Ch.	Latitudes N	Latitudes S	Departures E	Departures W	D.M.Ds.	Double Areas +	Double Areas −
A-B	S80°29½'W	34.46		5.69		33.99	61.81		351.89
B-C	S33 04 W	25.49		21.36		13.91	13.91		297.15
C-D	S33 46¾ E	33.93		28.20	18.87		18.87		532.06
D-E	N87 58¼ E	28.63	1.01		28.61		66.34	67.21	
E-A	N 0 27 E	54.24	54.23		0.43		95.38	5173.51	
								5240.72	1181.10
							Total	Area 202	.98 Ac.
					Area	A-B-C-D.			
A-B				5 69		33.99	61.81		351.89
B-C				21.36		13.91	13.91		297.15
C-D				28.20	18.87		18.87		532.06
D-A	N27 43 E	62.41	55.25		29.03		66.77	3688.78	
						Area	A-B-C-D	125 38 Ac	1181.10
					Area	A-B-C-F.			
A-B				5.69		33.99	61.81		351.89
B-C				21 36		13.91	13.91		297.15
C-F	S33°46¾'E	25.22		20.96	14.03		14.03		294 08
F-A	N35 12 E	58.75	48.01		33.87		61.93	2973.10	
									943.12
						Area	A-B-C-F	101.50 Ac.	Check

Formulas

$$F\text{-}D = \frac{2\ A\text{-}D\text{-}F}{A\text{-}D \sin \theta}$$

$$\tan \beta = \frac{F\text{-}D \sin \theta}{D\text{-}A - D\text{-}F \cos \theta}$$

$$F\text{-}A = \frac{F\text{-}D \sin \theta}{\sin \beta}$$

Area A-B-C-D = 125 38 Ac.
Half Park = 101.49
Area A-D-F ≐ 23.89 Ac.

C-D = 33.93
F-D = 8.712
C-F = 25.22

Line	D-A	Log 2 A-D-F	2.67928	Log D-A-D-F cos θ	1.76534	Line	C-F	F-A
Dist.	62.412	Log D-A	1.79527	Log D-F sin θ	0.88401	Departure	14.028	33.873
Log Dist.	1.79527	Log sin θ	9.94391	Log tan β	9.11867	Log Dep.	1.14687	1.52978
Log cos Br.	9.94704	Log D-F	0.94010	β	7°29'	Log sin	9.74509	9.76075
Log Lat.	1.74231	D-F	8.712	Bear. D-A	N27°43'E	Log Dist.	1.40148	1.76903
Log D.M.D.	1.82457	Log D-F sin θ	0.88401	Bear F-A	N35°12'E	Log cos	9.91969	9.91230
Log D. Ar.	3.56688	Log cos θ	9.67860	Log F-D	0.94010	Log Lat.	1.32147	1.68133
D. Area	3688.76	Log D-F cos θ	0.61870	Log sin θ	9.94391	Latitude	20.961	48.015
Log Dep	1.46291	D-F cos θ	4.156	Log sin β	9.11498	Log D.M.D.	1.14700	1.79189
Log tan Br.	9.72060	D-A	62.412	Log A-F	1.76903	Log D. Ar.	2.46847	3.47322
Bear.	N27°43'E	D-A-D-F cos θ	58.256	A-F	58.753	D. Area	294.08	2973.10

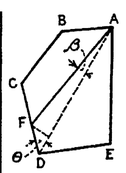


FIG. 283b.—Computations for partition of land.

Sometimes the tract in question will be of such shape that a line drawn from the given point in the boundary to any corner will cut off an area nowhere near that required. Under these circumstances, or when the traverse has a large number of sides, it is advisable to plot the traverse with protractor and scale and to establish a trial line of subdivision such as GH' in Fig. 283a. The planimeter (see Art. 157, p. 213) may be used to good advantage for finding the area cut off by this trial

line, and the line may be shifted until the area cut off agrees closely with that required. The scaled distance AH' may be used in the computations. It will be seen that the method of solution is now identical with that just described for the case where the trial line is drawn to a corner.

283a. Figure 283*b* shows the computations for the division of a tract into equal parts by a line passing through the corner A. Taking the calculations in order down the page, there are: (1) tabulations for finding the area of the entire tract by the D.M.D. method, (2) tabulations for finding the area cut off by the trial line AD, (3) tabulations for checking the computations by determining the area cut off by the true line of division AF, and (4) logarithmic computations for (*a*) calculating the length and bearing of the trial line DA, (*b*) solving the triangle ADF, and (*c*) computing the latitudes, departures, D.M.D.'s, and double areas for the lines CF and FA, which values are employed in determining the area $ABCF$ given in (3).

284. To Cut Off a Required Area by a Line Running in a Given Direction.—In Fig. 284, $ABCDEF$ represents a tract of land of known

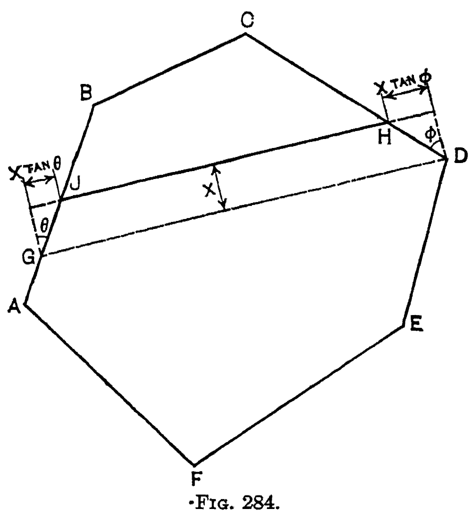


FIG. 284.

dimensions and area, which is to be divided into two parts, each of a required area, by a line running in a given direction. The figure is assumed to be drawn at least roughly to scale, and the corrected latitudes and departures are known.

Through the corner that seems likely to be nearest the line cutting off the required area, a trial line DG is drawn in the given direction. In the figure $ABCDG$, the latitudes and departures of AB, BC, and

CD and the bearings of DG and GB are known. $GBCDG$ therefore is a closed traverse for which the length of two sides GB and DG are unknown. By the methods of Art. 267e these unknown quantities are found, and the latitudes and departures of the courses are determined. The area cut off by the trial line is computed. The difference between this area and that required is represented in the figure by the trapezoid $DGJH$ in which the side DG is known. The angles at D and G may be calculated from the known bearings of adjacent sides, and in this way θ and ϕ are determined. Then

$$\text{Area of trapezoid} = GD \cdot x + \frac{x^2}{2}(\tan\theta + \tan\phi),$$

in which $\tan\theta$ or $\tan\phi$ is positive or negative according as θ or ϕ lies within or without the trapezoid, and x is the altitude of the trapezoid. (In the figure both angles lie without the trapezoid and hence both tangents are negative.) The value of x is found by solving this equation. Then $GJ = x \sec\theta$; $DH = x \sec\phi$; and $JH = GD + x(\tan\theta + \tan\phi)$ in which the signs of $\tan\theta$ and $\tan\phi$ are as given above.

In the field the points H and J are established on the lines CD and AB, at the calculated distances from the adjacent corners. The side JH is then measured. If this measured value agrees with the computed value, the field work and portions of the computations are verified. A further check on the computations is introduced by calculating the area of $BCHJ$ and comparing it with the required area of this figure.

285. Problems.

1. A square field contains 40 acres. What are its dimensions in chains, in rods, and in feet?

2. How many acres are there in a rectangular tract 50 × 100 ft.? in a tract 400 × 400 ft.? in a tract 2,640 × 2,640 ft.?

3. What is the area of a triangle when its three sides are 219.0, 317.2, and 301.6 ft.? When two sides are 1,167.1 and 392.7 ft. and the included angle is 39°46′?

4. Given the notes shown in Fig. 95e, page 105. Calculate the area of the field by using the two sides and included angle of each triangle. Check by using the three sides of each of the oblique triangles.

5. The mutually bisecting diagonals of a four-sided field are 480 and 360 ft. The angle of intersection between the diagonals is 100°. Find the interior angles and the lengths of the sides.

6. Given the notes at the top of page 415, for a closed traverse. Calculate the latitudes and departures, and balance the survey by the Compass Rule. Assume that the coordinates of A are 270.0N and 340.0E and compute the coordinates of all other corners. By the coordinate method (Art. 274) calculate the area.

Course	Bearing	Distance, feet
AB	N6°37′W	932.4
BC	N16°43′E	500.0
CD	N2°38′E	1,018.0
DE	N78°53′E	664.9
EF	S25°57′E	900.6
FG	S21°28′W	468.1
GH	S10°22′E	938.7
HI	S45°23′W	595.4
IA	N87°57′W	715.3

7. Given the following deflection-angle traverse:

Station	Deflection angle	Distance, feet
A	41°30′R	
		1,171.3
B	72°15′R	
		722.9
C	59°25′L	
		756.4
D	100°00′R	
		1,065.0
E	66°6.5′R	
		679.0
F	50°28.5′R	
		1,015.0
G	27°10′L	
		1,180.0
H	116°15′R	
		955.2
A		

Calculate the bearings of the several courses, assuming the bearing of *AB* to be N26°38′W. Balance the survey by the Compass Rule, and calculate the coordinates of the corners, assuming the coordinates of *A* to be 430.0N and 1,925.0E. Calculate the area by the coordinate method, following the rule of Art. 274*a*.

8. Find the error of closure of the following traverse, balance the survey by the Compass Rule, and compute the area in acres by the D.M.D. method using four-place tables of logarithms:

Course	Bearing	Length, feet
AB	S45¾°E	294.4
BC	N65½°E	262.4
CD	N35¼°E	313.6
DE	N64¼°W	392.0
EF	S59°W	196.0
FA	S26¼°W	240.0

9. Following are the notes for a transit traverse, distances being in Gunter's chains.

Course	Bearing	Length, chains
AB	S58°08′E	10.24
BC	S67°07′E	9.32
CD	S9°39′W	24.00
DE	S84°22′W	24.92
EF	N6°21′E	18.92
FA	N29°52′E	18.80

Calculate the latitudes and departures, balance the survey by the Transit Rule, and calculate the area in acres by the D.M.D. method. Use four-place logarithms.

10. Given the data of problem 7. Calculate bearings, compute latitudes and departures, balance the survey by the Crandall method, and find the area in acres by the D.M.D. method, using six-place logarithms.

11. A traverse *ABCD* is established inside a four-sided field and the corners of the field are located by angular and linear measurements from the traverse stations, all as indicated by the following data:

Course	Bearing	Length, feet
AB	S89°58′E	296.4
AE	N20°00′W	34.2
BC	S43°20′W	333.9
BF	N35°20′E	16.9
CD	S80°21′W	215.6
CG	S73°00′E	27.6
DA	N27°24′E	314.2
DH	S36°30′W	15.7

Calculate the latitudes and departures and balance by the Compass Rule. Calculate the coordinates of each transit point and of each property

corner, using D as an origin of coordinates. Calculate the length and bearing of each side of the field $EFGH$ and tabulate results. Calculate the area of the field by the coordinate method.

12. Given the following offsets from traverse line to irregular boundary measured at points 25 ft. apart:

Distance, feet	Offset, feet	Distance, feet	Offset, feet
0	0.0	175	81.7
25	57.5	200	95.2
50	87.3	225	97.4
75	90.8	250	85.6
100	93.2	275	73.4
125	79.2	300	24.1
150	60.0		

By the Trapezoidal Rule (Art. 278*a*) calculate the area between traverse line and boundary.

13. Given the data of problem 12. Calculate the required area by Simpson's One-third Rule (Art. 278*b*).

14. Following are offsets taken at intervals of 50 ft., to the right and to the left of a traverse line.

Offset left, feet	Distance, feet	Offset right, feet
74.8	0	32.9
84.2	50	26.1
111.5	100	18.6
101.1	150	32.7
71.3	200	49.8
32.7	250	86.9
18.5	300	47.2
12.1	350	22.3
6.7	400	9.0

By the Trapezoidal Rule (Art. 278*a*) calculate the area between boundaries thus defined.

15. Given the data of problem 14. Calculate the required area by Simpson's One-third Rule.

16. Following are offsets from a traverse line to an irregular boundary, taken at irregular intervals:

Distance, feet	Offset, feet	Distance, feet	Offset, feet
0	30.4	200	87.7
20	61.7	210	44.5
60	102.8	260	40.0
70	57.5	290	81.1
100	61.1	360	95.2
180	93.3	410	73.5

Compute the area between traverse line and boundary by means of the rule of Art. 278*d*.

17. Given the notes shown in Fig. 95*h*, p. 107. Calculate the area of the tract surveyed, including the irregular areas between traverse *ABCDE* and river or lake.

18. In Fig. 279*a*, what is the area of the circular segment *EFQ* if the length of the chord *L* is 817.2 ft. and the middle ordinate *M* is 89.17 ft.?

19. In Fig. 279*a*, what is the area of the circular segment *EFQ* if the chord length *L* is 600 ft. and the middle ordinate *M* is 7.85 ft.?

20. Solve problems 18 and 19 using the approximate expression (Eq. (12)) of Art. 279*a*. Compare the results with those of problems 18 and 19, and for each case calculate the per cent of error introduced through use of the approximate expression.

21. A curved corner lot is similar in shape to that shown in Fig. 279*b*. The tangent distances *T* are each 50.0 ft. and the intersection angle *I* is 40°. What is the area between the circular curve *ABC* and the tangents *AD* and *CD*? What should be the value of the external distance *E*?

22. Given the data of problem 9. Find the area north of a line running from *F* to a point *G* on the line *CD* and distant 10.00 chains from *C*. Calculate the length and bearing of *FG*.

CHAPTER XVII

PRINCIPLES OF FIELD ASTRONOMY

286. General.—To treat the subject of field astronomy in a comprehensive manner would require a volume in itself, and it is not within the province of this text to do more than discuss the more common problems (see references on p. 497. To the person familiar with the subjects of general astronomy and spherical trigonometry, the processes of astronomical observation and computation are at once clear. To those unfamiliar with these subjects, certain new conceptions must be acquired which can not be learned without thoughtful study. In the following articles considerable space is devoted to describing fundamentals which are applicable to all astronomical observations, with the thought that the surveyor should be familiar with the astronomical and trigonometric principles upon which the processes of observation and computation are based. What follows is intended to be applied to surveys of moderate precision. In the interest of simplicity of presentation, certain statements are made which, by strict interpretation, would need to be slightly modified if applied to surveys of high precision.

287. Astronomy.—The surveyor is directly interested in the science of astronomy in that it offers him a means of determining the absolute position of any point or the absolute position and direction of any line on the surface of the earth. The absolute position of a point is given by its longitude and latitude, and the absolute direction of a line is defined by the bearing which the line makes with the true meridian.

In *geodetic* surveying it is necessary to determine the latitude and longitude of certain points with great precision, the work involving observations on numerous stars, and requiring instruments of high precision. The requirements of *plane* surveying, in general, will be met if sufficient data be available so that the true azimuth or bearing of the survey lines may be established with a degree of precision at least equal to that with which the relative direction of survey lines is determined by direct angular measurement.

The azimuth of a line is established by angular observations to some celestial body. Incidental to the work of computing the azimuth of the line to the celestial body at the instant of sighting,

the latitude of the place must be known. Also for certain observations it is essential that the longitude be roughly determined in order that the position of the star at a given instant may be computed with some accuracy. If the survey is through a territory for which there is a reliable map, latitude and longitude may ordinarily be determined with sufficient precision by scaling from the map, but for surveys in more remote regions, it is usually necessary to determine, roughly, at least the latitude.

288. The Celestial Sphere.—In making observations of the sun and stars, the surveyor is not interested in the distance of these heavenly bodies from the earth, but is interested merely in their angular position. In defining their position it is convenient to imagine their being attached to the inside surface of a hollow sphere of infinite radius of which the earth is the center. This imaginary globe is called the *celestial sphere*. It is a helpful conception to imagine the earth as being fixed, and to consider the celestial sphere as rotating from east to west, its axis being the prolongation of that of the earth. The portion of the celestial sphere seen by the observer is the hemisphere above the plane of his own horizon.[1]

A vertical line at the position of the observer coincides with the plumb line and is normal to the observer's horizon plane. The point where this vertical pierces the celestial sphere above the head of the observer is called the *zenith*, and the corresponding point in the opposite hemisphere, directly below the observer, is called the *nadir*.

The *celestial equator* is the great circle formed by the intersection of the earth's equatorial plane with the surface of the celestial sphere.

The *celestial poles* are the points where the earth's axis prolonged pierces the celestial sphere.

Figure 288*a* represents the celestial sphere, the point O being the earth and $NES'W$ being the horizon of an observer, with letters standing for the points of the compass. Figure 288*b* may be taken as an enlarged view of the earth in the same position as that assumed in Fig. 288*a*. A is an observer in the Northern Hemisphere, the line NS being in his horizon plane and corresponding to NS' in Fig. 288*a*. Evidently he views everything above the horizon plane or that portion

[1] More properly speaking, the plane passes through the center of the earth parallel with the observer's horizon plane, but the radius of the earth is so small in comparison with the distance to any of the celestial bodies here considered that the error introduced is negligible for rough observations. In the case of the sun the error produced by this assumption is much larger than for any of the stars, amounting under certain conditions to about nine seconds of arc, and requiring an appropriate correction (see Art. 300).

of the celestial sphere (Fig. 288*a*) which is shown by full lines. *B* is an observer in the Southern Hemisphere, *N′S′* being in his horizon. Since the size of the earth is negligible as compared with that of the celestial sphere, it may be considered that *N′S′* in Fig. 288*b* coincides

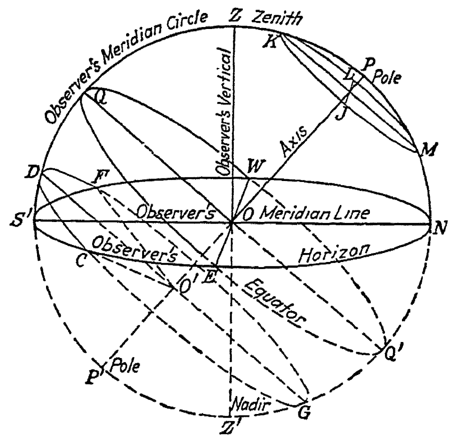


Fig. 288*a*.

with *NS′* in Fig. 288*a*. Therefore if *B* is at a point on the earth diametrically opposite *A*, the portion of the celestial sphere which he views above his horizon plane will be the opposite hemisphere to that seen by *A*, or that portion of Fig. 288*a* which is shown by dotted lines.

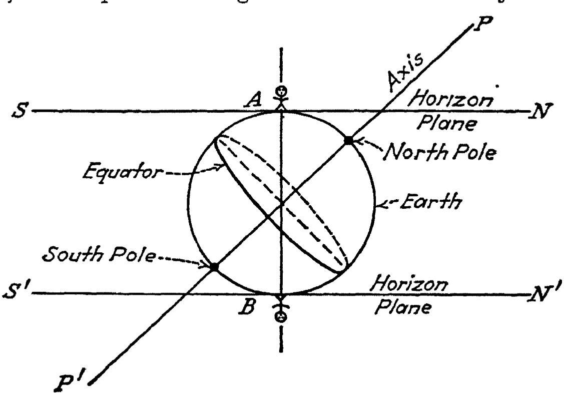


Fig. 288*b*.

Assuming the observer to be in the Northern Hemisphere (Fig. 288*a*), *Z* is the zenith and if *P* and *P′* mark the celestial poles, *P* is the *visible* or *elevated pole*, and *EQWQ′* is the *celestial equator*, of which the portion *EQW* is visible to the observer.

Obviously no two points on the earth have the same celestial horizon, except they be diametrically opposite; in that case, the celestial hemisphere in view from one point will be opposite to that viewed from the other. Since we are, for the sake of simplicity, assuming that the celestial sphere is rotating and the earth remains stationary, N, E, S', W, and Z are regarded as fixed points with respect to any given station on the surface of the earth. If $S'N$ is a meridian line in the plane of the horizon passing through the station of the observer, then a vertical plane of which this line is an element cuts the celestial sphere in the great circle $S'ZPNZ'P'$, which is called the meridian circle or more often, simply the meridian. Since the meridian passes through the fixed points, it is, for any station, a fixed circle, but at a given instant the meridian for one station does not occupy the same position in the celestial sphere as does the meridian for another station, except the two stations be at the same longitude.

Any celestial body which is on the celestial equator will first become visible at E, will pass over the meridian at Q, and will disappear from view at W, and the interval of time when the body is above the horizon will be the same as the interval when it is below, or the angle whose arc is EQW is 180°.

Any star which is below or south of the equator will follow some path as $CDFG$. It will become visible at C, will pass over the meridian at D, and will disappear from view at F. It will be above the horizon for a less length of time than it will be below, or the angle whose arc is CDF (angle $CO'F$) is less than 180°. From the figure it is evident that if any star is sufficiently far below the equator it will never appear above the observer's horizon.

Similarly, any star which is above or north of the equator will be above the horizon for a greater length of time than it is below. If it is far enough above the equator, it will be continuously visible to an observer in a northern latitude and will, during the course of a single revolution of the celestial sphere, follow some path as $JKLMJ$.

Consulting Fig. 288*b*, it is seen that when the station of the observer is at the equator his horizon plane is parallel to the axis of the earth or (since the radius of the earth is negligible as compared with the radius of the celestial sphere) as illustrated by Fig. 288*a*, the plane of the horizon includes the celestial axis. Under these conditions all heavenly bodies will come into view during each revolution of the celestial sphere, and each will be visible or above the horizon, and invisible or below the horizon, for the same period of time. If a star were exactly at either celestial pole, which is not the case, it would appear continuously on the observer's meridian and on the horizon.

Again, consulting Fig. 288*b*, if the observer were at either pole, his horizon would be parallel to the equator of the earth or, in Fig. 288*a*,

the observer's horizon $NES'W$ would coincide with the equator $QWQ'E$. It is evident that the elevated pole would be directly above the observer's head, and if the observer were at the north pole of the earth, all stars north of the celestial equator would be continuously visible and all those below or south of the equator would be continuously invisible.

289. Observer's Position; Latitude and Longitude.—The position of any point on the surface of a sphere may be fixed by angular measurement from two planes of reference at right angles to one another passing through the center of the sphere; these measurements are called the *spherical coordinates* of the point. The spherical coordinates of any station on the surface of the earth are designated as the *latitude* and *longitude* of the station. Figure 289 represents the earth, PP' being the axis and $QUVQ'$ being the equator. Let S be the station of an observer. Then $PSUP'$ is a *meridian circle* through the station. Also RSR' is a *parallel* passing through the station, the plane of RSR' being parallel to that of the equator.

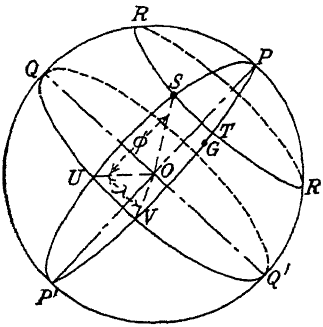


FIG. 289.

The latitude of a place may, for all practical purposes, be defined as the angular distance of the place above or below the equator. When the station is above the equator the latitude is north and its sign is positive; when below the equator, the latitude is south and its sign is negative. Hence in the figure the latitude of S is given by the angle ϕ or by the angular distance, measured along any meridian circle, between the equator and the parallel passing through S, such as US, VT, QR, etc., and the latitude is north or positive. The latitude of a place is stated in degrees. Thus the latitude of the equator is 0° and that of the north pole is +90°, or 90°N.

The longitude of a place is defined as the angular distance measured along the arc of the equator between a reference meridian and the meridian circle passing through the station. The reference meridian is called the *primary meridian*. The primary meridian most generally used is that of Greenwich, England. Hence in the figure if the point G represents Greenwich, PGP' is the primary meridian, and the longitude of S is given by the angle λ or by the angular distance VU. Longitudes are expressed either in degrees of arc or in hours of time (15° = 1 hr.) and are measured east or west of the Greenwich meridian.

290. Position of a Celestial Body; Right Ascension and Declination.—Just as the position of any place on the earth is conveniently fixed by its latitude and longitude, so may the position of any heavenly body be similarly expressed by spherical coordinates referred to the celestial equator and a great circle normal thereto. In Fig. 290 is shown the celestial sphere in a position similar to that of the earth in Fig. 289, S being a celestial body whose position is to be fixed by spherical coordinates. Comparable with the meridian circles or meridians of longitude of the earth are the *hour circles* of the celestial sphere, all of which converge at the celestial poles. The arc PSU is a portion of the hour circle passing through S.

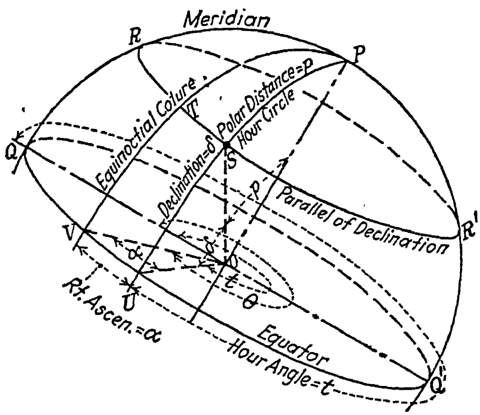


FIG. 290.

Comparable with the parallels of latitude of the earth are the *parallels of declination* of the celestial sphere. RSR' is the parallel of declination passing through S. And comparable with the prime meridian through Greenwich, is the *equinoctial colure* of the celestial sphere which passes through the *vernal equinox*, an imaginary point among the stars where the sun in its apparent upward path crosses the equator on March 21 of each year. In the figure, V represents the vernal equinox and PTV is the equinoctial colure. Comparable with the longitude and latitude of a station on the earth are the *right ascension* and *declination* of a heavenly body.

The right ascension of the sun or any star is defined as the angular distance measured along the celestial equator between the vernal equinox and the hour circle through the body. Right ascensions are measured eastward from the vernal equinox and may be expressed either in degrees of arc (from 0° to 360°) or in hours of time (0^h to

24^h). Thus, in the figure, the right ascension of S is given by the angle α in the plane of the equator or by the arc VU.

The declination of any celestial body is defined as the angular distance of the body above or below the celestial equator. If the body is above the equator its declination is said to be north and is considered as positive; if it is below the equator its declination is said to be south and is considered as negative. Declinations are expressed in degrees and cannot exceed 90° in magnitude. Thus in the figure, the declination of S is given by the angle δ or by the arc of any hour circle between the equator and the parallel of declination RSR', such as US, VT, QR, $Q'R'$, etc. Since S is above the equator, its declination is north and is considered as positive.

The *polar distance* of a star is the complement of the star's declination. In the figure, it is given by the angle $p = 90° - \delta$ or by the arc PS. In defining the position of a star near either pole, the polar distance is often given instead of the declination.

For present purposes it may be considered that the vernal equinox is a fixed point on the celestial equator, just as Greenwich is a fixed point on the earth. But while stations on the earth may be said to maintain practically an unvarying position with respect to the equator and the meridian of Greenwich, the coordinates of celestial bodies with respect to the celestial equator and the equinoctial colure change more or less with the passage of time. The fixed stars, or those outside the solar system, alter their positions in the celestial sphere but slightly from month to month and from year to year, the annual change being less than a minute of arc in either right ascension or declination.[1]

Since the earth actually travels about the sun, apparently the sun rapidly changes its position in the celestial sphere, making a complete circuit of the heavens once each year, its apparent path among the stars (called the *ecliptic*) twice cutting the equator during this interval. Its right ascension therefore changes 24^h (or 360°) during each year, being 0^h at the instant the sun's center apparently crosses the equator on March 21 and being 12^h when the sun in its downward path among the stars again cuts the equator on September 22. The annual change in the declination, caused by the fact that the axis of rotation of the earth is not at right angles to the plane of the earth's orbit about the sun, is nearly 47°, the declination varying from about N23½° on June 21 to about S23½° on December 21.

[1] These variations in position are due to (a) *precession* or the slow change in the direction of the earth's axis due to attraction of the sun, moon, and planets, and (b) *nutation* or small inequalities in the motion of precession, similar to the oscillation of a spinning top.

290a. Astronomical Tables Used by the Surveyor.—By means of astronomical observations and calculations, the positions of many of the heavenly bodies are predicted, and values of their right ascensions and declinations for various dates are available in various publications, the one most widely used by astronomers in the United States being the "American Ephemeris and Nautical Almanac" (800 pp.; price $1),[1] published two or three years in advance for each year by the Nautical Almanac Office, U. S. Naval Observatory. By means of the Ephemeris it becomes a simple matter to determine for any instant of time the celestial position of any heavenly body therein catalogued.

Most of the astronomical data used by the surveyor are concisely presented in the "American Nautical Almanac" (200 pp.; price 50 cents),[1] also published by the Nautical Almanac Office.

In still more condensed form is the "Ephemeris of the Sun and Polaris" (12 pp.; price 5 cents),[1] published by the General Land Office. This ephemeris lists for each day of the current year the position of the sun and of Polaris (the north star), by means of which the surveyor may compute from his field observations the latitude or longitude of the point of observation, the time of observation, or the azimuth of a reference line. The major points of difference between the arrangement of these tables and that of the tables published by the Nautical Almanac Office are explained in Arts. 297*d* and 307.

Useful condensed tables of data regarding the sun and Polaris are furnished to surveyors free of charge by various manufacturers of surveying instruments.

291. Hour Angle and Declination.—In many of the problems of field astronomy it is not only necessary that a star's position in the celestial sphere be known, but also that its position with respect to the meridian through a given station on the surface of the earth be determined. In Fig. 290 let $QRPR'Q'$ represent the meridian of some station on the earth, say that of the observer, and let S be some heavenly body, say a star, whose position with respect to the meridian $QRPR'Q'$ and the equator $QQ'UV$ it is desired to establish. The spherical coordinates of the star are given by the angular distance of the star above or below the equator, which in the figure is given by the arc US, defined in the preceding article as the declination, and the angular distance measured along the equator between the meridian and the hour circle through the star. When this angular measurement is from east to west it is called an *hour angle*.

The hour angle of any heavenly body may then be defined as the angular distance measured from east to west along the equator from

[1] Sold by Superintendent of Documents, Washington, D. C.

the meridian of reference to the hour circle through the body. Thus, in the figure, the hour angle of S is given by the angle t or by the angular distance $QQ'U$. Hour angles are expressed either in hours of time or in degrees of arc. In the figure the hour angle is more than 12^h or more than 180°. Without qualification, it is understood that an hour angle is measured from the upper branch of the meridian, that is, the branch above the station or above the observer's head, but in connection with the definition of civil time, hour angles are reckoned from the lower branch of the meridian. In the figure, if the hour angle were reckoned from the lower branch, it would be defined by the angular distance $Q'U$, and would be 12^h more or less than that given by the arc $QQ'U$, which is the hour angle reckoned from the upper branch. Sometimes the hour angles of stars east of the meridian are reckoned eastward from the upper branch of the meridian, rather than westward. When an hour angle is expressed in this way it is preceded by a minus sign. Thus if the hour angle of S (Fig. 290) were reckoned eastward, it would be given by the angular distance QU.

292. Equator Systems Compared.—The system of coordinates described in Art. 291 is seen to be similar to that described in Art. 290, with this difference, that in the system last considered, the angular distance along the equator is measured from a fixed meridian to the hour circle through the star, while in the method described in Art. 290 the angular distance along the equator is measured from the vernal equinox, which is a point on the celestial equator that rotates with the celestial sphere. Thus, while right ascensions of fixed stars have annual variations of but a few seconds, hour angles of the stars change as rapidly as the celestial sphere apparently rotates, which is 24^h or 360° for each $23^h\ 56^m$ of our civil time, and hour angles of the sun change approximately 24^h or 360° for each 24^h of our civil time.

The systems described in the two preceding articles are called *equatorial systems of coordinates*, being so named by reason of the fact that in each case the primary plane of reference is the celestial equator. The position of a heavenly body above or below the equator is given by the declination δ which is the same in one system as in the other. In the system described in Art. 290 the angular distance along the equator is the right ascension α and is reckoned from the vernal equinox; in the system described in Art. 291 the angular distance along the equator is the hour angle t and is measured from the meridian. Let θ be the hour angle of the vernal equinox represented in Fig. 290 by the angular distance $QQ'V$ measured along the equator. At any instant of time, if the hour angle of the vernal

equinox with respect to a given meridian is known and if the right ascension of a heavenly body S is known, then the hour angle of the body may be computed, since by the figure

$$t = \theta - \alpha \qquad \text{or} \qquad \theta = t + \alpha$$

This equation is therefore an expression by means of which the coordinates of one system may be transposed to those of the other.

293. Horizon System.—In the ordinary operations of engineering, when observations are made with the engineer's transit, measurements are in the form of horizontal and vertical angles. The construction of the surveying instruments is such that it would be inconvenient to take angular measurements otherwise. It follows that, in astronomical field work where observations are made

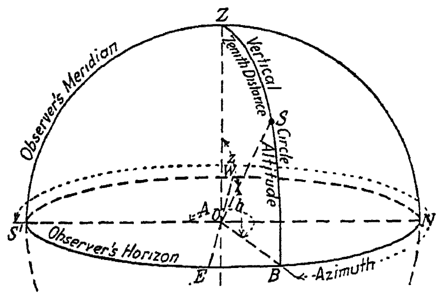


FIG. 293.

with the engineer's transit or similar instrument, angular measurements to celestial bodies must be of like character to those made on terrestrial objects. The *horizon system* of spherical coordinates is such a system extended to include the celestial sphere. In the horizon system the coordinates of an object are given by its *azimuth* and *altitude*. The system is illustrated by Fig. 293 which represents a portion of the celestial sphere. $NES'W$ is the observer's horizon, Z is the zenith, and $S'ZN$ is the meridian; S is a heavenly body and BSZ is part of a great circle, called a *vertical circle*, through the body and the zenith.

The azimuth of a heavenly body is defined as the angular distance measured along the horizon in a clockwise direction from the meridian to the vertical circle through the body. Azimuths may be reckoned either from the south or the north point of the meridian, but in astronomical work azimuths are customarily reckoned from south through 360°. An exception is often made in the case of circumpolar stars for which azimuths are reckoned from north. Also in trigonometric computations azimuths of stars west of north or east of south

are often expressed as the counter-clockwise angles from the meridian and are considered as negative values. In Fig. 293 the azimuth of S reckoned in the customary manner is given by the angle A or by the angular distance $S'NB$, an arc of the horizon. If the azimuth of S were reckoned from north, it would be given by the angle $(A - 180°)$ or by the angular distance NB. The negative azimuth reckoned from south is given by the arc $S'EB$.

The altitude of a heavenly body is defined as the angular distance measured along a vertical circle, from the horizon to the body. It is expressed in degrees of arc. The altitude of S is given by the vertical angle h and also by the angular distance BS, the arc of a vertical circle passing through the zenith. Except in rare instances, celestial objects are observed when above the true horizon, when the sign of the altitude is positive. It is seen that positive altitudes may vary between 0° and 90°.

The complement of the altitude is called the *zenith distance*. It is the angular distance from the zenith to the celestial body measured along a vertical circle. In the figure the zenith distance is given by the angle z or by the angular distance ZS. Thus $z = 90° - h$. Since the celestial sphere is apparently rotating about its axis, while the meridian, horizon, and zenith are imagined as remaining fixed in position, it is clear that in general both the azimuth and altitude of a star undergo a maximum change in the course of one rotation of the sphere or in the course of one day.

If the star is near the pole the daily range of both azimuth and altitude is small, regardless of the observer's latitude. If the observer is at the pole of the earth, all stars above the celestial equator go through a daily change in azimuth of approximately 360°, but their altitudes (neglecting slight secular changes in declinations) remain constant. If the observer is at the equator, a star on the celestial equator would traverse a vertical circle through the east and west points of the horizon and through the zenith. When the star came in view on the eastern horizon, its altitude would be 0° and its azimuth (from south) would be 270°. The azimuth would remain constant until the star crossed the meridian, when it would instantaneously change to 90°, at which value it would remain until the star disappeared below the horizon. During this interval the altitude of the star would change at a uniform rate (except for the effect of variable refraction) from 0° to 90° and then back to 0°. If there were a star at either celestial pole, its altitude would remain constantly at 0°, and its azimuth would be constantly 0° or 180° depending upon whether the star were at the south celestial pole or at the north celestial pole. At a station intermediate between the pole and the equator, any circumpolar star is continuously above the horizon, and since it crosses the meridian twice daily, once when above the pole and once when below, it follows

that (for the northern hemisphere) its azimuth (from south) is 180° twice daily. When it crosses the lower branch of the meridian, its altitude is a minimum; and when it crosses the upper branch, its altitude is a maximum. Obviously the mean of these two values (if variable refraction and slight changes in declination are to be neglected) is the altitude of the celestial pole. If any heavenly body arrives at a given altitude when it is east of the meridian, it will later arrive at the same altitude west of the meridian, and neglecting change in declination, the azimuth measured east of the meridian to the first position will equal the azimuth measured west of the meridian to the second position.

294. Relation between Horizon and Equator Systems.—Figure 294*a* represents a section of the earth through the poles and the

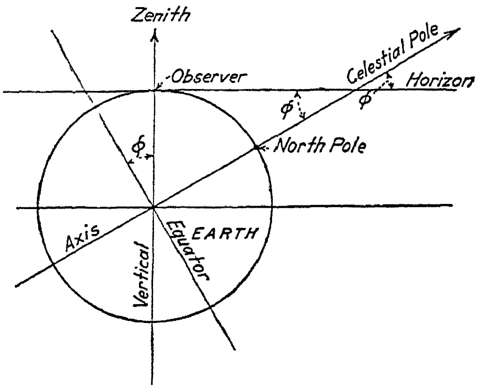


FIG. 294*a*.

station of an observer. Since the latitude of a place is its angular distance from the equator measured along a meridian of longitude, it is evident from the figure that the latitude of the observer is given by the angle ϕ between the equator and a vertical line through the observer's station, the angle being measured in the plane of the meridian. Also it appears, from similar triangles, that the latitude is given by the angle ϕ between the axis and the horizon, likewise measured in the plane of the meridian.

Similarly Fig. 294*b* represents a section of the celestial sphere through the celestial poles and observer's zenith. For reasons just explained,

$$\angle QOZ = \angle NOP = \phi = \text{latitude of place}$$

Hence the latitude of a place is given by the angular distance NP which is the altitude of the pole, or by the angular distance QZ which is the zenith distance or coaltitude of the equator. The angular

distance from the pole to the zenith is $90° - \phi = c$ which is the colatitude of the place.

In a northern latitude if any heavenly body S whose declination is δ is on the meridian and south of the zenith, then from Fig. 294*b*

$$\phi = 90° - h + \delta = z + \delta$$

and for any star north of the zenith

$$\phi = h \pm (90° - \delta) = h \pm p,$$

in which the sign of p, the polar distance, is positive or negative according as the star is below or above the pole. When a heavenly body S is on the meridian and south of the zenith, both its hour angle (reckoned from the upper branch of the meridian) and its

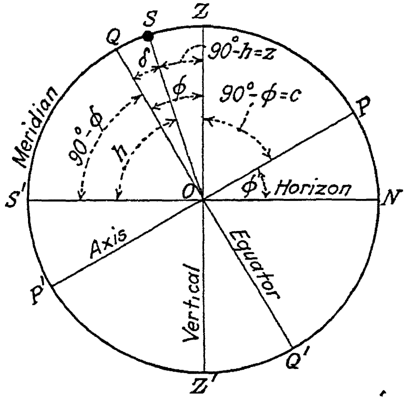


FIG. 294*b*.

azimuth (reckoned from south) are zero; but if a star is at upper culmination north of the zenith its hour angle (reckoned from the upper branch) is zero but its azimuth (reckoned from south) is 180°.

294a. The relation between the coordinates of the horizon system and those of the equator system is shown for a star S not on the meridian, by Fig. 294*c*. The place of observation is assumed to be north of the equator at a latitude ϕ, as shown by the angle between the horizon plane and the celestial axis, and the star is in a position east of the meridian and above the celestial equator.

In the horizon system, the coordinates of S are A, the azimuth measured from the south point of the horizon, and h, the altitude measured above the horizon. In the equator system the angular position is fixed by t, the hour angle measured from the upper branch of the meridian and δ, the declination measured above the equator. Most of the problems of field astronomy involve transposing from one

system of spherical coordinates to the other, that is, certain coordinates in one or both systems are known or observed and calculations are made to determine the remaining unknowns; and this makes necessary the solution of the spherical triangle defined by the mutually intercepted arcs of the meridian, the vertical circle through the star, and the hour circle through the star. The vertices of this triangle are the pole P, the zenith Z, and the star S. The triangle is called the *astronomical triangle* or the *PZS triangle.*

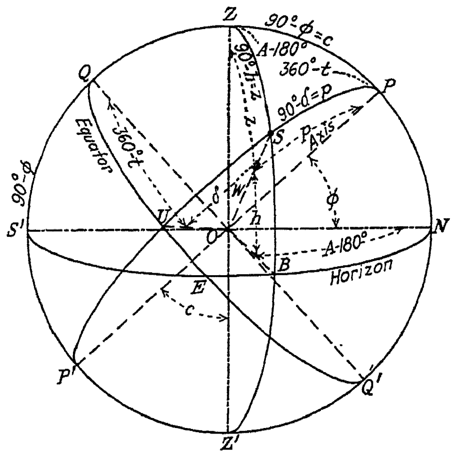


FIG. 294c.

Referring again to the figure, the sides of the PZS triangle are seen to be the angular distances PS, PZ and ZS.

$$PZ = 90° - \phi = c = \text{colatitude},$$
$$PS = 90° - \delta = p = \text{codeclination} = \text{polar distance},$$
$$ZS = 90° - h = z = \text{coaltitude} = \text{zenith distance}.$$

If the star is below the equator, the above expression for PS still holds true when due account is taken of the algebraic sign of δ.

When the star is east of the meridian, as in the figure, and azimuths are reckoned from the south point of the horizon and hour angles are reckoned from the upper branch of the meridian, then the angles of the PZS triangle formed by the intersecting planes of the meridian, the hour circle, and the vertical circle are:

$$Z = A - 180° = (\text{azimuth} - 180°)$$
$$P = 360° - t = (360° - \text{hour angle})$$

If the star is west of the pole it can be readily shown by a sketch that the angles of the PZS triangle are

$$Z = 180° - A = (180° - \text{azimuth})$$
$$P = t = \text{hour angle.}$$

295. Spherical Trigonometry.—The solution of a spherical triangle depends upon the principles of spherical trigonometry, of which the surveyor should have some knowledge. A derivation of the fundamental equations of spherical trigonometry follows:

In Fig. 295*a*, let OX, OY, and OZ be the X, Y, and Z axes of rectangular coordinates, and let ABC be a spherical triangle on the surface of a sphere of unit radius of which O is the center, the side c being in the XY plane.

Since the radius of the sphere is unity, each of the distances OA, OB, and OC is unity and the arcs a, b, and c are measures respectively of the central angles BOC, COA, and AOB.

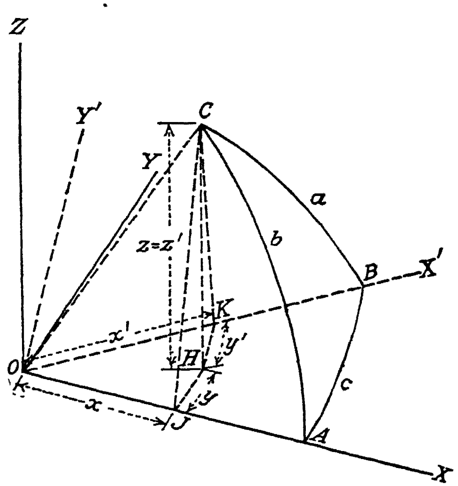


FIG. 295*a*.

Let H mark the projection of C on the XY plane and let JH be constructed parallel to OY. Then, since the plane of the plane triangle is parallel to the YZ plane, $\angle CJH$ is equal to $\angle A$ of the spherical triangle ABC. The coordinates of C are $x = OJ$, $y = JH$, and $z = HC$. Then, since the radius of the sphere is unity

$$x = \cos b \tag{1}$$
$$y = \sin b \cos A \tag{2}$$
$$z = \sin b \sin A \tag{3}$$

Let the ZX and the ZY planes be rotated about the Z-axis through $\angle AOB = \angle c$, the new position of the Y-axis being OY' and the new position of the X-axis being OX', passing through B. As

before, the projection of C on the XY plane is H. Construct HK parallel to OY'. Then $\angle CKH$ of the plane triangle equals $\angle B$ of the spherical triangle. The coordinates of C with respect to the new position of the axes are $x' = OK$, $-y' = KH$, and $z = HC$. Then

$$x' = \cos a \tag{4}$$
$$y' = -\sin a \cos B \tag{5}$$
$$z' = \sin a \sin B \tag{6}$$

From Fig. 295*a*, it is clear that $z = z'$. In Fig. 295*b* is shown a plan view of the XY plane of Fig. 295*a*. The angle between the

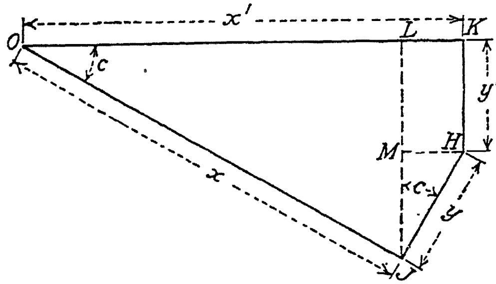


FIG. 295*b*.

two positions of the X-axis is c, and the projection of the point C upon the XY plane is H. From the figure,

$$x' = OL + LK = OJ \cos c + JH \sin c = x \cos c + y \sin c$$

Also

$$-y' = JL - JM = OJ \sin c - JH \cos c = x \sin c - y \cos c$$

The three equations for transforming the coordinates given by Eqs. (4) to (6) to those given by Eqs. (1) to (3) are therefore

$$x' = x \cos c + y \sin c \tag{7}$$
$$y' = y \cos c - x \sin c \tag{8}$$
$$z' = z \tag{9}$$

Substituting in Eqs. (7) to (9) the values of the coordinates given by Eqs. (1) to (6), there results:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{10}$$
$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A \tag{11}$$
$$\sin a \sin B = \sin b \sin A \tag{12}$$

Since these equations connect the values of all angles and sides in any spherical triangle, they may be regarded as the fundamental expressions which give the relations that exist in any spherical triangle, regardless of its size, shape, or position.

296. Solution of the *PZS* Triangle.—It is evident that equations similar to those of the preceding article could be derived by considering either of the other two vertices of the spherical triangle of Fig. 295*a* as the one for which coordinates are written. Thus, three similar sets of fundamental equations may be written for the astronomical triangle, P, Z, or S, as desired, being substituted in place of C in Fig. 295*a*.

In surveying, the solution of the astronomical triangle is made in connection with determinations of azimuth. Usually observations are made on the sun or on some star that can be readily identified. In this case the altitude of the body is measured, its declination is determined from tables, and the latitude of the place of observation is known. Hence, the three sides of the astronomical triangle are known. The determination of azimuth of the heavenly body involves the computation of the angle at Z; and determinations of longitude or time involve the computation of the angle at P, as a measure of the hour angle.

In Fig. 296*a*, let PZS be the astronomical triangle of Fig. 294*c*, for which the sides are $90° - \phi$ (the colatitude), $90° - h$ (the coaltitude or zenith distance), and $90° - \delta$ (the codeclination or polar distance). Imagine that the spherical triangle of Fig. 295*a* is rotated in position so that its vertices A, B, and C coincide respectively with Z, S, and P of the astronomical triangle; then $a = 90° - \delta$, $b = 90° - \phi$, $c = 90° - h$, and $A = Z$. Substituting these values in Eq. (10), there results

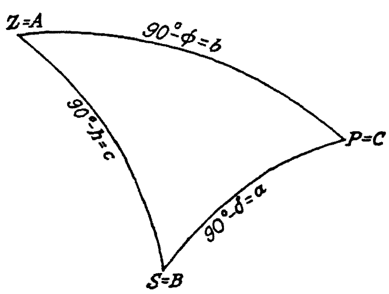


FIG. 296*a*.

$$\cos Z = \frac{\sin \delta}{\cos h \cos \phi} - \tan h \tan \phi \tag{13}$$

which is a general expression for determining azimuth when the three sides of the astronomical triangle are known, Z being the azimuth from the north point of the horizon, considered positive if the star is east of the meridian and negative if the star is west of the meridian, and being less or greater than 90° according as the sign of $\cos Z$ is positive or negative.

The preceding equation may be expressed as

$$\text{vers } Z = \sec \phi \sec h[\text{vers } p - \text{vers } (\phi - h)]$$

in which $p = 90° - \delta$ = polar distance. This is a convenient form when tables of versed sines are available.

An equation similar to Eq. (13) may be developed for the unknown angle at P. By assuming the vertices A, B, and C of the spherical triangle of Fig. 295a respectively to coincide with P, S, and Z of the astronomical triangle of Fig. 294c, then as shown by Fig. 296b, $90° - h = a$, $90° - \phi = b$, and $90° - \delta = c$. Making these substitutions in Eq. (10) and letting $P = t =$ hour angle in either direction from the meridian,

$$\cos t = \frac{\sin h}{\cos \delta \cos \phi} - \tan \delta \tan \phi \tag{14}$$

which is a general expression for determining the hour angle of any celestial body when the three sides of the astronomical triangle are known.

The above equation may be expressed in the form

$$\text{vers } t = \sec \phi \sec \delta[\text{vers } z - \text{vers } (\phi - \delta)]$$

in which $z = 90° - h =$ zenith distance.

296a. While in any case Eq. (13) may be used to determine azimuth and Eq. (14) may be used to determine hour angle, these equations in the form given are not always as suitable nor as convenient as some other forms. Where logarithmic computations are employed, the solution of either of these expressions involves the use of both logarithmic and natural trigonometric functions. Also when the unknown angle is small or is near 180°, since the magnitude of the cosine is changing slowly, a relatively small error in the computed value of the cosine will produce a relatively large error in the angle itself. The values of trigonometric and logarithmic functions given in tables are not absolute quantities, and if a trigonometric function is found from a combination of several values, each of which is not absolute, the accumulated error is likely to be one or more in the last place. For this reason, in so far as errors of computation are involved, the above equations are not suitable for accurately computing azimuth and hour angle when the observed celestial body is near the meridian. On the other hand, when the unknown azimuth or hour angle is near 90° or 270°, its cosine is changing rapidly and hence Eqs. (13) and (14)

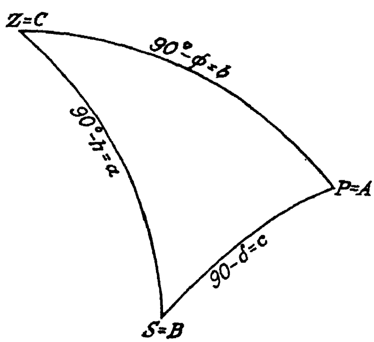


FIG. 296b.

are most suitable for accurately computing these large angles. The following examples illustrate the point under discussion. It is seen that the error in the computed angle of example 1 is nearly eight times that of the angle of example 2.

Example 1: In determining the azimuth of a star at a given instant, by Eq. (13) the errors of the computations are such that the ratio of precision of the computed cosine is $\frac{1}{5{,}000}$. It is desired to know the error in the corresponding angle, the azimuth being approximately 20°. By Fig. 21*a*, p. 19, the angular error is approximately 01′50″, found by extrapolation at the intersection of the line representing $\frac{1}{5{,}000}$ and that for 20° for cosines.

Example 2: Same conditions as example 1, but azimuth approximately 70°. The angular error in the azimuth is 15″, being found by interpolation at the intersection of the $\frac{1}{5{,}000}$ line with the line for 70° for cosines.

296b. By a series of substitutions which will not be given here but which may be found in any treatise on spherical trigonometry, Eq. (13) may be changed to the form

$$\tan^2\frac{1}{2}Z = \frac{\sin(s-h)\sin(s-\phi)}{\cos s\cos(s-p)} \tag{15}$$

and Eq. (14) may be changed to the form

$$\tan^2\frac{1}{2}t = \frac{\cos s\sin(s-h)}{\cos(s-p)\sin(s-\phi)}. \tag{16}$$

In these two equations $p = 90° - \delta$ = polar distance, $s = \frac{1}{2}(h + \phi + p)$, and the remaining letters have the same significance as in Eqs. (13) and (14).

For a given angular value the tangent changes more rapidly than the cosine. Thus for a given error of computation of the trigonometric function, Eqs. (15) and (16) will always render a closer determination of azimuth and hour angle than will Eqs. (13) and (14). For angles near 90° and 270° the difference between the rate of change of the tangent and of the cosine is not large, hence within this range Eqs. (13) and (14) do not render results which are materially greater in error than those obtained by using Eqs. (15) and (16). But when the object is near the meridian, that is, when the azimuth is near 0° or 180°, Eqs. (15) and (16) will for given errors of computation render possible closer determinations of angles than will Eqs. (13) and (14). This is illustrated by the following examples, a continuation of examples 1 and 2 of the preceding article.

Example 3: In determining the azimuth of a star by Eq. (15) it is desired to know what angular error will be introduced if the ratio of precision of the computed quantity $\tan^2 \frac{1}{2}Z$ is $\frac{1}{5,000}$. If the error in $\tan^2 \frac{1}{2}Z$ is $\frac{1}{5,000}$, then the error in $\tan \frac{1}{2}Z$ is approximately $\frac{1}{10,000}$.

If Z is 20°, then $\frac{1}{2}Z$ is 10°. Consulting the diagram of Fig. 21*b*, the corresponding angular error in $\frac{1}{2}Z$ is approximately 03½″; hence the error in the calculated value of Z is about 07″.

If Z is 70°, then $\frac{1}{2}Z$ is 35°. From Fig. 21*b*, the corresponding angular error in $\frac{1}{2}Z$ is 09″; therefore the error in the calculated value of Z is 18″.

It should be understood that the preceding discussion regarding the relative advantages of the tangent and cosine forms of expressions for determining azimuth and hour angle refers to the errors of computation. In effect, it means that the required precision of angle might be obtained with the tangent form with, say, a five-place table, when to obtain the same precision of computation it might be necessary to use, say, a six- or seven-place table if the cosine form were used, all depending upon the magnitude of the calculated angle. It does not mean that a given error in an observed quantity, say the altitude, will have any less effect upon the computed value if the tangent form of equation is employed; obviously since the fundamental trigonometric relations are the same for one form of equation as for the other, the error introduced in a computed value on account of a given error in an observed quantity will be the same, regardless of the form of the equation used in determining the angle.

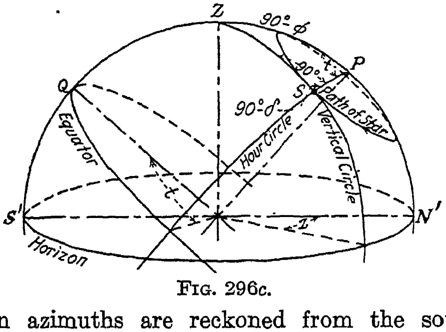


FIG. 296*c*.

296c. When azimuths are reckoned from the south point of the horizon, Eqs. (13) and (15) take the following forms, A being the azimuth measured either clockwise or counter-clockwise from the south point:

$$\cos A = \tan h \tan \phi - \frac{\sin \delta}{\cos h \cos \phi} \tag{13'}$$

$$\cot^2 \frac{1}{2} A = \frac{\sin (s - h) \sin (s - \phi)}{\cos s \cos (s - p)} \tag{15'}$$

When cot $\frac{1}{2}A$ has been determined, the computations for hour angle are somewhat reduced if Eq. (16) is modified as follows:

$$\tan \frac{1}{2} t = \frac{\sin (s - h)}{\cot \frac{1}{2}A \cos (s - p)} \tag{16'}$$

296d. Azimuth at Elongation.—The most favorable position for determining azimuth by observation on a circumpolar star (or on any star which crosses the upper branch of the meridian north of the zenith) occurs when it is farthest east or farthest west of the pole, when the star appears to be traveling vertically for some time. In this position it is said to be at eastern or western *elongation* according as it is east or west of the meridian. At the instant of elongation, since the star appears to be traveling vertically, its apparent path in the celestial sphere is tangent to the vertical circle through the zenith, as is illustrated by Fig. 296*c*. Therefore the angle between the plane of the hour circle and the plane of the vertical circle, or in other words, the angle at S, is 90°. For azimuth determinations of this sort, the latitude of the place of observation is known and the declination of the star for the given date is obtained from tables. At the instant of elongation, there are then known in the astronomical triangle the side $ZP = 90° - \phi$, the side $PS = 90° - \delta$, and the angle $S = 90°$.

If the spherical triangle ABC of Fig. 295*a* is made to coincide with the astronomical triangle of Fig. 296*c*, so that $a = 90° - \delta = p$, $b = 90° - \phi$, and the vertices A, B, and C coincide respectively with Z, S, and P, then remembering that the sine of S is unity when S is 90°, by substituting in Eq. (12) Art. 295, there is obtained

$$\sin Z = \frac{\sin (90° - \delta)}{\sin (90° - \phi)}$$

or

$$\sin Z = \frac{\sin p}{\cos \phi} \tag{17}$$

which is the expression employed for determining the azimuth of a circumpolar star when at elongation, the latitude of the place and the star's polar distance being known, Z being the azimuth reckoned east or west of north according as the star is at eastern or western elongation.

By considering the spherical triangle ABC of Fig. 295*a*, as taking the position PSZ in Fig. 296*c*, and substituting the proper values in Eq. (11), Art. 295, there is derived

$$\cos t = \tan \phi \tan p \tag{18}$$

which is an expression for finding the hour angle of a star at the instant of elongation, the hour angle t being reckoned east or west of the upper branch of the meridian, depending upon the position of the star, and the latitude ϕ and the polar distance $p = 90° - \delta$ being known. The equation is useful in determining the time at which elongation will occur on any given date.

296e. Altitude of a Star.—When a star cannot be readily identified through the transit telescope, the process of bringing it into the field of view is considerably expedited if its approximate altitude is computed prior to the observation, and this value is laid off on the vertical circle of the transit. Also a check on the correctness of observations and calculations for azimuth and hour angle is obtained if the computed value of the altitude agrees with the observed value.

Again referring to the derivation of the fundamental equations of Art. 295, if the PZS triangle is substituted for Fig. 295a, in such manner that $a = 90° - h$, $b = 90° - \phi$, $c = 90° - \delta$, and the vertices P, S, and Z lie respectively at A, B, and C, then by substituting in Eq. (10) there is obtained

$$\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t, \tag{19}$$

in which t is the hour angle at a given time, and h is the altitude at the same instant.

By trigonometric substitutions there may be derived from Eq. (19) the expression

$$\sin h = \cos (\phi - \delta) - \cos \phi \cos \delta \operatorname{vers} t \tag{20}$$

a form more suitable for accurate determinations of the altitude.

296f. Azimuth of a Circumpolar Star.—If the spherical triangle ABC of Fig. 295a be made to coincide with the astronomical triangle of Fig. 294c, so that $a = 90° - h$, $b = 90° - \delta$, $c = 90° - \phi$, and the vertices A, B, and C coincide respectively with P, Z, and S, then by substituting in Eqs. (12) and (11) (Art. 295), dividing Eq. (12) by Eq. (11), and dividing the result thereof by $\sin \delta$, there results

$$\tan Z = \frac{\sin t}{\cos \phi \tan \delta - \sin \phi \cos t} \tag{21}$$

This expression is commonly used in finding the azimuth of Polaris or any other circumpolar star when the star is not at elongation and when the hour angle of the star is accurately known.

297. Time.—As the earth rotates about its axis in its travel through space, all celestial bodies apparently rotate about the earth (or about its axis) from east to west with regularity. We have imagined that

the earth is the center of the celestial sphere—a hollow sphere of infinite radius to which are fastened the sun and stars—which rotates about the celestial axis regularly. It has been stated that owing to the fact that the earth in its orbit travels about the sun but does not travel about the fixed stars which are far outside its orbit, once each year the sun apparently encircles the celestial sphere along a path called the *ecliptic* which twice cuts the celestial equator during this interval. The point among the stars where the sun in its apparent travel northward cuts the celestial equator on March 21 of each year is called the *vernal equinox*, and it has been stated that the vernal equinox is a point of reference whose position on the celestial sphere is unchanging. There is no star at that point, but it is helpful to imagine that the vernal equinox is an invisible celestial body rigidly fastened in its position on the celestial sphere, while each of the so-called "fixed" stars slowly moves along a path of extremely small compass on the surface of the sphere, and the sun travels rapidly along the ecliptic in a direction opposite to that of the rotation of the celestial sphere.

Because the sun is apparently traveling from west to east among the stars, while the rotation of the celestial sphere about the earth is apparently from east to west, the angular velocity of the sun about the axis of the celestial sphere is less than that of the fixed stars or of the vernal equinox, just as the angular velocity of a passenger walking towards the rear of a train upon a circular track is less than that of the locomotive. This being the case, at a given meridian the hour angle of the sun and of the vernal equinox will agree at some instant on March 21, but thereafter it will be less for the sun than for the vernal equinox. Six months later, on September 22, when the sun has covered one half of its annual journey, the hour angle of the sun will be 180° or 12^h less than that of the vernal equinox; and 1 yr. later the hour angle of the sun will be 360° or 24^h less than that of the vernal equinox, and hence the hour angles will again numerically agree.

In the course of a tropical year as measured by the time taken by the sun apparently to make a complete circuit of the ecliptic, there actually occur 366.2422 revolutions of the earth, or apparently a like number of revolutions of the vernal equinox about the earth. For reasons just explained, the sun during this interval will have traveled through a total hour angle 360° or 24^h less than that traversed by the vernal equinox, hence during a tropical year the sun apparently revolves about the earth 365.2422 times.

The interval of time occupied by one apparent revolution of the

sun about the earth is called a *solar day*, the unit of time with which we are all familiar. The interval of time occupied by one apparent revolution of the vernal equinox is called a *sidereal day*, a unit of time which is much used by astronomers. Since 366.2422 sidereal days occupy the same period of time as 365.2422 solar days, the sidereal day is a shorter time interval than the solar day.

When any celestial body, real or imaginary, apparently crosses the upper branch of a meridian, it is said to be at *upper transit* or *upper culmination;* when any celestial body crosses the lower branch of the meridian it is said to be at *lower transit* or *lower culmination.*

The beginning of a sidereal day at a given place occurs at the instant the vernal equinox is at upper transit.

The solar day is considered as beginning at the instant of lower transit (midnight), as does the civil day.

Both sidereal and solar days are divided into 24 hr. each of 60 min. duration. The hours are reckoned consecutively from 0 to 24.

297a. True and Mean Suns.—On account of the elliptical shape of the earth's orbit, the apparent angular velocity of the sun we see, or the *true sun*, is not constant, hence the days as indicated by the apparent travel of the true sun about the earth are not of uniform length. To make our solar days of uniform length, astronomers have invented the *mean sun*, a fictitious body which is imagined to move at a uniform rate along the celestial equator, making a complete circuit from west to east in one tropical year.

The time interval as measured by one apparent revolution of the *true sun* about the earth is called an *apparent solar day*. The time interval as measured by one revolution of the *mean sun* is called a *mean solar day*, which is the same as the civil day.

297b. Apparent (True) Solar Time.—Beginning with the 1925 volume, the American Ephemeris and Nautical Almanac has considered the apparent solar day at any place as the time interval between two successive lower transits of the true (apparent) sun for the meridian of that place. The solar day therefore begins at midnight and the *apparent solar time* at any place is given by the hour an gle of the true sun plus 12^h. Thus, if the hour angle of the sun is $45° = 3^h$ at a given place and at a given instant, then the apparent solar time for the place is $3^h + 12^h = 15^h$. *Apparent time* has the same meaning as *apparent solar time.*

Local apparent time is the apparent solar time for the meridian of the observer. Apparent solar time for any other meridian is designated by name. Thus apparent solar time for the meridian of Greenwich is called *Greenwich apparent time.*

297c. Mean Solar (Civil) Time.—Beginning with the 1925 volume, the American Ephemeris and Nautical Almanac has considered the mean solar day at any place to be the time interval between two successive lower transits of the mean sun for the meridian of that place. The mean solar day therefore begins at midnight, as does the civil day, and the *mean solar time* is given by the hour angle of the mean sun plus 12^h. Thus, if the hour angle of the mean sun is $-15° = -1^h$, the mean solar time is $-1^h + 12^h = 11$ hr. *Mean time* has the same meaning as *mean solar time* or *civil time* and is the time in general use by the public.

Local mean time is the mean solar time for the meridian of the observer. Mean solar time for any other meridian is designated by name. Thus mean solar time for the meridian of Greenwich is called *Greenwich mean time.*

297d. Equation of Time.—The difference between mean time and apparent time at any instant is called the *equation of time.* When the true sun is ahead of the mean sun, apparent time is faster than mean time, and *vice versa.* The equation of time may be obtained from either of the two types of solar ephemeris, which differ in arrangement as follows:

1. In the American Ephemeris, the equation of time is given for each day at the instant of 0^h Greenwich civil time (midnight Greenwich mean time). In the Nautical Almanac, the equation of time is given for each day for the even hours, Greenwich civil time. If the sign of the equation of time is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ it indicates that the true sun is $\left\{\begin{matrix}\text{ahead of}\\ \text{behind}\end{matrix}\right\}$ the mean sun; hence, when using this type of ephemeris and when the apparent time is desired, the equation of time is applied to the published values of mean time in accordance with the sign as given.

2. In the Ephemeris of the Sun and Polaris, published by the General Land Office, the equation of time is given for each day at the instant of Greenwich apparent noon. The column headings state directly whether the equation of time is to be added to, or subtracted from, the apparent time when the mean time is desired, using this type of ephemeris.

3. The American Ephemeris contains, in addition to the table described in (1), a table giving the equation of time for each day at the instant of apparent noon at the meridian of Washington. The signs there given are to be applied to the published values of apparent time, when mean time is desired; these signs are the opposite of those given in (1) above.

To find the equation of time at any instant other than that for which a value is tabulated, it is necessary to interpolate, adding to or subtracting from the given value the change that has taken place in the equation of time since the instant to which the tabulated value applies.

Example 1: It is desired to determine by use of the Nautical Almanac the equation of time at the instant of $3^h30^m45^s$ p.m. Greenwich mean time on Dec. 15, 1927. Greenwich civil time $= 12^h + 3^h30^m45^s = 15^h.51$.

From the Nautical Almanac the equation of time at 16^h G.C.T. is $+5^m5^s.2$. The hourly change in the equation of time (H.D.) is $1^s.2$

$$(16^h - 15^h.51)1.2 = 0^s.6$$

The equation of time at the given instant is

$$+5^m5^s.2 + 0^s.6 = +5^m5^s.8.$$

Example 2: It is desired to determine by use of the American Ephemeris the Greenwich apparent time (G.A.T.) at the instant of $3^h30^m45^s$ p.m. Greenwich mean time on Dec. 15, 1927. Greenwich civil time $= 12^h + 3^h30^m45^s = 15^h.51$.

The equation of time at 0^h is $+5^m24^s.3$. The rate of change per hour is $-1^s.19$. The change since 0^h is $-1^s.19 \times 15.51 = -18^s.5$. The equation of time at $15^h30^m45^s$ is $+5^m24^s.3 - 18^s.5 = 5^m5^s.8$

$$\begin{aligned} \text{G.A.T.} = \text{G.C.T.} + \text{Eq. time} &= 15^h30^m45^s + 5^m5^s.8 \\ &= 15^h35^m50^s.8 \text{ after midnight} \\ &= 3^h35^m50^s.8 \text{ after noon} \end{aligned}$$

Example 3: It is desired to determine from the Ephemeris of the Sun and Polaris (General Land Office) the Greenwich mean time (G.M.T.) at the instant of $9^h00^m15^s$ Greenwich apparent time (G.A.T.) on Oct. 10, 1926. The time that will elapse before G.A. noon is $3^h.00$.

The equation of time at G.A. noon is $12^m47^s.6$, to be subtracted from apparent time. The change in one day is $12^m47^s.6 - 12^m31^s.4 = 16^s.2$. The change per hour is $\frac{16.2}{24} = 0^s.67$.

The change since G.A. noon is $-3.00 \times 0.67 = -2^s.0$
Eq. time for $9^h00^m15^s$ G.A.T. $= 12^m47^s.6 - 2^s.0$
$= 12^m45^s.6$

G.A.T. $= 9^h00^m15^s$
Eq. time $= 12^m45^s.6$, to be subtracted from apparent time.
G.M.T. $= 8^h47^m29^s.4$

By inspecting the tabulated values of the equations as given in the ephemerides it will be seen that in February the true sun is as much as 14^m behind the mean sun and that in November the true sun is more

than 16^m ahead of the mean sun, while on about the dates April 15, June 15, September 1, and December 25, the equation of time is zero and hence the hour angle of the true sun is for an instant the same as that of the mean sun.

297e. Sidereal Time.—The *sidereal time* at any place is the hour angle of the vernal equinox at that place; and the beginning of the sidereal day, occurring when the vernal equinox crosses the upper branch of the meridian, is called *sidereal noon.* Twenty-four-hour clocks regulated to keep sidereal time are called *sidereal clocks.* The vernal equinox being an imaginary point cannot be observed, as can the sun, but since it is the point of reference to which the right ascensions of stars are referred, the sidereal time may be obtained by determining the hour angle of any star the right ascension of which is known. Then if θ be the sidereal time, $\theta = t + \alpha$, as explained in Art. 292. The sidereal day is shorter than the mean solar day by $3^m55^s.9$ mean solar time, or $3^m56^s.6$ sidereal time. The sidereal hour is shorter than the mean solar hour by $9^s.830$ mean solar time, or $9^s.856$ sidereal time. In a sidereal hour there are $60^m - 9^s.830 = 59^m50^s.170$ mean solar time; in a mean solar hour there are $60^m + 9^s.856 = 1^h00^m09^s.856$ sidereal time. Apparent right ascensions of the sun and stars are given in the Nautical Almanac and also in the American Ephemeris. The following example illustrates the use of the American Ephemeris, in which example the solar ephemeris (for 0^h Greenwich civil time) is employed for computing the sidereal time at a given place at a given instant Greenwich civil time (G.C.T.), the hour angle of the true sun having been determined for the given instant. The column headed "Apparent Right Ascension" gives for each day of the month the right ascension of the true or apparent sun at the instant of 0^h Greenwich civil time, and the column headed "Variation per Hour" gives the rate of change in the right ascension at the same instant.

Example 1: At a given place the hour angle of the true sun at 4 p.m. Greenwich civil time July 3, 1927, is $-32°15'45''$. It is desired to know the sidereal time at the given instant. By the American Ephemeris the apparent right ascension α for 0^h G.C.T. is $6^h43^m52^s.0$. The hourly change in α is $+10^s.33$. The change during the time elapsed since 0^h G.C.T. is

$$16 \times 10.33 = 165^s.3 = 2^m45^s.3$$

$$\begin{aligned} \alpha \text{ at 4 p.m. G.C.T.} &= 6^h43^m52^s.0 + 2^m45^s.3 = 6^h46^m37^s.3 \\ t &= -32°15'45'' \qquad\qquad\quad = -2^h09^m03^s \\ \hline \theta &= \qquad\qquad\qquad\qquad\qquad\quad = 4^h37^m34^s \end{aligned}$$

This is the sidereal time at the given place at the instant of 4 p.m. Greenwich civil time.

The following example illustrates one method of determining by the use of the American Ephemeris the Greenwich sidereal time (G.S.T.) corresponding to a given instant for which the Greenwich civil time (G.C.T.) is known.

Example 2: It is desired to know the Greenwich sidereal time corresponding to $15^h30^m15^s$ G.C.T. August 1, 1927. Mean solar time interval since 0^h G.C.T. = $15^h30^m15^s.0 = 15^h.504$. Gain of sidereal on solar time in $15^h.504 = +15.504 \times 9^s.856 = +2^m32^s.8$. Sidereal time interval since 0^h (G.C.T.) is $15^h30^m15^s.0 + 2^m32^s.8 = 15^h32^m47^s.8$.

From the solar ephemeris the sidereal time of 0^h G.C.T. (which is the right ascension of the mean sun plus 12 hr.) is $20^h34^m25^s.3$.

$$\begin{aligned}\theta &= \text{G.S.T. of } 0^h \text{ G.C.T.} + \text{sidereal interval since } 0^h \text{ G.C.T.}\\ &= 20^h34^m25^s.3 + 15^h32^m47^s.8 - 24^h = 12^h07^m13^s.1\end{aligned}$$

In the Solar Ephemeris for 0^h G.C.T., in the column headed "Sidereal Time" is given for each day the G.S.T. of 0^h G.C.T. If at any given instant the G.C.T. is known, then the interval of sidereal time since 0^h G.C.T. may be computed, and this quantity added to the G.S.T. of 0^h G.C.T. is the G.S.T. at the given instant. In the example the sidereal time is in excess of one day, hence 24^h is deducted.

The preceding examples may be solved in a similar manner by using the Nautical Almanac.

Tables given in both the American Ephemeris and the Nautical Almanac are useful in converting sidereal time into mean solar time, and *vice versa*.

298. Relation between Longitude and Time.—It has been shown that as the earth rotates upon its axis, the hour angle of the sun referred to any meridian on the earth undergoes a change of 360° or 24^h during the interval of one solar day of 24 hr. Since the longitudes of the earth range from 0° to 360°, it follows that the mean solar time interval taken for the mean sun apparently to travel from the meridian of one place to that of another is equal to the difference in longitude between the two places. The same statement applies equally well to the sidereal time interval and the vernal equinox. It is therefore evident that at any instant, the *difference in local time* between two places, whether the time under consideration be sidereal, mean solar, or apparent solar, is equal to the *difference in longitude* between the two places, expressed in hours. This relation makes it possible to determine the time when difference in longitude between two places is known, or to determine longitude when difference in time between two places is given.

Most of the solar ephemerides are for the meridian of Greenwich, and a problem of frequent occurrence is to find the local time corresponding to a given instant Greenwich time. As stated in Art. 289, the longitude of a place is reckoned east or west according to whether the place is east or west of Greenwich. The local time (L.T.) of a place at a given instant is obtained by adding to or subtracting from the Greenwich time (G.T.) the difference in longitude ($\Delta\lambda$) between the two places; if the place is east of Greenwich, the difference in longitude is added, and if the place is west, the difference in longitude is subtracted.

$$\text{L.T.} = \text{G.T.} \pm \Delta\lambda$$

The following examples illustrate the process of changing from the time of one place to that of another.

Example 1: An observation on the sun is taken at $9^h52^m56^s$ local apparent time (L.A.T.). The longitude of the place is $7^h12^m36^s$ west of Greenwich. What is the Greenwich apparent time (G.A.T.)?

$$\begin{aligned}\text{G.A.T.} = \text{L.A.T.} + \Delta\lambda &= 9^h52^m56^s + 7^h12^m36^s \\ &= 17^h05^m32^s\end{aligned}$$

Example 2: On Nov. 20, 1927, the mean sun crosses the lower branch of the Greenwich meridian at $3^h52^m2^s.80$, Greenwich sidereal time. At that instant it is desired to find the local sidereal time at a place whose longitude is $5^h12^m24^s.2$ west of Greenwich.

$$\begin{aligned}\text{L.S.T.} = \text{G.S.T.} - \Delta\lambda &= 3^h52^m2^s.8 - 5^h12^m24^s.2 + 24^h \\ &= 22^h39^m38^s.6 \text{ Nov. 19}\end{aligned}$$

Example 3: At the instant of $18^h48^m15^s$ Greenwich civil time, the local civil time of a place is $10^h37^m42^s$. It is desired to determine the longitude of the place with respect to Greenwich.

$$\Delta\lambda = 18^h48^m15^s - 10^h37^m42^s = 8^h10^m33^s \text{ west of Greenwich.}$$

299. Standard Time.—In order to eliminate the industrial confusion attendant upon the use of local time, the United States has been divided into belts, each of which occupies a width of approximately 15° or 1^h of longitude; in each belt the watches and clocks which control civil affairs all keep the same time. These belts are called standard time belts, and the time kept in each belt is *standard time.* The time in each belt differs from that of its neighbors by one mean solar hour, the standard time in the belt to the east being 1^h faster and that to the west being 1^h slower. The time in any belt is a whole number of hours slower or faster than Greenwich mean time.

In the North American continent there are five time belts, as follows:

Atlantic Standard.—Time kept in the Maritime Provinces of Canada, 4 hr. slower than Greenwich civil time. It is the local mean time of the meridian whose longitude is 60° west of Greenwich (the 60th meridian).

Eastern Standard.—Time kept in the Eastern States from Maine to Central Ohio, 5 hr. slower than Greenwich civil time. It is the local mean time of the 75th meridian.

Central Standard.—Time kept in Central States from Central Ohio to Central Nebraska, 6 hr. slower than Greenwich civil time. It is the local mean time of the 90th meridian.

Mountain Standard.—Time kept in Western States from Central Nebraska to Western Utah, 7 hr. slower than Greenwich civil time. It is the local mean time of the 105th meridian.

Pacific Standard.—Time kept in Pacific States west of Utah, 8 hr. slower than Greenwich civil time. It is the local mean time of the 120th meridian.

The exact boundaries of the time belts are irregular in character and can be determined only from a map.

The Greenwich mean time is found by adding to the standard time the longitude (expressed in hours) of the meridian for which standard time is also local mean time.

Example 1: At a given instant the Central standard time is 9^h00^m a.m. It is desired to find the Greenwich mean time. The longitude of the meridian to which Central standard time is referred is 90° or 6^h west of Greenwich. The Greenwich mean time is $9^h00^m + 6^h00^m = 15^h00^m$, or 3^h00^m p.m.

If the longitude of a place is known, the standard time of the belt in which the place is situated may be determined by algebraically adding to the local mean time the difference in longitude (expressed in hours) between the given place and the meridian for which standard time is also local mean time.

Example 2: By observation on a star, the local mean time at a given instant is found to be $18^h37^m46^s$. The longitude of the place is $\lambda_l = 89°49'30'' = 5^h59^m18^s$. The standard time at the given instant is to be found. The place is evidently in the Central time belt for which the standard time (C.S.T.) is local time for the 90th meridian. The longitude of this meridian expressed in hours is $\lambda_s = \frac{90°}{15} = 6^h$.

$$\begin{aligned} \lambda_l &= 5^h59^m18^s \\ \lambda_s &= 6^h00^m00^s \\ \hline \Delta\lambda &= -0^m42^s \end{aligned}$$

$$\text{C.S.T.} = \text{L.M.T.} + \Delta\lambda = 18^h37^m46^s + (-0^m42^s) - 12^h = 6^h37^m04^s \text{ p.m.}$$

300. Parallax Correction.—The origin of the several systems of coordinates for locating the position of a heavenly body has been taken as the center of the earth, and in the previous discussion it has been assumed that the celestial sphere was of infinite radius. In point of fact, angular measurements to a heavenly body are

made from stations on the surface of the earth, and the distance to any celestial body is not infinite. In Fig. 300, $S_1S_2S_3$ represents the path of any celestial object, O being the center of the earth and T being the station of an observer on the earth's surface. Then, h' represents the true altitude of S_1 above the observer's horizon and h represents the true altitude measured above the celestial horizon. The difference between these two angles which is $\angle OS_1T = C_p$ is called the *parallax correction*. Hence the true altitude above the celestial horizon $h = h' + C_p$.

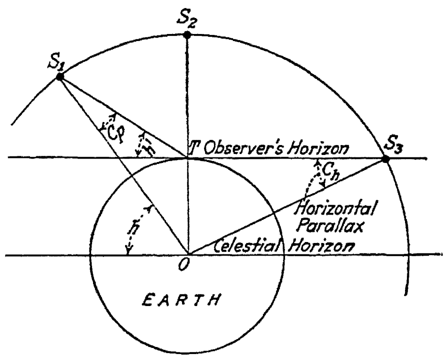


FIG. 300.—Parallax.

From the figure it is evident that the parallax correction can be calculated, if the distance to the heavenly body is known. Also it is obvious that the correction depends upon the altitude, being zero for S_2 when the body is directly overhead, and being a maximum when the body is on the observer's horizon at S_3. When the body is in the latter position, the correction is called the *horizontal parallax*. If the horizontal parallax be designated by C_h, it can be readily demonstrated that the parallax correction for any observed altitude is

$$C_p = +C_h \cos h.$$

For the fixed stars the parallax correction is too small to be of consequence. For the sun the parallax correction is sufficiently large to be of consequence in other than rough observations (see Art. 301*a*). The horizontal parallax of the sun (given for each day of the year in the American Ephemeris) varies somewhat from month to month, being always slightly less than 09″.

301. Refraction Correction.—When a ray of light emanating from any heavenly body enters the atmosphere of the earth, it is bent downward. Hence the sun and stars appear to be higher above the observer's horizon than they actually are. The angle of deviation

of the ray from the direction that it takes on entering the earth's atmosphere, to the direction it takes at the surface of the earth, is called the *refraction* of the ray. Since celestial objects appear too high, their observed altitudes above the observer's horizon are greater than their true altitudes above the observer's horizon. The *refraction correction* C_r is applied to the observed altitude (h'') to determine the true altitude above the observer's horizon (h'). Since the observed altitude is always too great, the refraction correction is always subtracted.

$$h' = h'' - C_r$$

The magnitude of the refraction correction depends upon the temperature and barometric pressure of the atmosphere and upon the altitude of the ray. It does not depend upon the distance to the body from which emanates the ray. Under normal conditions the refraction correction amounts to about 34′ when the sun or star is on the observer's horizon, but for low altitudes it varies greatly with changes in atmospheric conditions. For an altitude of 45°, the correction is about 01′. Table II gives values of refraction corrections for a barometric pressure of 29.5 in., for various temperatures, and for altitudes between 10° and 90°. Between these altitudes it can be shown that, other things remaining constant, the refraction correction varies as the cotangent of the altitude. Hence $C_r = 0$ when $h = 90°$.

Owing to the uncertainties of the refraction correction for small altitudes, observations for accurate determinations are never taken upon a body which is near the horizon.

301a. Refraction and parallax corrections for the sun are usually considered together. Table I gives corrections for the combined effect of refraction and parallax, to be subtracted from observed altitudes of the sun to determine the true altitudes above the celestial horizon.

302. Problems.

1. When the local apparent time is $8^h17^m12^s$ at a place whose longitude is 96°15′10″W, what is the Greenwich apparent time?

2. On a given date 0^h Greenwich civil time occurs at $4^h17^m32^s$ Greenwich sidereal time. At that instant what is the local sidereal time at a place whose longitude is $7^h17^m43^s$?

3. When it is $15^h31^m12^s$ Greenwich civil time, it is $10^h16^m37^s$ local civil time. What is the longitude of the place?

4. What is the Greenwich civil time when it is 3^h15^m p.m. Central standard time?

5. If the local civil time at a place is $16^h23^m22^s$ and the longitude of the place is 78°36′20″, what is the Eastern standard time?

6. From an ephemeris find the equation of time for the instant of $4^h15^m0^s$ Pacific standard time on April 21 of the current year. If the longitude of the place is $7^h46^m3^s$W, calculate the local mean and local apparent times.

7. From an ephemeris, find the equation of time for the instant of $3^h19^m30^s$ p.m. apparent time July 4 of the current year, at a place whose longitude is $6^h15^m30^s$. Calculate the corresponding local mean time.

8. At a given place the hour angle of the true sun at 11^h30^m p.m. Greenwich civil time on January 12 of the current year is 42°36′30″. What is the local sidereal time?

9. The mean radius of the earth is 3,956 miles and the mean distance to the sun is 92,900,000 miles. What is the sun's mean horizontal parallax? What is the parallax correction when the altitude is 30°?

10. The observed altitude of a star is 23°15′20″. The temperature is 90°F. By Table II, find the refraction correction, and compute the true altitude of the star.

11. The observed altitude of the sun's center is 15°07′30″. The temperature is 15°F. By Table I find the parallax and refraction correction and calculate the true altitude of the sun.

CHAPTER XVIII

DETERMINATION OF AZIMUTH, LATITUDE, LONGITUDE, AND TIME

303. General.—The field methods of determining azimuth, latitude, longitude, and time described in the succeeding articles are the rough methods that are in common use on surveys of ordinary precision within latitudes of the United States, where the engineer's transit or the repeating theodolite is the instrument with which angular measurements are made. The two heavenly bodies upon which the surveyor takes most observations are the sun and Polaris, the pole star. For this reason the following discussion is concerned chiefly with these two bodies, but the principles involved are the same for any star, as is clear from the preceding chapter.

Measurements to the sun cannot be made with the same degree of precision as can measurements to the stars, hence the probable error in computed values is larger than when a fixed star is chosen. The sun may be viewed at times convenient to the surveyor, however, and solar observations are suitable for determinations of azimuth, latitude, and longitude which, while in general of relatively low precision, are sufficiently accurate for the majority of surveys.

Polaris, being near the pole, changes its position slowly. It is the bright star among all others which is most favorably located for precise determinations of latitude and azimuth, but owing to its slow change in azimuth it is not suitable for longitude or time observations.

For more precise methods, such as those necessary on precise geodetic surveys, the reader is referred to texts on geodesy and engineering astronomy (see references, p. 497).

304. Angular Measurements.—The effect of instrumental errors upon the precision of horizontal angles was discussed in Art. 211, p. 284. Whenever observations are made to determine azimuth, a part of the field work consists in measuring the horizontal angle between the celestial body and a reference mark on the earth's surface. Since the sights to the celestial body are in general steeply inclined, it is highly important that the horizontal axis be in adjustment with respect to the vertical axis and that the transit be carefully leveled. Attention is again called to the fact that, even though the horizontal axis be in perfect adjustment, it will be inclined unless the vertical axis

is truly vertical, and the error due to such inclination will in general not be eliminated by a reversal of the telescope between sights. For accurate observations the transit should be equipped with a sensitive striding level by means of which the horizontal axis may be leveled prior to each sight. With the ordinary transit not so equipped, the plate may be leveled with greater refinement by means of the telescope level than it can be by means of the plate levels, and this method of leveling the plate should be employed when other than rough observations are being made. Also sights should be taken with the telescope in both the direct and reversed positions in order that the mean of horizontal angles may be free from other instrumental errors.

When altitudes are observed, the index error of the vertical circle should be determined just after each observation (see Art. 198, p. 266). It is to be noted that the error due to the line of sight not being parallel with the axis of the level tube will be eliminated by taking the mean of altitudes with the telescope in the direct and in the reversed positions, but any error due to the inclination of the vertical axis will not be eliminated by this procedure. When the transit has a full vertical circle, the index error may be most accurately found by taking vertical-circle readings to some well-defined point in the direction of the star or sun with the telescope in both the direct and reversed positions. The difference between the two readings is double the index error. This method of determining the index error does not take into account the error due to the vertical axis not being truly vertical, but does determine the errors due to the vernier being displaced and to the line of sight not being parallel to the axis of the level tube.

305. Prismatic Eyepiece.—When altitudes are high, it is impossible to view the image by looking directly into the eyepiece of the transit telescope. The *prismatic eyepiece* is a device which, when attached to the telescopic eyepiece, reflects the rays through an angle of 90° with the axis of the telescope. By means of one type the sun and stars may be observed when at altitudes of more than 80°. The reflecting medium is a prism, hence the image appears upside down from that seen through the telescopic eyepiece but it is not reversed horizontally.

306. Observations on the Sun.—The sun has an angular diameter of about 32′. Since the sun is so large, observations with the ordinary transit cannot be made with any degree of precision by sighting the intersection of the cross-hairs at the estimated position of the sun's center, and it is the practice to bring the cross-hairs tangent to the sun's image. When the horizontal cross-hair is brought tangent

to the lower edge of the sun, the sight is said to be taken to the sun's *lower limb* and this is indicated in the notes by the symbol $\underline{\odot}$. Similarly the symbol $\overline{\odot}$ indicates a sight to the sun's *upper limb*, $\odot\!|$ a sight with the vertical cross-hair to the sun's *right limb*, and $|\!\odot$ a sight to the sun's *left limb.*

When a single observation is taken it is necessary to correct an observed vertical angle to the sun's limb by the amount of the sun's semidiameter. The American Ephemeris and other solar ephemerides give values of the semidiameter of the sun for each day of the year. The semidiameter varies from about 15′46″ in July to about 16′18″ in January. For rough calculations the semidiameter may be taken as 16′ at all seasons of the year. The semidiameter is added to or subtracted from an observed altitude according as the lower limb or upper limb of the sun is observed.

To reduce to the sun's center an observed horizontal angle taken to one of the sun's limbs, a correction equal to the semidiameter times the secant of the altitude is applied. Thus if the altitude is 60° and the semidiameter is 16′, the correction to a horizontal angle is $16' \sec h = 32'$. As the sun approaches the zenith, the correction becomes very large (approaching 90°), hence the surveyor should not depend upon single readings when the sun is at a high altitude.

For most solar observations, an equal number of sights is taken to opposite limbs of the sun, in which case the average of observed horizontal and vertical angles gives the horizontal and vertical angles to the sun's center at the mean of the times, and no correction for semidiameter is necessary.

The sun is too bright to be viewed directly through the telescopic eyepiece; in fact to observe the sun in this manner may result in serious injury to the eye. Some transits are equipped with a piece of colored glass, called the *sun glass,* which, when screwed to the eyepiece end of the telescope cuts off many of the light rays and makes it possible to sight at the sun with comfort. All prismatic eyepieces are equipped with such a glass.

In case a prismatic eyepiece is not available and the sun is not at too great an altitude, good observations can be made by bringing the sun's image to a focus on a white card held several inches in the rear of the telescopic eyepiece.

A rough pointing on the sun is made by sighting over the telescope. The eyepiece is then drawn back and the objective is focused until the sun's image and the cross-hairs are clearly seen on the card. If the eyepiece of the telescope is erecting, the image on the card will be inverted; if the eyepiece is inverting, the image will appear erect. The cross-hairs are visible only on the image of the sun. Since the

angle between lines of sight defined by the stadia hair is 34′, while the diameter of the sun is but 32′, all three horizontal hairs are not at the same time visible on the sun's image. A common blunder is to mistake one of the stadia hairs for the middle cross-hair. This mistake may be avoided by rotating the telescope slightly about the horizontal axis until all three hairs have been seen.

The so-called "solar screen" is a device utilizing the principle of the card. It consists of a piece of ground white glass fixed to a metal arm which is screwed or clamped to the eyepiece end of the telescope. The sun and cross-hairs are brought to a focus on the ground glass as described in the preceding paragraph.

307. Declination of the Sun.—In connection with the determinations of azimuth, latitude, or longitude by solar observations, it is necessary to determine the declination of the sun at a given instant, and this is done by interpolating between values given in a solar ephemeris for the current year. There are two types of ephemeris in common use, each of which is designed to be employed under certain conditions, but either of which it is possible to use under any circumstances.

The first of these types gives the apparent declination for each day of the year at the instant of 0^h *Greenwich civil time* and is especially adapted for use when the standard time or the Greenwich civil time is known. The solar ephemeris for the Greenwich meridian as given in the American Ephemeris and Nautical Almanac is an example of an ephemeris of this kind. The American Nautical Almanac gives declinations for the even hours of Greenwich civil time.

The second type gives the apparent declination for each day of the year at the instant of *Greenwich apparent noon* and is especially adapted for use when the longitude of the place and the local apparent time of the observation are known. The Ephemeris of the Sun and Polaris published by the General Land Office contains an ephemeris of this sort. The American Ephemeris also gives the apparent declination for each day of the year at the instant of Washington apparent noon.

Abbreviated solar ephemerides, either for civil time or for apparent time, are published in the form of pamphlets by various manufacturers of surveying instruments and are furnished to surveyors free of charge.

The following examples illustrate the use of each of these types of ephemeris.

Example 1: An observation is taken on the sun at 10^h00^m a.m. Eastern standard time, on Dec. 15, 1927. It is desired to determine the declina-

tion at the given instant. By ephemeris for 0^h Greenwich civil time the declination for 0^h on Dec. 15 is $\delta_0 = -23°12'28''$. The Greenwich civil time is $10.00 + 5.00 = 15^h.00$. The average hourly variation is $-8''.8$. The change in declination since 0^h G.C.T. is $-8.8 \times 15 = -02'12''$. Declination at instant of observation $= -23°14'40''$.

Example 2: An observation is taken on the sun as it crosses the meridian on Nov. 16, 1926, at a place whose longitude is 87°49'30'' west of Greenwich. It is desired to determine the apparent declination at the given instant. G.A.T. $= \lambda = \dfrac{87°49'30''}{15} = 5^h51^m18^s$ (after apparent noon) $= 5^h.86$. From the Ephemeris of the Sun (General Land Office), the declination at Greenwich apparent noon is S18°37'21''. The average difference for $1^h = -37''.8$. The change in declination since Greenwich apparent noon is $-37''.8 \times 5.86 = 03'42''$. Declination at local apparent noon at place $=$ S18°41'03''.

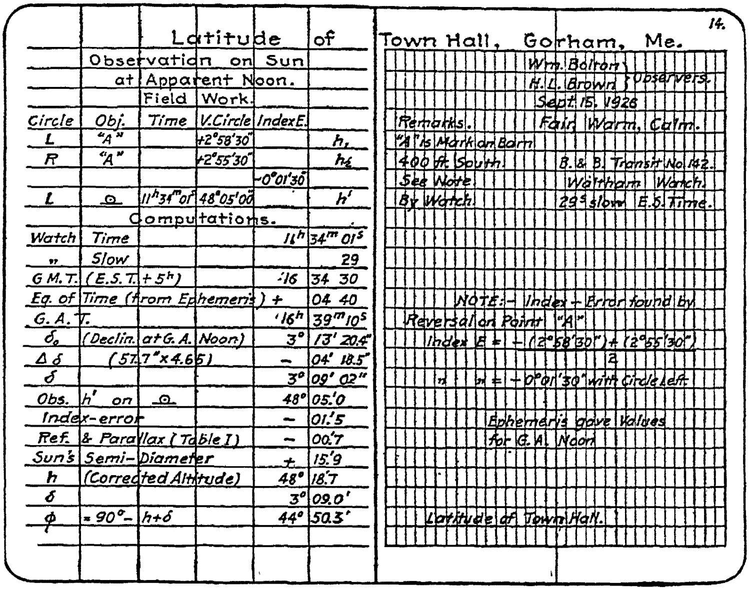
Example 3: It is desired to determine the apparent declination of the sun at the instant of 1^h00^m p.m. Eastern standard time, on Nov. 18, 1926, from a solar ephemeris giving values for Greenwich apparent noon. The difference between Eastern standard time and Greenwich mean time is 5^h; hence it is 6^h00^m after Greenwich mean noon. At Greenwich apparent noon the equation of time as given in the ephemeris is $-14^m52^s.4$. Since this is to be subtracted from Greenwich apparent time to give Greenwich mean time, it follows that apparent time is faster than mean time, and the Greenwich apparent time is roughly $6^h00^m + 15^m = 6^h.25$. The daily rate of change in the equation of time is given by the difference between the equation of time for Nov. 18 and that for Nov. 19 or $14^m52^s.4 - 14^m39^s.8 \doteq 12^s.6$.

The change in the equation of time since Greenwich apparent noon is $\dfrac{6.25 \times 12.6}{24} = 3^s.3$. The equation of time is decreasing and hence the equation of time for the given instant is $14^m52^s.4 - 3^s.3 = 14^m49^s.1$. The interval since Greenwich apparent noon is $6^h00^m + 14^m49^s.1 = 6^h14^m49^s.1 = 6^h.247$. At Greenwich apparent noon the apparent declination is S19°06'59''; the average difference for 1^h is $36''.1$. The change in apparent declination since Greenwich apparent noon is $36''.1 \times 6.25 = 3'46''$. South declinations are increasing, hence the apparent declination at the given instant is $\delta = 19°06'59'' + 3'46'' =$ S19°10'45''.

In example 3 the equation of time has been determined for the given instant. For all practical purposes the equation of time for Greenwich apparent noon might have been employed, since the small error of $3^s.3$ in time would have no effect upon the computed change in the declination unless declinations were carried out to tenths of seconds.

If the equation of time were neglected entirely, the error introduced in the computed value of the apparent declination would be but 09'', not sufficiently large to be of consequence in rough calculations.

308. Latitude by Observation on Sun at Noon.—The latitude of a given station may be determined with a fair degree of precision by observing with the engineer's transit the altitude of the sun at local apparent noon. If the longitude of the place is roughly known, it is unnecessary to observe the time, but if the longitude is unknown, the standard time of the observation must be taken. The problem consists in determining the true altitude h of the center of the sun above the celestial horizon and computing the apparent declination



Latitude of Town Hall, Gorham, Me. 14.

Observation on Sun at Apparent Noon.

Field Work.

Circle	Obj.	Time	V.Circle	Index E.	
L	"A"		+2°58'30"		h_1
R	"A"		+2°55'30"		h_2
				−0°01'30"	
L	⊙	11h34m01s	48°05'00"		h'

Computations.

Watch	Time			11h	34m 01s
"	Slow				29
G.M.T.	(E.S.T. + 5h)			16	34 30
Eq. of	Time (from Ephemeris)			+	04 40
G.A.T.				16h	39m 10s
δ_0	(Declin. at G.A. Noon)			3°	13' 20.4"
$\Delta\delta$	(57.7" × 4.65)			−	04' 18.5"
δ				3°	09' 02"
Obs.	h' on ⊙			48°	05'.0
Index-error				−	01'.5
Ref.	& Parallax (Table I)			−	00'.7
Sun's	Semi-Diameter			+	15'.9
h	(Corrected Altitude)			48°	18'.7
δ				3°	09.0'
ϕ	= 90° − h + δ			44°	50.3'

Wm. Bolton }
H. L. Brown } Observers.
Sept. 15, 1926
Remarks. Fair, Warm, Calm.
"A" is Mark on Barn
400 ft. South. B. & B. Transit No. 142.
See Note. Waltham Watch.
By Watch. 29s slow E.S. Time.

NOTE:—Index-Error found by Reversal on Point "A".

Index E = $-\dfrac{(2°58'30") + (2°55'30")}{2}$

" " = −0°01'30" with Circle Left.

Ephemeris gave Values for G.A. Noon

Latitude of Town Hall.

FIG. 308.—Latitude by observation on sun at noon.

δ of the sun at the instant of sighting. Then as explained in Art. 294, the latitude is

$$\phi = 90° - h + \delta$$

The accuracy obtainable ordinarily depends upon the precision of the instrument. Since the maximum rate of change of declination is about 01′ per hour, a considerable error in time will affect the declination but slightly. With the ordinary transit having a vertical circle reading to single minutes the latitude may be determined in this manner with an error not greater than 01′. The mean of a series of observations on different days will, of course, render a much closer result.

When the time of observation is observed, an ephemeris giving values of apparent declination for 0^h Greenwich civil time is prefer-

able. When the direction of the meridian is unknown, which is the usual case, the procedure is as follows: The transit is set up and carefully leveled; the horizontal cross-hair is sighted either on the lower or upper limb of the sun and is kept continuously in this position until a maximum altitude has been reached and the sun begins its apparent descent. At that instant the watch time is observed. The altitude and time are recorded. With the telescope still in the meridian plane its index error is determined, preferably by the method of double-sighting on an index mark as described in Art. 304. The watch is compared with a clock keeping correct standard time and the error is noted. The Greenwich civil time of the observation is calculated, and the declination is found in the solar ephemeris as illustrated by example 1, Art. 307. The true altitude of the sun's center is determined by applying to the observed altitude, corrections for semidiameter (see ephemeris) and refraction and parallax (see Table I). The latitude is then determined by the equation given above. It should be noted that the sign of the refraction and parallax correction as given in Table I is always negative. The sign of the declination is negative from September to March, and is positive for the remainder of the year.

When desired, a second sight may be taken on the opposite limb of the sun with the telescope reversed. The mean of the two vertical angles is taken as the altitude of the sun's center at the mean of the two times of observation, no correction for index error being necessary. If time between sights does not exceed 3 or 4 min., the mean altitude may be considered as the altitude at apparent noon. The latitude is then calculated as described in the preceding paragraph.

308a. When an ephemeris giving values at Greenwich apparent noon is to be used, the longitude of the place being unknown and the standard time being known, the Greenwich apparent time of the observation is determined and the declination at the given instant is found, as illustrated by example 3, Art. 307. Figure 308 is an example illustrative of the method here described. The column headed "Circle" indicates the position of the vertical circle, as right or left, and therefore shows whether the telescope is in the direct or reversed position.

308b. When the longitude of the place is known, it is assumed that the instant of observation is local apparent noon. Hence the Greenwich apparent time is taken as the longitude of the place. The procedure in the field is identical with that described in Art. 308, except that time is not observed. The declination is most conveniently found from an ephemeris giving values for Greenwich apparent noon, as illustrated by example 2, Art. 307.

309. Azimuth by Direct Solar Observation.—The azimuth of a line may be determined by a single observation of the sun at any time when it is visible, provided the latitude of the place is known. (See Art. 312 for simultaneous determination of azimuth and longitude.)

At a known instant of time the sun is observed, and the altitude of the sun and the horizontal angle from the sun to the given line are measured. The declination of the sun at the given instant is found from a solar ephemeris. With the declination δ, latitude ϕ, and altitude h known, the *PZS* triangle may be solved as described in Art. 296, the azimuth of the sun from north being given by any of several expressions, one of which is

$$\cos Z = \frac{\sin \delta}{\cos h \cos \phi} - \tan h \tan \phi$$

Knowing the azimuth of the sun and the horizontal angle from its position to the line, the azimuth of the line is readily computed.

The accuracy with which azimuths may be determined depends upon the precision of field observations, but also the accuracy is affected by the shape of the astronomical triangle and by the exactitude with which corrections to the observed altitude can be determined. In the light of previous discussion, it is evident from inspection that the *PZS* triangle becomes weak as the sun approaches the meridian and that the solution becomes indeterminate at the instant of apparent noon. On the other hand, although the strength of the *PZS* triangle increases as the hour angle measured in either direction from the meridian approaches $90° = 6^h$, the refraction correction becomes large and very uncertain for low altitudes, particularly for those less than 10°. For these reasons, when possible, observations within the latitudes of the United States are usually taken between the hours of 8 to 10 a.m. and 2 to 4 p.m. In no case should reliable results be expected for altitudes of less than 10°.

The effect of errors in the sides of the astronomical triangle upon the precision of the computed azimuth of the sun at the instant of observation is given by the following table in which there are shown the changes in azimuth of the sun due to changes of 01′ in the latitude, declination, and altitude at the latitude of 40°, which is about the mean latitude of the United States. The changes in azimuth have been computed by means of the equation given in the second paragraph of this article. The values are approximate and are given for comparative purposes only. The months named are those during which the declination is not greatly different from the value given in the second column. Thus during the period of November, December, and January, the declination varies from −15° to −23°. The hour angles give the approximate time interval before or after local apparent noon. It will

be noted that when the hour angle is 1^h30^m, a 01′ error in latitude, declination, or altitude produces an error of about 03′ in the azimuth. When the hour angle has increased to 3^h, an error of 01′ in latitude or altitude produces an error of about 01′ in the azimuth. It will also be noted that under the given conditions, the effect of an error in declination is, in general, greater than is the effect of an error of the same magnitude in either latitude or altitude.

Approximate Change in Calculated Azimuth of Sun for 01′ Change in Latitude, Declination, and Altitude, for Latitude 40°

Months	Declination	Hour angle	Altitude	Azimuth	Change in azimuth for 01′ change in: Latitude	Declination	Altitude
November, December, January	−20°	3^h15^m	15°	46½°	1′10″	1′45″	1′25″
March, September	0°	4^h40^m	15°	77°	35″	1′25″	55″
	0°	3^h20^m	30°	61°	1′10″	1′45″	1′20″
	0°	1^h30^m	45°	33°	3′00″	3′20″	3′00″
May, June, July	+20°	5^h45^m	15°	104°	04″	1′20″	45″
	+20°	4^h30^m	30°	88°	35″	1′25″	50″
	+20°	3^h10^m	45°	77½°	1′10″	1′45″	1′00″
	+20°	1^h45^m	60°	56°	2′40″	2′55″	2′10″

Azimuths calculated by Eq.: $\cos Z = \dfrac{\sin \delta}{\cos h \cos \phi} - \tan h \tan \phi$

When the azimuth of a given line is to be determined by this method, the transit is set up and carefully leveled over one end of the line. The *A* vernier is set at zero on the horizontal circle, and the reading of the *B* vernier is noted. A sight is taken along the given line with the telescope, say, in the direct position, and the lower motion is clamped. The upper motion is loosened and a sight is taken at the sun, the vertical and horizontal cross-hairs being brought tangent with the sun in the upper left-hand quadrant of the field of view if the observation is in the morning, or in the upper right-hand quadrant if the observation is in the afternoon. At the instant of tangency, the time is observed. As quickly as consistent with accuracy, vertical and horizontal circle readings are taken on all verniers. The upper motion is then loosened, the telescope is plunged, and a second pointing is made, this time with the sun in the

diagonally opposite quadrant from that of the first observation; that is, the vertical and horizontal cross-hairs are brought tangent to the sun in the lower right-hand quadrant if in the forenoon, and in the lower left-hand quadrant if in the afternoon. The time of tangency is observed and the vertical and horizontal angles are measured as before. The upper motion is loosened, and the field work is completed by again sighting along the line and reading the horizontal circle, this time with the telescope still in the reversed position. The watch is compared with a timepiece keeping correct standard time, and the correction is noted. The mean of the two observed altitudes is taken to be the apparent altitude of the sun's center at the mean of the two observed times. Similarly the mean of the two horizontal angles is taken as the horizontal angle to the sun's center at the mean of the observed times.

309a. The process of bringing the cross-hairs tangent to the sun is illustrated by Fig. 309*a*, which shows the position of the sun in the

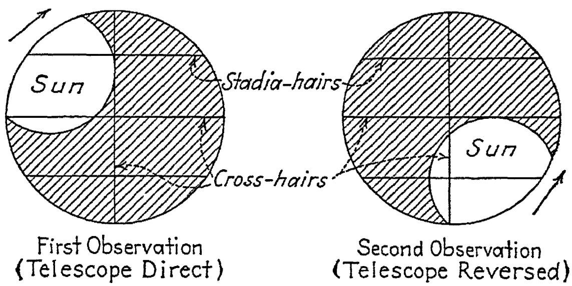


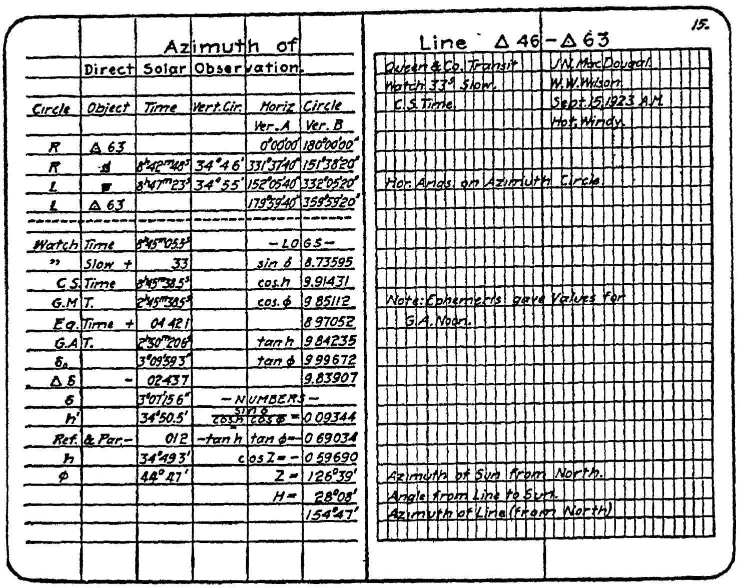
Fig. 309*a*.—Position of sun in field of view just prior to tangency in morning.

field of view of the telescope just prior to each observation taken in the morning. The portion of the field of view covered by the sun in the position shown depends upon the angle of the field of view. For telescopes of low magnifying power, the full disk may be visible. If the position of the sun is determined by a card held in the rear of the eyepiece, the image on the card will include that portion of the sun's disk shown in the figure, and the cross-hairs on the shaded portion of the field of view will not be visible.

During the process of taking the first observation, the horizontal cross-hair is sighted a short distance above the sun's lower limb as illustrated. Since the altitude of the sun is increasing, the horizontal cross-hair approaches tangency due to the sun's apparent movement. At the same time, the vertical cross-hair is kept continuously on the sun's western limb by means of the upper-motion tangent-screw. The instant when the vertical and horizontal cross-hairs are simultaneously tangent to the sun's disk is the instant of observation, and

the horizontal and vertical circle readings are taken with the direction of the line of sight unchanged from the position at that instant. For the second observation, in the lower right-hand quadrant, the vertical cross-hair is set a short distance to the right of the sun's eastern limb. Since the sun is traveling westward, the vertical cross-hair approaches tangency due to the sun's apparent movement. At the same time the horizontal cross-hair is kept continuously on the sun's upper limb. As before, observations are taken for the instant when both horizontal and vertical cross-hairs are simultaneously tangent to the sun's disk. It will be noted that the procedure is such that the final setting for either observation requires the manipulation of only one tangent-screw, the cross-hairs always being visible upon the sun's disk.

A similar demonstration may be made to show the advantage of sighting at the sun first in the upper right-hand quadrant and then in the lower left-hand quadrant for afternoon observations in northern latitudes.


Azimuth of Direct Solar Observation.

Circle	Object	Time	Vert. Cir.	Horiz Circle Ver. A	Ver. B
R	△ 63			0°00′00″	180°00′00″
R	☉	8h42m48s	34°46′	331°37′40″	151°38′20″
L	☉	8h47m23s	34°55′	152°05′40″	332°05′20″
L	△ 63			179°59′40″	359°59′20″
Watch	Time	8h45m05.5s		— LO	GS —
"	Slow +	33		sin δ	8.73595
C.S.	Time	8h45m38.5s		cos h	9.91431
G.M.	T.	2h45m38.5s		cos φ	9.85112
Eq.	Time +	04.421			8.97052
G.A.	T.	2h50m20.6s		tan h	9.84235
δ_0		3°09′59.3″		tan φ	9.99672
Δδ	−	02.437			9.83907
δ		3°07′15.6″	— N	UMBERS	—
h′		34°50.5′	$\frac{\sin\delta}{\cos h}$	$\cos\phi$ =	0.09344
Ref.	& Par. −	01.2	− tan h	tan φ = −	0.69034
h		34°49.3′	c	os Z = −	0.59690
φ		44°47′		Z =	126°39′
				H =	28°08′
					154°47′

15.

Line △ 46 − △ 63

Queen & Co. Transit | J.N. MacDougal
Watch 33s Slow. | W.W. Wilson
C.S. Time | Sept. 15, 1923 A.M.
Hot, Windy.

Hor. Angles on Azimuth Circle.

Note: Ephemeris gave Values for G.A. Noon.

Azimuth of Sun from North.
Angle from Line to Sun.
Azimuth of Line (from North)

FIG. 309b.—Azimuth by observation on sun.

309b. To the mean of the observed altitudes are applied corrections for parallax and refraction (Table I). The sun's apparent declination is found from a solar ephemeris as described in Art. 307. Attention is again directed to the two types of solar ephemeris, one giving declinations for 0^h Greenwich civil time, the other giving declinations for

Greenwich apparent noon. If the former type (such as the American Ephemeris) is employed, the mean time interval since 0^h G.C.T. must be determined (see example 1, Art. 307). If an ephemeris of the latter type (such as the Ephemeris of the Sun, General Land Office) is used, the apparent time interval since Greenwich apparent noon must be found, and this involves computing the equation of time (see example 3, Art. 307). The azimuth of the sun at the given instant can be computed from Eq. (13), p. 435, either by using a calculating machine or by logarithms. When observations are of ordinary precision, as those taken with a transit reading to minutes, five places are sufficient for calculations. In solving the above equation, it should be noted that when declinations are south, the sign of sin δ is negative. The azimuth of the line is calculated by algebraically subtracting the mean of the observed horizontal angles from the azimuth of the sun, angles taken in a clockwise direction from line to sun being considered positive.

309c. Figure 309*b* illustrates a suitable form for notes, the observations being taken with a transit having a vertical circle reading to 01′ and a horizontal circle reading to 20″. It is seen that horizontal angles are read from the azimuth circle. An ephemeris giving declinations for Greenwich apparent noon was used, hence the central standard time was transformed into Greenwich apparent time in order that the change in declination since Greenwich apparent noon could be determined.

309d. Great care should be taken in leveling the instrument, for reasons explained in Art. 304. Unless it is equipped with a striding level, the final test prior to observing the sun should be to see that the telescope bubble remains centered as the instrument is revolved about the vertical axis. While reversal of the telescope between observations will eliminate from the mean of the observed altitudes errors due to lack of parallelism between line of sight and axis of telescope level tube, and will eliminate from the mean of the horizontal angles the error due to the horizontal axis not being perpendicular to the vertical axis, it will not eliminate the effect of inclination of the vertical axis.

If the transit is not equipped with a full vertical circle, it can not, of course, be reversed between sights, but otherwise the procedure may be as just described. If the transit has not a full vertical circle, care should be taken to see that it is in good adjustment.

If for any reason it is desired to obtain the azimuth of a line by a single pointing, the sun may be brought tangent in any of the quadrants. The altitude and azimuth corrections for the sun's center will then be made as described in Art. 306.

When it is assumed that at the mean of the two times the sun is at a position given by the mean of the vertical and the mean of the horizontal angles, it is equivalent to saying that the sun is apparently traveling in a straight line, which of course is not true. Within a period of 10 min., however, the error introduced is so small as to be of no consequence.

310. Time by Observation on Sun at Noon.—If the longitude of a station is known and the direction of the meridian has been established, the standard time may be determined quite accurately by observing the sun as it crosses the meridian at local apparent noon. In determining time in this manner, the transit is set up and carefully leveled over the north end of the meridian line, and a sight is taken along the meridian. The line of sight is elevated to intercept the path of the sun, and at the instant of tangency between the west limb and the vertical cross-hair, the time is noted. The telescope is quickly plunged and reversed in azimuth, and a second sight is taken

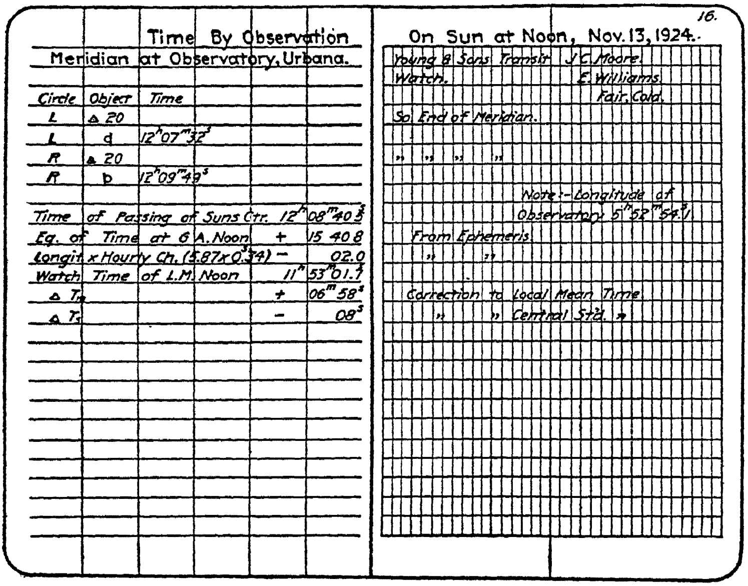


Fig. 310.—Time by observation on sun at noon.

along the meridian. The telescope is again rotated about the horizontal axis until the line of sight intercepts the path of the sun, and the time of tangency between the vertical cross-hair and the east limb of the sun is observed. The mean of the two times thus observed is the watch time of upper transit of the sun's center, which is local apparent noon.

The longitude of the place expressed in hours, for reasons explained in Art. 298, is the Greenwich apparent time reckoned from noon. From the Solar Ephemeris of the General Land Office, or other ephemeris giving values for Greenwich apparent noon, the equation of time at the instant of observation is determined as illustrated

by example 3, Art. 297*d*. Since the equation of time is the difference between mean and apparent time at the instant of observation, it is clear that 12^h plus or minus the equation of time is the local mean time of local apparent noon, the sign being plus or minus according as mean time is faster or slower than apparent time, as indicated by the ephemeris. Then if T be the observed time, E the equation of time, and ΔT_m be the correction of the timepiece to local mean time,

$$\Delta T_m = 12^h \pm E - T$$

Since the difference between standard time and local mean time at a given place is the difference in longitude in hours between the place of observation and the meridian at which standard time and local mean time agree, it follows that the correction to give standard time is

$$\Delta T_s = \Delta T_m + \Delta\lambda$$

in which $\Delta\lambda$ is the difference in longitude between the given meridian and the standard meridian. If the place of observation is east of the meridian, the sign of $\Delta\lambda$ is negative; and if west, positive. Figure 310 is an example where the correction for Central standard time is found.

310a. If the ephemeris used gives values for 0^h Greenwich civil time, an exact determination of the equation of time necessitates changing the Greenwich apparent time of observation (reckoned from noon) to Greenwich civil time (reckoned from 0^h) by means of an approximate equation of time. The true equation of time is then found by interpolation between values given in the ephemeris. Since the equation of time is always less than 17^m and since its rate of change is small, for rough determinations of time the error will be negligible if for the purpose of calculating the equation of time, mean and apparent time are assumed to agree, as the following example illustrates.

Example: It is desired to determine the equation of time at 6^h G.A.T. (reckoned from noon) on Oct. 6, 1926.

By the Ephemeris of the Sun (General Land Office), the equation of time at Greenwich apparent noon is $11^m40^s.18$ and the daily change is $+17^s.47$. The increase since noon is $+17.47 \times 6/24 = +4^s.37$, and the exact equation of time at the given instant is $11^m40^s.18 + 4^s.37 = 11^m44^s.55$.

By the American Ephemeris the equation of time at 0^h G.C.T. is $11^m31^s.44$; neglecting the difference between mean and apparent time, the approximate time interval since 0^h is $12^h + 6^h = 18^h$. The daily change is $+17^s.66$, and the increase since 0^h is $+17.66 \times 18/24 = +13^s.25$. The approximate equation of time at the given instant is therefore

$11^m44^s.69$, differing from the exact value of the preceding paragraph by only $0^s.14$.

From the ephemeris it is seen that the mean time is slower than apparent time, hence the Greenwich civil time is slower than that assumed above by approximately $11^m44^s.69$ (the approximate equation of time), or is approximately $18^h - 11^m44^s.69 = 17^h48^m15^s.31 = 17^h.804$. The increase in the equation of time since 0^h G.C.T. is more exactly $17.66 \times \frac{17.80}{24} = +13^s.11$. The exact equation of time is then $11^m31^s.44 + 13^s.11 = 11^m44^s.55$, which is the same as that found by use of the ephemeris giving values for Greenwich apparent noon.

310b. The length of time taken by the sun in crossing the meridian depends somewhat upon the sun's declination and semidiameter, but it is approximately 2⅓ min. It is therefore clear that no time can be lost in reversing the transit for a second sight along the meridian preparatory to observing the east limb of the sun. If for any reason it is impracticable to observe but one limb, the time in seconds earlier or later than the sun's center passes the meridian is approximately $\frac{4S}{\cos \delta}$ in which S is the sun's semidiameter in minutes of arc (approximately 16′) and δ is the sun's declination. In the American Ephemeris for Washington apparent noon will be found for each day the sidereal time of passing of the semidiameter of the sun, from which the mean solar time interval may be computed from the known relation between sidereal and mean solar time (see Art. 297*e*).

311. Longitude by Observation on Sun at Noon.—If the standard time is accurately known and the meridian has been established, the longitude of a place may be determined by an observation upon the sun at local apparent noon, the field procedure being in all respects identical with that just described for finding time.

With the standard time of passage of the center of the true sun (local apparent noon) known, the Greenwich civil time of local apparent noon can be computed, and the equation of time at the instant of local apparent noon may readily be found from an ephemeris giving values for 0^h Greenwich civil time. Then if E is the equation of time, T_m is the standard time of local mean noon, and T_a is the standard time of local apparent noon, the difference in longitude (expressed in time) between the given place and the meridian at which mean time and standard time agree is

$$\Delta\lambda = (T_a \pm E) - 12^h = T_m - 12^h$$

in which the sign of the equation of time is positive or negative according as apparent time is faster or slower than mean time, indicated by the ephemeris. If the result is negative, the meridian of the

place is east of the standard meridian; if the result is positive, the meridian of the place is west of the standard meridian. Following is an example illustrating the method.

Example: The sun's center is observed to pass the meridian at a given place at $11^h30^m12^s.2$ a.m. Pacific standard time, Dec. 2, 1927. The longitude of the place is desired.

The difference between Pacific standard time and Greenwich civil time is 8^h, hence the G.C.T. is $19^h30^m12^s.2 = 19^h.503$. From ephemeris giving values for 0^h G.C.T. the equation of time at 0^h is $10^m59^s.63$. The difference for one day is $-22^s.64$. The change since 0^h is $-22.64 \times \frac{19.503}{24} = -18^s.30$. The equation of time at the instant of local apparent noon is $10^m59^s.63 - 18^s.30 = 10^m41^s.33$. The ephemeris indicates that apparent time is faster than mean time, hence local mean noon occurs at a standard time later than that for local apparent noon by an amount equal to the equation of time, and the Pacific standard time of local mean noon is $T_m = T_a + E = 11^h30^m12^s.2 + 10^m41^s.3 = 11^h40^m53^s.5$. Then $\Delta\lambda = 11^h40^m53^s.5 - 12^h = -19^m6^s.5 = -4°46'38''$. Pacific standard time is local mean time for the 120° meridian, hence the longitude of the place is $120° - 4°46'38'' = 115°13'22''$.

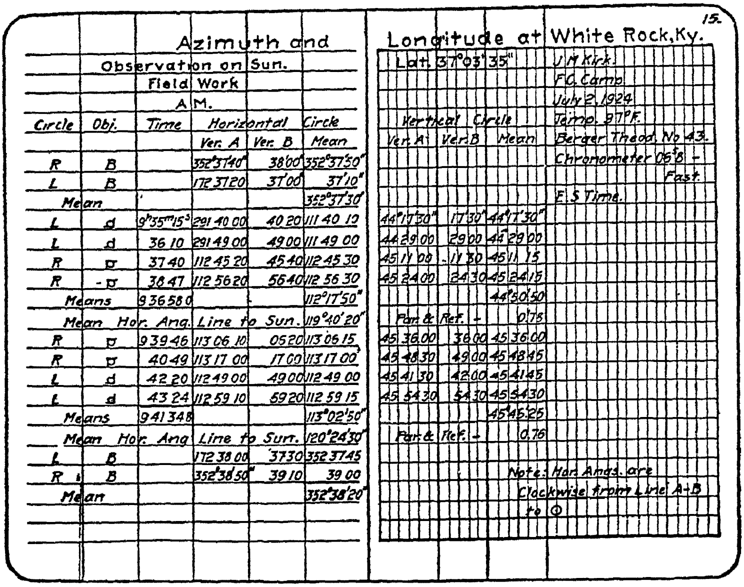
312. Azimuth and Longitude by Solar Observation.—This method is a modification of that described in Art. 309, the field procedure being essentially the same except that usually two series of observations are taken, one in the forenoon and another in the afternoon of the same day. The method may properly be employed when it is desired to obtain azimuth closer than single minutes, and is well adapted for use with a repeating theodolite. The time may be determined by a watch, but for accurate results it is usually observed with a chronometer. The method as here described is employed in connection with the magnetic observations of the U. S. Coast and Geodetic Survey.

The terrestrial mark, from which horizontal angles are measured to the sun, should be a sharply defined point at least 300 ft. from the instrument, and the instrument should be in good adjustment. If the instrument is equipped with a striding level for the horizontal axis, the bubble should be carefully centered when the telescope is pointed in the general direction of the sun. What was said in Art. 309 regarding favorable hours for making observations also applies here. For most accurate results the observations are planned so that the sun is approximately as far west of the meridian for the afternoon series as it is east of the meridian for the morning series.

Figure 312*a* shows the notes for a series of morning observations, and serves to make clear the field procedure. The column headed "Circle" indicates whether the vertical circle was on the observer's right or on his left, when the sight was taken, and hence indirectly gives the position of the telescope. The temperature is taken in order that refraction corrections may be made accurately.

For the morning series the theodolite is set up at one end of the line whose azimuth is to be determined, a sight is taken to the other end

(as *B* in notes) and both verniers of the horizontal circle are read (circle right). The upper motion is loosened, the telescope is reversed, a second sight is taken along the line, and the horizontal circle is read as before (circle left). The upper motion is loosened and a sight is taken on the sun, bringing it tangent to the cross-hairs in the upper left-hand quadrant (circle left). The time is observed at the instant of tangency, and vertical and horizontal circles are read. This will be called *Solar Observation* 1. The upper motion is loosened, a sight is again taken on the sun, and the procedure just outlined for Solar Observation 1 is



		Azimuth and				Longitude at			White Rock, Ky. 15.
	Observation on Sun.					Lat. 37°03'35"			J. M. Kirk.
		Field Work							F. C. Camp
		A.M.							July 2, 1924
Circle	Obj.	Time	Horizontal		Circle	Vertical	Circle		Temp. 97°F.
			Ver. A	Ver. B	Mean	Ver. A	Ver. B	Mean	Berger Theod. No 43.
R	B		352°37'40"	38'00"	352°37'50"				Chronometer 06ˢ8 –
L	B		172 37 20	37'00"	37'10"				Fast.
Mean					352°37'30"				E. S. Time.
L	☉	9ʰ35ᵐ15ˢ	291 40 00	40 20	111 40 10	44°17'30"	17'30"	44°17'30"	
L	☉	36 10	291 49 00	49 00	111 49 00	44 29 00	29 00	44 29 00	
R	☉	37 40	112 45 20	45 40	112 45 30	45 11 00	11 30	45 11 15	
R	-☉	38 47	112 56 20	56 40	112 56 30	45 24 00	24 30	45 24 15	
Means		9 36 58 0			112°17'50"			44°50'50	
Mean Hor. Ang. Line to Sun.					119°40'20"	Par. & Ref. –		0'78	
R	☉	9 39 46	113 06 10	05 20	113 05 15	45 36 00	36 00	45 36 00	
R	☉	40 49	113 17 00	17 00	113 17 00	45 48 30	49 00	45 48 45	
L	☉	42 20	112 49 00	49 00	112 49 00	45 41 30	42 00	45 41 45	
L	☉	43 24	112 59 10	59 20	112 59 15	45 54 30	54 30	45 54 30	
Means		9 41 34 8			113°02'50"			45°46'25	
Mean Hor. Ang. Line to Sun.					120°24'30"	Par. & Ref. –		0.76	
L	B		172 38 00	37 30	352 37 45				
R	B		352°38'50"	39 10	39 00			Note: Hor. Angs. are	
Mean					352°38'20"			Clockwise from Line A-B	
								to ☉	

FIG. 312*a*.—Observations for azimuth and longitude.

repeated (circle left). This will be called *Solar Observation* 2. The upper motion is loosened, the telescope is reversed, a sight is taken at the sun in the lower right-hand quadrant (circle right), and the time and circle readings are observed. This will be called *Solar Observation* 3. The upper motion is loosened, a fourth sight is taken on the sun (circle right), it being a repetition of Solar Observation 3. It will be called *Solar Observation* 4. The observations so far described constitute a set which for convenience will be called *Set* 1.

Without again turning to the line, four more observations are taken on the sun, the procedure for the first two being identical with that for Solar Observations 3 and 4 (that is, circle right and sun in lower right-hand quadrant) and the procedure for the last two being identical with that for Solar Observations 1 and 2 (that is, circle left and sun in upper left-hand-quadrant). The upper motion is then loosened,

a sight is taken along the terrestrial line (circle left) and the horizontal circle is read. The upper motion is loosened, the telescope is reversed, a sight is taken along the line (circle right) and the horizontal circle is read. This completes another group of observations, which for convenience will be called *Set* 2. Sets 1 and 2 make up the morning series.

The procedure followed in making the afternoon series of observations is the same as described above except that the sun is observed first in the upper right-hand, then in the lower left-hand quadrant.

Azimuth and Longitude at White Rock, Ky. 17.

Computations.

	Set	1	2	3	4
	h	44°49′.7	45°44′.5	44°05′.3	41°28′.5
	ϕ	37 03.6	37 03.6	37 03.6	37 03.6
	p	66 55.5	66 55.5	66 56.7	66 56.8
	$2s$	148°48′.8	149°43′.6	148°05′.6	145°28′.9
	s	74 24.4	74 51.8	74 02.8	72 44.4
	$s-p$	7 28.9	7 56.3	7 06.1	5 47.6
	$s-h$	29 34.7	29 07.3	29 57.5	31 15.9
	$s-\phi$	37 20.8	37 48.2	36 59.2	35 40.8
Check	$2s$	148°48′.8	149°43′.6	148°05′.6	145°28′.7
colog	cos s	0.57056	0.58316	0.56090	0.52767
colog	cos $(s-p)$	0.00371	0.00418	0.00334	0.00222
log	sin $(s-h)$	9.69339	9.68723	9.69842	9.71516
log	sin $(s-\phi)$	9.78293	9.78743	9.77933	9.76586
log	cot² ½A	0.05059	0.06200	0.04199	0.01091
log	cot ½A	0.02530	0.03100	0.02100	0.00546
Az. of	☉ from N	93°20′.2	94°05′.2	267°13′.8	269°16′.8
Ang. ☉	to Line	240 19.7	239 35.5	66 27.1	64 23.4
Az. of	A-B	333°39′.9	333°40′.7	333°40′.9	333°40′.2
Mean	Az	333°40′.4			
Cal.	Bear.	N 26°19′.6 W			

For Field Notes See P. 15 & 16.
Computed by J. C. Kirk.
Checked by F. C. Camp.
July 2, 1924.

Set	1	2	3	4
colog cos $(s-p)$	0.00371	0.00418	0.00334	0.00222
log sin $(s-h)$	9.69339	9.68723	9.69842	9.71516
colog cot ½A	9.97470	9.96900	9.97900	9.99454
log tan ½t	9.67180	9.66041	9.68076	9.71192
$t°$	50°19′00″	49°10′12″	51°13′51″	54°30′32″
t^h	−3h21m16.0s	−3h16m40.8s	3h24m55.8s	3h38m02.1s
E	3 40.2	3 40.2	3 43.4	3 43.5
Local M. T.	8 42 24.2	8 46 59.4	3 28 39.2	3 41 45.6
Obs. Time	9 36 58.0	9 41 34.8	4 23 10.8	4 36 22.8
ΔT_m	−54 33.8	−54 35.4	−54 31.6	−54 37.2
ΔT_s	−06.8	−06.8	−06.9	−06.9
$\Delta\lambda$	54m27.0s	54m28.6s	54m24.7s	54m30.3s
Mean	54m27.6s = 13°36′.9			
λ	88°36′.9			

FIG. 312*b*.—Computations for azimuth and longitude.

The mean horizontal and vertical angle and time for each set are calculated as shown in the notes (Fig. 312*a*) and each mean altitude is corrected for refraction and parallax for the observed temperature (Table I). The declination for the mean time of each set is found as explained in Art. 307. As explained in Art. 296*c*, the azimuth and hour angle of the sun at each of the mean times can be determined by the following equations:

$$\cot^2 \frac{1}{2}A = \frac{\sin (s - \phi) \sin (s - h)}{\cos s \cos (s - p)}$$

$$\tan \frac{1}{2}t = \frac{\sin (s - h)}{\cot \frac{1}{2}A \cos (s - p)}$$

The computations may be made by means of five-place logarithms or by use of a calculating machine. Figure 312*b* shows logarithmic computations for four sets of observations comprising the morning and afternoon series, the notes for sets 1 and 2 being shown in Fig. 312*a*.

The azimuth of the line is found independently for each set by combining the mean horizontal angle with the computed azimuth of the sun. If the work were without error, the azimuth of the line should be the same by one set as by the others. The closeness of agreement between the several values is an indication of the precision of the measurements. The mean of the four independent determinations is taken as the most probable value. In the computations of Fig. 312*b*, the maximum difference between values is 01′, and the probable error of the mean is about 00′.2.

The hour angle computed for each of the four sets, expressed in hours, is the local apparent time of observation. This is converted into local mean time by applying the equation of time, the latter being added or subtracted according as mean time is faster or slower than apparent time. Thus for set 1, the hour angle is $-3^h21^m16^s.0$, which is equivalent to $8^h38^m44^s.0$. The observed Eastern standard time is $9^h36^m58^s.0$ and the G.C.T. is 5^h greater or $14^h36^m58^s.0$. The equation of time by ephemeris giving values for 0^h G.C.T. is found to be $3^m40^s.2$, mean time being faster than apparent time. Hence the equation of time is added to L.A.T. to give $8^h42^m24^s.2$, the local mean time. The difference between local mean time and the correct standard time is the difference in longitude between the place of observation and the meridian at which standard time is also local mean time. If local mean time is faster than standard time, the place is east of the standard meridian and the longitude is therefore less; if slower, the place is west and the longitude is greater. If the work were without error, the four independent determinations of the longitude given by the four sets of observations should yield the same result. The reliability of the observations may be judged by the closeness of agreement between the several computed values. The mean is taken as the most probable value. In the example, the observed time, which is Eastern standard time, is faster than local mean time, hence the longitude is greater than 75°. The difference in longitude $\Delta\lambda$ as determined by computations varies from $54^m24^s.7$ to $54^m30^s.3$, and the range in values is therefore $5^s.6 = 01'.4$. The mean value of $\Delta\lambda$ is $54^m27^s.6 = 13°36'.9$, for which the probable error is about $\pm 00'.2$. The longitude of the place is 88°36′.9.

313. Solar Attachments.—A solar attachment is a device which, when attached to the transit, furnishes a means of determining the direction of the meridian by mechanically solving the *PZS* triangle. There are several varieties of solar attachment differing widely in appearance, but being alike in principle. Each type has a polar axis and a line of collimation. The solar attachment may be rotated about the polar axis just as the transit may be rotated about its vertical axis. Means are provided for laying off the colatitude of the place and the declination of the sun. When these quantities have been laid off, and the line of collimation of the solar attachment

is directed to the sun, the line of sight of the transit telescope is in the plane of the meridian.

The use of a solar attachment makes it possible to determine the azimuth of a line more quickly than by direct observation and numerical computation, but the precision of the azimuth determination is likely to be much lower when an observation is made with a solar attachment than when the sun's azimuth is computed from direct measurement of its altitude. At the present time the solar attachment is little used except on certain mining and public-land surveys of moderate precision, where several azimuth determinations must be made daily. Most surveyors agree that its cost and the additional time required to adjust it more than offset any advantage gained, unless it is to be used very frequently. Because of the inaccuracies of the device, a solar attachment is useless for the more refined surveys, and it seems not improbable that before many years its use will be no more widespread than is that of the surveyor's compass. The three types of solar attachment which have been most popular are the Smith, the Saegmuller, and the Burt, each of which will be briefly described. The Smith is in most general use.

All that has previously been said regarding favorable hours for taking direct observations, applies equally well to observations with a solar attachment. Also errors in latitude and declination settings will have the same effect as when the azimuth is computed mathematically.

314. Smith Solar Attachment.—This attachment consists of a telescope of low magnifying power, called the *solar telescope,* mounted in collar bearings which are attached to a graduated vertical arc, called the *latitude arc.* This assembly is mounted on a horizontal axis, called the axis of the latitude arc, which is attached to one of the standards of the transit as shown in Fig. 314. The solar telescope can be rotated about its own axis in the collar bearings and can also be revolved about the axis of the latitude arc. The latitude arc is read by a vernier which is fixed to the standard of the transit. In front of the objective is a plane mirror, called the *reflector,* which has its axis normal to the line of sight of the solar telescope and which can be rotated (in bearings) about its axis. From the axis of the mirror an arm carrying a vernier leads to a graduated arc, called the *declination arc,* attached to the barrel of the telescope. Movements of both latitude and declination arcs are controlled by clamps and tangent-screws. Near the eyepiece end of the solar telescope is an *hour circle,* the index of which is on one of the collars. The solar telescope is equipped with cross-hairs and with a set of *equatorial hairs* parallel to the axis of the reflector and at a distance apart approximately equal to the apparent diameter of the sun.

When the latitude is laid off on the latitude arc and the main telescope is pointed in the direction of north, the axis of the solar telescope points towards the pole. Further, for the position just specified, if the declination of the sun, corrected for refraction, is laid off on the declination arc, the mirror is so tilted that the course of the sun may be followed by rotating the solar telescope about its

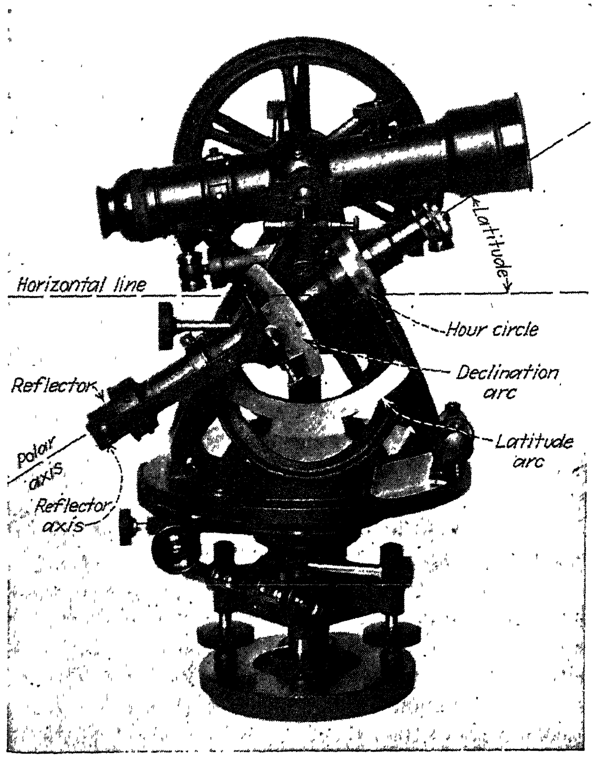


FIG. 314.—Smith solar attachment.

own axis, the image of the sun being reflected from the mirror to the objective, and thence to a focus at the cross-hairs of the solar telescope. When the sun is viewed in this manner, the index of the hour circle reads the local apparent time.

The advantage of the Smith solar attachment lies in a construction which permits the use of the main telescope without disturbing the latitude and declination settings. Since the latitude is constant for a given locality and the declination changes but slowly, this is a decided advantage when frequent observations are necessary, and for this

reason the Smith solar attachment may be regarded as superior to either of the other attachments described herein. It has been adopted by the General Land Office as the standard instrument for use on the public-land surveys.

314a. Azimuth with Smith Solar Attachment.—In using the Smith solar attachment, the latitude and declination settings are made on the appropriate arcs, with the declination setting corrected for refraction. The transit is set up and carefully leveled, and the horizontal circle is set at zero. The local apparent time is then set off on the hour circle, this being done by rotating the solar telescope in its collar bearings. The transit is revolved on the lower motion until the image of the sun appears between the equatorial hairs of the solar telescope, when the line of sight of the transit telescope lies in the plane of the meridian, and the axis of the solar telescope points towards the celestial pole. Since the declination changes slowly, if no mistake has been made it will be possible to keep the image of the sun between the equatorial hairs for some time by simply rotating the telescope in its collar bearings, keeping the hour circle set at the apparent time. When a sight is taken along the line with the transit telescope, the reading of the clockwise scale gives the azimuth.

This type of solar attachment is particularly convenient for checking azimuth traverses, since the latitude and declination settings may be maintained without interfering with the normal functions of the transit telescope. At any station the azimuth can be checked simply by setting the horizontal circle back to zero, rotating the solar telescope until the hour circle reads the apparent time, and making any slight change in the declination setting that may have occurred in the declination since the preceding observation.

314b. Adjustments of Smith Solar Attachment.—In adjusting the Smith solar attachment, the latitude of the place should be accurately known, and the instrument should be set up in a position where objects a mile or more away may be viewed with the telescope level. The following relations should exist:

1. *The equatorial hairs should be parallel to the axis of the reflector.* With all settings made as for a solar observation for azimuth, the transit is turned about the vertical axis until the sun's image is accurately spaced between the equatorial hairs, the vertical axis is clamped in this position, and the solar telescope is rotated about its own axis, causing the sun's image to travel across the field of view. If the limbs of the sun follow the equatorial hairs, the latter are parallel to the axis of the reflector. If the sun's limbs depart sensibly from the hairs, adjustment is made by loosening the cross-hair ring and rotating it through a small angle, as for the corresponding adjustment of the dumpy level.

2. *The line of sight of the solar telescope should coincide with the axis of the collar bearings.* The test and adjustment is performed in a manner identical with the corresponding adjustment of the wye level. The mirror is swung to give an unobstructed view through the solar telescope, and the intersection of the cross-hairs is focused on some distant point. The telescope is then rotated about its axis through an hour angle of 12^h (180°) and if the intersection moves away from the point, it is brought halfway back to its original position by means of the adjusting screws controlling the cross-hair ring.

3. *The line of sight of the solar telescope, and hence the polar axis, should be perpendicular to the axis of the latitude arc.* Further, when the solar telescope is rotated about the axis of the latitude arc, its line of sight should generate a plane parallel to that generated by the line of sight of the main telescope when it is revolved about the horizontal axis. Some patterns have no provision for this adjustment, the desired relation being established by the manufacturer and being assumed to remain permanently fixed. In the General Land Office pattern, provision is made for this adjustment, the axis of the latitude arc being of considerable length, and a striding level for this axis being furnished with the instrument. For the General Land Office pattern, the transit is very carefully leveled, the main telescope is sighted at a distant point, and the solar telescope is sighted towards the same point. The striding level is placed on the latitude axis and the latter is made level by means of the lower pair of capstan nuts on the base frame of the attachment. If the line of sight of the solar telescope falls to one side of the distant point, it is brought to the point by means of the left-hand upper pair of capstan nuts on the base frame of the attachment. When these two relations have been established, both telescopes are plunged, the transit is revolved 180° in azimuth, and a sight is taken to the distant point as before. If the line of sight falls to one side of the point, one-half of the apparent error is due to the line of sight not being perpendicular to the latitude axis, and the correction is made by bringing the line of sight halfway to the point by means of the capstan nuts at one end of the telescope and the remaining distance by means of the left-hand upper pair of capstan nuts on the base frame of the attachment. This procedure should be repeated until the adjustment is verified.

4. *The latitude arc should read zero when the line of sight of the solar telescope, or the polar axis, is horizontal.* The transit is leveled very carefully, and the main telescope is leveled and sighted at some distant point of the landscape. The solar telescope is revolved about the latitude axis until its line of sight strikes the same point (the polar axis is now horizontal), when the vernier of the latitude arc should read zero. If it does not read zero, either the index error may be observed and applied to future latitude settings or the vernier may be loosened and moved laterally until it does read zero. If the solar telescope is equipped with a striding level, the index error may be observed by leveling the solar telescope, and a sight to the distant point is unnecessary.

5. *The declination arc should read the true declination of the sun, corrected for refraction.* A short time before apparent noon the transit is set up and carefully leveled over one end of an established meridian. The latitude of the place is laid off on the latitude arc, the hour circle is set at 12^h, and the main telescope is sighted along the meridian. At the instant of apparent noon, the image of the sun is brought between the equatorial hairs of the solar telescope by rotating the reflector about its axis, and the reading on the declination arc is observed. The difference between this reading and the declination of the sun at the given instant (as determined from the solar ephemeris) corrected for atmospheric refraction, is the index error of the declination arc. This may be applied to future declination settings, or the error may be corrected by loosening the vernier and moving it along the arc until it reads the calculated declination setting.

On public-land surveys where the Smith solar attachment is in constant use, it is the practice to make the above test daily, since it serves to check not only the adjustment of the declination vernier but also the adjustment of the latitude vernier and the collimation of the solar telescope.

6. *The reading of the hour circle at any instant should give the local apparent time.* The watch time of local apparent noon is determined by observing with the main telescope the instant when the sun's center crosses the meridian, as described in Art. 311. At any convenient time thereafter, the main telescope is pointed along the meridian, the latitude and declination are set off on their respective arcs, the sun's image is brought to the center of the field of view by rotating the solar telescope about its own axis, and the watch time is observed. If the time interval since the observed passage of the sun over the meridian, which is the correct local apparent time, does not agree with the reading of the hour circle, the set-screw clamping the hour circle to the barrel of the telescope is loosened, and the hour circle is rotated until the index reads the correct value.

315. Saegmuller Solar Attachment.—This attachment consists of a telescope of low power, called the *solar telescope*, mounted between standards which revolve about a polar axis. The whole is mounted on top of the transit telescope, the polar axis being normal to the plane defined by the line of sight of the main telescope and the horizontal axis of the transit. The solar telescope is equipped with cross-hairs defining the line of sight as do those of the main telescope, and in addition is provided with four hairs forming a square the sides of which are approximately equal to the apparent diameter of the sun. Attached to the solar telescope is a level tube the axis of which is parallel to the line of sight of the solar telescope. The solar telescope is equipped with a prismatic eyepiece. The movement of the solar telescope about the polar axis and about the axis of the standards, or the equatorial axis, is controlled by clamp- and tangent-screws.

When the image of the sun appears inside the square formed by the four hairs, the line of sight of the solar telescope is directed towards the

sun's center. When the main telescope is elevated through an angle equal to the colatitude of the place and is pointed along the meridian in a southerly direction, it is clear from the discussion of Art. 294 that the line of sight of the main telescope is in the plane of the equator, and that the polar axis of the solar attachment points to the celestial pole. Further, with the main telescope in this position, if the line of sight of the solar telescope makes an angle with the polar axis equal to the sun's codeclination or polar distance, then any time when the sun is above the horizon and the solar telescope is pointed in the direction of the sun, it should be possible to bring the line of sight to the sun's center simply by rotating the solar telescope about the polar axis. Also, when the lines of sight of both telescopes are in the same vertical plane, it is clear that the angle between them is equal to the sun's declination.

The methods of adjusting the Saegmuller solar attachment are similar to those governing the adjustment of the transit.

316. Burt Solar Attachment.—The Burt solar attachment is attached to the main telescope in the same manner as is the Saegmuller, but the solar telescope is replaced by bi-convex lenses and metallic screens in duplicate, these being rigidly mounted at opposite ends of a bar which forms the vernier arm for a graduated arc on which declinations may be set off. The line of collimation of the attachment is defined by the optical center of one of the bi-convex lenses and the intersection of lines etched upon the opposite metallic screen, these lines corresponding to cross-hairs in the transit telescope.

When the line of collimation is pointed at the sun's center, the sun's unmagnified image appears at the intersection of the cross-hairs. With a magnifying glass the line of collimation can be directed more exactly at the sun's center by bringing the image inside a square etched on the metallic screen, this square corresponding to that formed by the four hairs in the solar telescope of the Saegmuller attachment. Regardless of whether the sun is above or below the equator, declinations are laid off in the same direction on the arc. When the attachment is in use, if the declination is north or positive the line of collimation is pointed at the sun with declination arc nearer the sun; if the declination is south or negative the line of collimation is pointed with the declination arc nearer the observer. Surrounding the base of the polar axis is an *hour circle* graduated in hours and reading to 5 min. of time. If the colatitude is laid off on the vertical circle of the transit and the main telescope is pointed south, the index of the hour circle reads the local apparent time when the line of collimation of the attachment points at the sun. The use of the Burt attachment is identical with that of the Saegmuller attachment, except as indicated by the preceding description.

317. Declination Settings.—If any of the varieties of solar attachment is to be used frequently, a table is prepared giving the declinations to be laid off when using the solar attachment at various hours of the day. The time may be either local apparent or standard time,

but usually is the former. The apparent declination for a given time is found from a solar ephemeris, as described in Art. 307. Since the sun appears to be higher than it really is, due to atmospheric refraction, it is evident from a study of the *PZS* triangle (Fig. 294*c*) that the declination setting for a sight to the apparent position of the true sun at a given instant must be algebraically greater than the true declination of the true sun, which is the value obtained from the ephemeris. If the refraction correction in altitude is known, the refraction correction in declination for a given latitude, hour angle, and declination may be computed by solving the spherical triangle. Table III gives such values throughout the year for latitude 40°. Any number in the second column gives the hour angle of the true sun on either side of the meridian—or in other words, gives the local apparent time before or after local apparent noon—to which the refraction correction given in the third column applies. For a latitude other than 40°, the correction of Table III is multiplied by the appropriate latitude coefficient of Table III*a*.

Example: Below is a table of declination settings computed for Nov. 3, 1926, at a place where the latitude is 34°30′ and the longitude is 7^h48^m west of Greenwich. The hours are local apparent time.

From the Ephemeris of the Sun (General Land Office), the apparent declination at Greenwich apparent noon is $-14°54'51''$, and the change for one hour is $-47''$. At 8 a.m. local apparent time it is $8^h + 7^h48^m - 12^h = 3^h48^m = 3.8^h$ G.A.T. Hence the declination at 8 a.m. local apparent time is $-14°54'51'' - 3.8 \times 47'' = -14°57'50''$.

From Table III, at latitude 40° the refraction correction for Nov. 3 and an hour angle of 4^h (equivalent to 8 a.m. or 4 p.m. local apparent time) is $+3'21''$. By Table III*a* the latitude coefficient for latitude 34°30′ is 0.80. The declination correction for 8 a.m. at the given latitude is therefore $3'21'' \times 0.80 = +2'41''$.

The declination setting at 8 a.m. L.A.T. is therefore $-14°57'.8 + 2'.7 = 14°55'.1$.

Local apparent time	Declination	Refraction correction	Declination setting
8^h a.m.	−14°57′.8	+2′.7	−14°55′.1
9	−14°58′.6	+1′.7	−14°56′.9
10	−14°59′.4	+1′.3	−14°58′.1
11	−15°00′.2	+1′.1	−14°59′.1
12 m.	−15°01′.0	+1′.0	−15°00′.0
1 p.m.	−15°01′.8	+1′.1	−15°00′.7
2	−15°02′.6	+1′.3	−15°01′.3
3	−15°03′.4	+1′.7	−15°01′.7
4	−15°04′.1	+2′.7	−15°01′.4

Declination settings for other hours are found in a similar manner The declination for noon is found from Table I, the altitude being calculated from the equation $h = 90° - \phi + \delta$. The example shows how such a table is prepared for a given date.

When an observation is to be made, the watch time is noted and the local apparent time is roughly calculated. The declination setting is then taken from the table, interpolating if necessary. Even by careful estimation the declination may be set only to half minutes of arc, hence values of the settings do not need to be interpolated with great accuracy. In fact under certain conditions the declination setting may not change appreciably for several hours, as illustrated by the afternoon values in the example.

318. Observations on Stars.—In general, the methods of determining azimuth, latitude, longitude, and time by direct solar observations are, with slight modifications, applicable to observations on the stars. Usually it is expected that a higher degree of precision will be obtained by stellar than by solar observations; consequently a corresponding degree of refinement is necessary and special care is taken to eliminate systematic errors. Since there are fixed stars in all parts of the heavens, it is an easy matter to select a star or stars in a celestial region favorable to an accurate determination of the quantity sought.

Thus, conditions favorable to an accurate determination of latitude by measuring the altitude of a star at culmination are a fairly high altitude, in order that the uncertainty of the refraction correction be small, and a rate of apparent movement that is small, in order that a series of observations may be taken without an appreciable change in the altitude. Within the latitudes of the United States, stars near the pole satisfy these conditions.

Likewise for accurate determination, an observation for azimuth by measured altitude and known declination and latitude should be taken upon a star in the east or west far enough above the horizon to eliminate the uncertain refraction, but not so near the meridian as to produce a weak astronomical triangle; and an observation for azimuth, knowing the hour angle, declination, and latitude, should be taken upon a circumpolar star, the nearer the pole the better, since the azimuth of such a star changes more slowly in a given length of time than does the azimuth of a star near the equator, and hence any error in time will have less effect. Also, for longitude or time, stars should be chosen near the equator because they are apparently traveling more rapidly than those near the pole.

The right ascensions and declinations for a large number of stars are given in the American Ephemeris and also in the Nautical Almanac.

Since for the fixed stars these coordinates change very slowly, it is not necessary to determine values for the hour of observation, as with the sun. In the ephemeris of the sun the sidereal time of 0^h G.C.T. is given, from which the sidereal time corresponding to any given solar time can be found, and the hour angle can be computed by the expression $t = \theta - \alpha$ as explained in Art. 297*e*.

Stars may be identified by means of charts which show the various constellations. For many stellar observations, however, the published direction and altitude of the star can be set off on the transit with sufficient accuracy so that the star will be brought into the field of view at a given time, and to be able to distinguish the star from among its neighbors is not essential. In fact, observations may be and frequently are taken on stars before dark, when they are still invisible to the naked eye.

During the hours of darkness, artificial illumination is required to make visible the cross-hairs of the transit. Some instruments are equipped with a reflector sleeve which slips over the objective as does the sun shade. When a flashlight is held to one side of the reflector, the field of view is faintly illuminated and both the cross-hairs and the star can be seen. With the transit not so equipped, the cross-hairs can be illuminated sufficiently by holding the light a few inches in front of the objective and a little to one side of the telescope barrel, thus causing the rays to enter the telescope diagonally. There will be found a position of the light where both cross-hairs and star are visible. A better diffusion of light will be given by a drop of paraffin wax at the center of the objective lens, the wax being shaved to a thin layer. A piece of thin paper with a hole in the middle, the paper being secured over the objective by means of a rubber band, answers the same purpose.

The position of any terrestrial mark that may be used in observing must of course be indicated by a light. Often the mark is a slit in the side of a box in which there is a lamp. The mark may be a strongly illuminated target, the source of illumination being shielded from the observer.

In sighting on a star, the objective should be focused until the star appears as a fine, brilliant point of light. Prior to looking for a star just before sunset or after sunrise, the objective may by properly focused by sighting at a distant object in the landscape. The proper position of the objective slide for focus on a star may be permanently marked on the barrel of the transit telescope.

319. Polaris.—The pole star, Polaris, is the star more than all others, on which observations for latitude and azimuth are taken in the latitudes of the United States. Its distance from the pole is

only a little more than a degree. Its annual change in declination is less than a minute of arc, and its maximum daily change in declination is less than a half second of arc. It is a second-magnitude star whose position is readily identified by the neighboring constellations of Ursa Major and Cassiopeia. Figure 319 shows the position of Polaris with respect to the pole and to these constellations. The seven most brilliant stars in the constellation of Ursa Major are known as the *Great Dipper;* and the two stars forming the part of the bowl farthest from the handle are called the *pointers* because a line through these stars points very nearly to the celestial north pole. It will be noted that the constellation of Cassiopeia is on the same side of the pole as Polaris, so that when Cassiopeia is above the pole, Polaris is near upper culmination; when Cassiopeia is west of the pole, Polaris is near western elongation; and so on. The position of Polaris relative to the pole may be quite closely estimated by noting the positions of δ Cassiopeia and ζ Ursa Major. A line joining these two stars passes nearly through the pole and Polaris. The line is nearly vertical when the star is at either culmination, and nearly horizontal when the star is at either elongation.

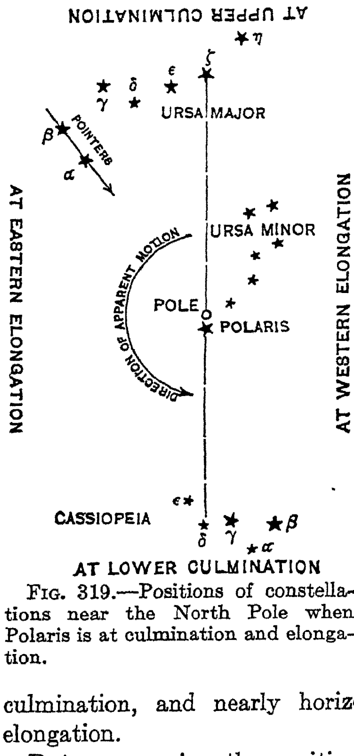


FIG. 319.—Positions of constellations near the North Pole when Polaris is at culmination and elongation.

Data concerning the position of Polaris can be found from a variety of sources. In the American Ephemeris the declination and right ascension for each day are given in the table of circumpolar stars for the upper transit at Washington. From the relation between sidereal and solar time which is given in the solar ephemeris for Greenwich, the hour angle of the star may be computed as already described. In the American Ephemeris are given, for dates at intervals of 10 days, the declination and right ascension of Polaris and the civil time of its upper culmination for the meridian of Greenwich. Since Polaris, in common with other fixed stars, travels at an angular rate more rapid than that of the sun, it follows that at

a given meridian it arrives at culmination at a little earlier mean solar time each day than it did the day before, the amount earlier being approximately equal to the gain of sidereal time on mean solar time for a 24^h interval. In the column headed "variation per day" this daily gain in time is given. Also, for the same reason, on any given date the star arrives at culmination at a local mean time which becomes greater or less, according to whether one travels easterly or westerly, the increase or decrease in time between two places being approximately equal to the gain of sidereal on solar time within the mean time interval represented by the difference in longitude between the two places. In the column headed "variation per hour" is given the change in local mean time of culmination per hour of longitude. To determine the local mean time of upper culmination at any given meridian on any given date, a value is taken from the table for the date nearest that given, and this is reduced to the given date by means of the "variation per day," and to the longitude of the place by means of the "variation per hour." This is illustrated by the following example:

Example: It is desired to find from the American Ephemeris the Eastern standard time of the upper culmination of Polaris on Dec. 8, 1927, at a place whose longitude is $5^h15^m45^s$ west of Greenwich.

On Dec. 5, 1927, U.C. at Greenwich occurs at	$20^h41^m41^s$ G.C.T.
The decrease in civil time for 3 days is $-3 \times 3^m55^s.9$	$= \quad -11^m48^s$
The G.C.T. of U.C. at Greenwich on Dec. 8	$= 20^h29^m53^s$
Change in time for $\Delta\lambda$ is $-5.25 \times 9^s.83$	$= \quad 52^s$
Local civil time of U.C. at place	$= 20^h29^m01^s$
$\Delta\lambda = 5^h15^m45^s - 5^h$	$= \quad +15^m45^s$
E.S.T. of Upper Culmination at place	$= 20^h44^m46^s$
	$= \quad 8^h44^m46^s$ p.m.

The Ephemeris of the Sun and Polaris (General Land Office) also gives values of the declination of Polaris and the Greenwich mean time of culmination at the meridian of Greenwich.

In Table IV herein are given data by means of which the approximate standard time of culmination may be determined for any place and date. Accompanying the table is an explanation of its use. The reasons for the various steps are clear when it is remembered that the hour angle of Polaris is changing at a faster rate than that of the mean sun, this gain being approximately the gain of sidereal time on mean time.

320. Latitude by Observation on Polaris at Culmination.—As shown in Art. 294, if h is the true altitude of any circumpolar star as it crosses the meridian, then the latitude is

$$\phi = h \pm p$$

in which the sign preceding p is positive or negative according to whether the star is at lower or upper culmination. By this method the latitude of a station is determined by measuring the altitude of Polaris when at either upper or lower culmination, and by applying to this altitude, corrected for refraction, the star's polar distance as given in Polaris tables. Inasmuch as the star is apparently traveling in a horizontal line when at either of these two positions, it is not essential to the accuracy of the latitude determination that the time of culmination be found, but in any case it facilitates the work of observing if the approximate time is known. For all ordinary observations the time of culmination may be found with sufficient accuracy from Table IV.

Further, it is not essential that the altitude of the star be observed at the instant it crosses the meridian. For some minutes before and after culmination the star travels in so nearly a horizontal line that, with the ordinary transit, vertical movement cannot be detected. Within the period 6^m before to 6^m after culmination, the maximum change in altitude is only $01''$, and within a half hour of culmination the change in altitude is only about $\frac{1}{2}'$.

The procedure to be employed in making an observation depends somewhat upon the precision with which the latitude is to be determined and upon the accuracy with which time is known. For an observation with the ordinary transit having a vertical circle reading to minutes, when the watch time or the longitude of the place may be in doubt by a few minutes, the time of culmination at the given station is roughly determined by Table IV herein, or by use of an ephemeris as described in the preceding article. A few minutes before the estimated time of culmination the transit is set up and very carefully leveled (as a final test the telescope bubble should remain centered as the transit is revolved about the vertical axis). The star is found with the naked eye by noting its position with respect to the neighboring constellations shown in Fig. 319. The objective is focused for a star. If the latitude is approximately known, its estimated value, plus or minus the star's polar distance, is set off on the vertical circle, and the telescope is sighted at Polaris. When the star has been brought within the field of view, the cross-hairs are illuminated and the star is continually bisected with the horizontal cross-hair. When after an interval of 3 or 4 min. the star no longer appears to move away from the hair but moves directly along it, Polaris is practically at culmination. The vertical angle is read with dispatch, the transit is carefully releveled, the telescope is plunged, and a second observation on the star is taken with the telescope in the reversed position. The mean of the two observed altitudes,

corrected for refraction (Table II), is taken as the true altitude of the star. The polar distance can be found approximately by Table VIII herein, in which table is given the mean polar distance for each year, or the polar distance can be found more exactly for the given date from any ephemeris giving declinations of Polaris for days of the current year. Finally the latitude is computed by applying to the true altitude, the polar distance with proper sign.

320a. When it is desired to determine the latitude within a few seconds and the standard time and longitude of the place are known within a minute or so, the watch time of culmination may be accurately computed as illustrated in the example of Art. 319, and a series of observations may be taken when the star is near culmination. To find the time of lower culmination, 12^h minus one half of the variation per day (practically one half of 3^m56^s) is added to or subtracted from the time of upper culmination. The time of lower culmination may also be found from Table IV herein without error of consequence. The number of observations will depend upon the accuracy with which the latitude is to be determined. The observing program is usually arranged so that an equal number of observations will be taken before and after culmination. The observations are begun at a given time interval (usually not more than 10^m) before

Corrections to Be Applied to Altitudes of Polaris Near Culmination to Give Altitude at Culmination

Interval from culmination, minutes of time	Change in altitude from culmination, seconds of arc
3	00
6	01
9	03
12	06
15	09
18	12
21	17
24	22
30	34

the calculated time of culmination, and at each sighting of the star, the watch time and altitude are observed. Half of the observations are taken with the telescope in the direct and half with it in the reversed position, and between pairs of observations the telescope is carefully releveled. The observed altitudes of the star for positions other than culmination are reduced to the altitude at culmination by

applying a correction which, for latitudes within the United States, is given approximately by the table on page 483.

When the star is at lower culmination, the correction is subtracted, and when at upper culmination, the correction is added. The mean of the altitudes reduced to culmination is corrected for refraction (Table II) and the polar distance is found and the latitude is computed as for the case described in the preceding article.

321. Azimuth by Observation on Polaris at Elongation.—As shown in Art. 296*d*, the azimuth of any star when at elongation is given by the expression

$$\sin Z = \frac{\sin p}{\cos \phi}$$

in which Z is the angle east or west of north according as the star is at eastern or western elongation, p is the star's polar distance, and ϕ is the latitude of the place. By this method, it is necessary that the latitude of the place and the polar distance of the star be known, in order that the azimuth of the star may be computed. In the field, the direction of Polaris from a given station is established by projecting a vertical plane from the star to the earth at the time of either eastern or western elongation. The terrestrial line thus established has the same azimuth as does the star at the given position, hence the azimuth of any connecting line can be found if the horizontal angle between the two lines is measured. The star's polar distance is found approximately by Table VIII or more exactly by the American Ephemeris or other ephemeris giving values of the declination of Polaris for the days of the year in which the observation is made. The latitude is determined by observation, as explained in preceding articles. Inasmuch as the star is apparently traveling in a vertical line when at either elongation, it is not essential to the accuracy of azimuth determination that the time of elongation be found, but in any case it facilitates the work of observing if the approximate time is known. For all ordinary observations, the time of elongation may be found with sufficient accuracy from Table IV of this book. The time of elongation may be determined with greater accuracy from an ephemeris for the current year.

Further, it is not essential that the direction to the star be observed at the instant of elongation. For some minutes before and after elongation the star travels in so nearly a vertical line that, with the ordinary transit, horizontal movement cannot be detected. For latitudes of the United States, within the period 4^m before elongation to 4^m after elongation, the maximum change in the azimuth of Polaris is less than $01''$ and within 10^m of elongation the change in azimuth is not more than $06''$.

The procedure to be followed depends somewhat upon the precision with which azimuth is to be determined and upon the accuracy with which the time of elongation is known. When the watch time or the longitude of the place may be in doubt by a few minutes, the standard time of elongation is roughly determined by Table IV. A few minutes before the estimated time of elongation the transit is set up over a given point and very carefully leveled. The telescope is focused for a star, the latitude of the place is laid off on the vertical circle to facilitate finding the star, and the transit is revolved about the vertical axis until Polaris comes within the field of view. The transit is clamped, the cross-hairs are illuminated, and the star is continuously observed until during a period of 2 or 3 minutes Polaris no longer appears to move horizontally but moves vertically directly along the cross-hair.

The telescope is depressed, and the position of the line of sight is marked on a stake or other reference monument 300 ft. or more away. The telescope is then plunged and another sight is taken upon Polaris. The line of sight is again depressed, and a second point is set on the stake beside the first. Usually the transit is releveled and another set of observations such as just outlined is made, so that there are at least four points on the stake beneath the star.

By daylight the mean of the points of each set is found, and the point marking the average of these is established on the stake. The line joining the occupied station with the established mean point defines the direction of Polaris when at elongation. Its azimuth is either computed by the expression given in the early part of this article, or it is found directly from tables. It is given within a few seconds by Table V herein, to seconds by the tables in the yearly Ephemeris of the Sun and Polaris (General Land Office), and to tenths of seconds in the yearly American Ephemeris. The azimuth of any other line joining the station occupied can be accurately found by measuring the angle between the two lines by repetition (Art. 206, p. 275) or a true meridian can be accurately established by a perpendicular offset from the established point on the stake, as illustrated in the following example:

Example 1: In taking an observation on Polaris at western elongation, the reference point marking the azimuth of the star is 400 ft. from the transit. The azimuth of the star at elongation is $-1°40'45''$. A point on the true meridian through the transit station is to be established by a perpendicular offset from the reference mark.

$$\begin{aligned}
\text{Log } 400 &= 2.60206 \\
\text{Log tan } 1°40'45'' &= \underline{8.46710} \\
\text{Log offset} &= 1.06916 \\
\text{Offset} &= 11.725 \text{ ft.}
\end{aligned}$$

The precision of azimuth determination by this method necessarily depends upon the quality of the instrument, the care and skill of the observer, and the number of observations, but for the procedure described, under ordinary conditions the error should not exceed 10″.

It should be noted that a given error in latitude produces a relatively small error in azimuth. For latitudes of the northern part of the United States, an error of 01′ in latitude produces an error of about 02″ in azimuth, and for lower latitudes the effect is less. Since the latitude can easily be determined within 20″ with the ordinary transit, it is evident that the principal error in azimuth is likely to be due, not to errors in the computed value of the azimuth of Polaris, but to the field operations of projecting the direction of the star to the earth. If the transit is equipped with a full vertical circle, the procedure is such that practically all instrumental errors of projecting the direction of the star to the ground, except that due to the vertical axis not being truly vertical, are eliminated. For reasons explained in Art. 211, if precise observations are to be obtained, it is important that the transit be leveled with great care, and in order that the error may be of an accidental rather than of a systematic nature, the instrument should be releveled at least for each set of two observations. When the transit is equipped with a striding level for the horizontal axis, the bubble should be centered prior to each sight on the star.

321a. When azimuth of a line is to be established within two or three seconds, a high-grade transit with a telescope of large magnifying power and with a sensitive striding level for the horizontal axis should be employed, and the standard time and longitude should be known within a minute or so in order that the time of elongation may be calculated with precision. A series of several observations may then be taken, at known times, during the interval just before and just after the instant of elongation, and the observations for times other than that of elongation may be reduced to elongation by applying a small correction.

The hour angle of the star when at elongation may be precisely determined as explained in Art. 296*d* by the equation $\cos t = \tan \phi \tan p$. The hour angle t expressed in time is practically the sidereal time interval between upper culmination and eastern or western elongation. The corresponding mean solar time interval is found by deducting from the computed hour angle a correction of $9^s.83$ per hour which, as explained in Art. 297*e*, is the difference in solar time between the sidereal hour and the mean solar hour. In the American Ephemeris this mean time interval is given for various degrees of latitude. The standard time of upper culmination of the star is found as shown in the example of Art. 319. The standard time of eastern or western elongation is then determined by adding to or subtracting from the time of culmination the mean time interval

between upper culmination and elongation. Following is a numerical example:

Example 2: It is desired to find accurately the Eastern standard time of western elongation of Polaris occurring in the early morning hours of Dec. 9, 1927, at a place where the longitude is $5^h15^m45^s$ west of Greenwich and the latitude is 50°00′00″ north.

From American Ephemeris

$$
\begin{aligned}
\delta &= 88°55'16'' = 90° - p \\
p &= 1°04'44'' \\
\log \tan p &= 8.274905 \\
\log \tan \phi &= \underline{10.076186} \\
\log \cos t &= 8.351091 \\
t &= 88°42'50''
\end{aligned}
$$

Hour angle at elongation $= t$	$= 5^h54^m51^s.4 = 5^h.92$
$-5.92 \times 9^s.83$	$= -\quad 58^s.2$
Mean time interval from U.C.	$= 5^h53^m53^s.2$
From example, Art. 319,	
E.S.T. of U.C. on Dec. 8, 1927	$= 8^h44^m46^s$ p.m.
E.S.T. of western elongation on Dec. 9, 1927	$= 2^h38^m39^s$ a.m.

In the Ephemeris of Sun and Polaris (General Land Office), the mean time of elongation for the meridian of Greenwich and latitude 40° is given for each day. To find the standard time of elongation for any other meridian and latitude 40°N, the procedure is the same as explained in Art. 319 for finding the mean or standard time of culmination.

For latitudes other than 40°, add to the time of western elongation $0^m.10$ for every degree south of 40°, and subtract from the time of western elongation $0^m.16$ for every degree north of 40°. Reverse these operations to determine time of eastern elongation.

When it is desired to determine azimuths with precision and the watch time of elongation of Polaris is accurately known, a series of observations is taken on the star as described in Art. 321, the program being so timed that approximately one half of the observations will occur in an interval of a few minutes before the instant of elongation, and the remainder will occur in a like period after elongation. If the transit is equipped with striding level, the horizontal axis is leveled each time the star is sighted. If not so equipped, the transit is carefully leveled with the telescope bubble prior to each set, which consists of two observations, one with the telescope in the direct and the other with it in the reversed position.

For each set of points marked on the distant reference monument, a mean is taken, and the azimuth of the star at the mean of the times

of the two observations comprising each set is found by applying a slight correction to the azimuth at elongation, this correction being found in Table V*a*. It will be noted that for the average latitude of the United States this correction amounts to less than 01″ when the star is 4^m from elongation, and about 05″ when the star is 10^m from elongation.

The distance from reference monument to transit station is measured and the mean mark for each set of two observations is corrected to give the equivalent mark at the instant of elongation, this being done by calculating the linear offset (for the distance from transit to mark) corresponding to the angular correction found in Table V*a*. If it were not for the accidental errors connected with the observations, the points thus determined would coincide. The mean of the group is taken as the point which gives the most probable direction of the star when at elongation. By measuring the linear variations from the mean and transforming these into angular variations, the probable angular error in the direction of the line defined by the transit station and the mean mark at the reference monument can be calculated, and thus the reliability of the observations can be ascertained.

For observations of this character the stability of the transit during the course of the measurements is of the greatest importance. Preferably the transit should be removed from the tripod and placed upon a concrete pier. Changes in temperature may also seriously affect the relations between the fundamental lines of the instrument, and hence the transit should be allowed to come to the temperature of the air before observations are begun. Also, the transit should be protected from wind.

Example 3: The azimuth of Polaris is to be computed for the time of western elongation at latitude 50°00′00″N and longitude $5^h15^m45^s$W on Dec. 9, 1927. From the American Ephemeris (Table VII) the declination δ for the given date is 88°55′15″.

$$\begin{aligned} p &= 90° - \delta = 1°04'45'' \\ \sin p &= 8.274940 \\ \cos \phi &= 9.80\dot{8}067 \\ \hline \sin Z &= 8.466873 \end{aligned}$$

Azimuth from North = −1°40′45″. Table V, American Ephemeris, gives 1°40′44″.5.

Example 4: With conditions as given in example 3, an observation is taken 10^m after western elongation. What is the azimuth of Polaris at the given instant?

By example 3 the azimuth at western elongation is	−1°40′44″.5
By Table V*a* herein, the correction is	5″.7
The azimuth of the star at given instant is	−1°40′38″.8

Example 5: For the observation of example 4 a point is marked directly below the star on a monument which is 400 ft. from the transit. What is the amount and direction of a linear offset to be measured from this point to establish an equivalent point for Polaris when at western elongation?

The angular correction is $5''.7$ as stated in example 4. The offset is $400 \tan 5''.7 = 0.0000276 \times 400 = 0.011$ ft.

Since the star has reached its most westerly position and is traveling east, the offset is made to the west.

322. Azimuth by Observation on Polaris at Any Time.—Although elongation is the most favorable time for accurate determination of the azimuth of Polaris, it is often inconvenient or impossible to view the star when in this position. Under these circumstances, if the standard time and the longitude of the place are accurately known, the hour angle of the star at any instant can be found and the azimuth of the star at any instant can be determined, as described in Art. 296*f*, by the expression

$$\tan Z = \frac{\sin t}{\cos \phi \tan \delta - \sin \phi \cos t}$$

in which Z is the azimuth east or west of north, and t is the hour angle reckoned from 0° to 180° before or after upper culmination. It should be noted that for angles between 90° and 180° the sign of $\cos t$ is negative. Table VII gives the azimuths for a declination of 88°51′ and various hour angles. By interpolation the azimuth for any hour angle and declination can be found.

The field observation consists in measuring the horizontal angle between a terrestrial mark and the position of the star. For a rough determination correct say within ½′, a single set of two observations, one with the telescope direct and the other with it reversed, should be taken, the time of the passing of the star across the vertical cross-hair at each setting being noted. The hour angle of the star is then found for the mean of the two times of observation, and the azimuth of the star (for this hour angle and the proper declination) is computed by means of the preceding equation or is found in Table VII. This azimuth combined with the mean of the two observed horizontal angles gives the azimuth of the reference line.

322a. To find the hour angle of Polaris at any observed watch time, the watch is compared with a timepiece keeping correct standard time, and the observed time is corrected accordingly. The correct standard time is changed to local mean time by adding or subtracting the difference in longitude expressed in hours between the place of observation and the standard meridian (meridian where standard time is also local mean time), adding, if the place of observation is

east of the meridian and subtracting, if west. The local mean time of the culmination nearest the time of observation is determined from the ephemeris as illustrated by the example of Art. 319, or is found with less precision by Table IV herein. The difference between this value and the local mean time of observation gives the mean solar time interval before or after upper culmination. For reasons previously explained (Art. 297*e*) Polaris is gaining on the mean sun at practically the same rate that sidereal time is gaining on mean solar time. Since in a mean solar hour there are $60^m + 9^s.856$ of sidereal time, it follows that the hour angle of the star expressed in time is greater than the mean solar time interval by $9^s.856$ or nearly 10^s per solar hour.

Example 1: What is the hour angle of Polaris Dec. 8, 1927 at $10^h30^m15^s$ p.m. Eastern standard time at a place whose longitude is $5^h15^m45^sW$? The standard meridian is 5^hW. The place is west of the standard meridian and hence local time is slower than E.S.T. by the difference in longitude. The local mean time of observation is therefore

$$
\begin{array}{rcl}
10^h30^m15^s - 15^m45^s & = & 10^h14^m30^s \text{ p.m.} \\
 & & \underline{12^h} \\
\text{The civil time from } 0^h & = & 22^h14^m30^s.
\end{array}
$$

By the example of Art. 319, the local civil time of U.C.	=	$20^h29^m01^s$
Mean time interval since U.C.	=	$1^h45^m29^s = 1^h.76$
Gain of sidereal on mean $= 1.76 \times 9.86$	=	17^s
Hour angle of Polaris	=	$1^h45^m46^s$ west of meridian.

Example 2 shows the use of Table VII for finding the azimuth of Polaris.

Example 2: By use of Table VII find the azimuth of Polaris for the observation of example 1 and latitude 50°N.

From an ephemeris the declination of the star for this date (Dec. 8, 1927) is 88°55′15″.

$$
\begin{array}{lr}
\text{For } t = 1^h45^m \text{ and } \delta = 88°51', & Z = 0°48'31'' \\
\text{Change in } Z \text{ in } 46^s = \dfrac{46 \times 378''}{60 \times 15} & = \quad 19'' \\
\text{Azimuth for declination of } 88°51' & = 0°48'50'' \\
\text{Change in } Z \text{ for change in } \delta = 4.25 \times 43'' & = -03'03'' \\
\text{Azimuth west of North} & = 0°45'47''
\end{array}
$$

In the Ephemeris of the Sun and Polaris (General Land Office), azimuths of Polaris at all hour angles are given, the argument being mean time interval before or after upper culmination. When using this table it is unnecessary to determine the actual hour angle of the star.

322b. When the azimuth of a line is to be accurately determined by observation on Polaris when the star is not at elongation, a series of observations is taken. Each angle from the star to the line is added to the sum of the preceding angles as when measuring an angle by repetition, and sights are taken first with the telescope in the direct and then with it in the reversed position. The total angle turned divided by the number of observations gives the horizontal angle from the mean position of the star to the line. The azimuths of Polaris for the mean of the times for the two observations of each set are determined from Table VII or from similar tables in an ephemeris. The average of the azimuths thus determined for the several sets forming the series is considered to be the azimuth of the mean position of the star, and this value, combined with the mean horizontal angle from the star to the line, gives the azimuth of the line.

The precision to be obtained depends upon the position of the star, the accuracy of observations of time, the number of observations, the quality of the instrument, and the care and skill of the observer. From an inspection of Table VII it will be noted that when the star is near upper or lower culmination, the azimuth changes at a relatively rapid rate, this change amounting to about 01′ of arc in $2\frac{1}{2}^m$ of time for latitude 40°N. For this reason the method should not be expected to give precise results when Polaris is near culmination unless the time is accurately observed. It is also important that systematic errors due to inclination of the vertical axis be eliminated by carefully releveling the instrument. If the transit is equipped with striding level, the horizontal axis is leveled just before each observation when the telescope is pointed in the direction of the star. The ordinary transit not so equipped should be carefully leveled before each set, making the final test with the telescope bubble as described elsewhere. The stability of the instrument likewise plays an important part in determining the accuracy, and for precise measurements the transit should be set on a pier or other substantial object.

323. Observations on Other Stars.—Latitude and azimuth by observation on any circumpolar star can be determined by methods identical with those for Polaris. The other stars near the pole are of lesser magnitude than Polaris and are therefore not as readily identified, but if the approximate direction of the meridian is known and the hour angle and declination of the star are known, the approximate altitude can be laid off and the telescope can be pointed so that the star will come within the field of view.

Latitude, azimuth, time, and longitude can be determined by observation on stars distant from the pole by methods similar to those described for the sun.

In the American Ephemeris for each year are tables giving the right ascensions and declinations of stars for the upper transit at Washington, for which place the longitude is $5^h 8^m 15^s.8$ West of Greenwich. Also there is given in the ephemeris of the sun for Washington apparent noon the sidereal time of 0^h civil time.

Since the right ascension of the fixed stars changes but a small fraction of a second per day, through the relation $\theta = t + \alpha$ it is seen that the hour angle of a star at any meridian other than Washington is readily found if the sidereal time is known, and that the sidereal time of upper culmination or upper transit of the star is equal to its right ascension. Further, knowing the longitude of the place of observation and having given the sidereal time of 0^h Washington civil time, the relation at a given instant between sidereal time and local civil or standard time can be determined as explained in the preceding chapter. It is therefore possible to find (by aid of an ephemeris) not only the declination of a star but also its hour angle at any instant of mean solar or standard time. This is illustrated in the following examples.

Example 1: What is the Pacific standard time of the upper transit of Betelgeux on Feb. 27, 1927 at a place whose longitude is $8^h 9^m 2^s.8$?

From the American Ephemeris the sidereal time of upper transit at the

meridian of Washington is		$5^h 51^m 13^s.1$
On Feb. 28 (by ephemeris) the sidereal time of 0^h civil time at the meridian of Washington is		$10^h 28^m 6^s.4$
The sidereal time interval (at upper transit) preceding 0^h is		$-4^h 36^m 53^s.3$
$\Delta\lambda = 8^h 9^m 2^s.8$ (place) $- 5^h 8^m 15^s.8$ (Washington)	=	$3^h 0^m 47^s.0$
Sidereal time interval before 0^h Washington civil time when upper transit occurs at place	=	$-1^h 36^m 6^s.3 = 1^h.60$
To change to mean solar time, subtract $9^s.83 \times 1.60$	=	$15^s.7$
Mean time interval before 0^h Washington civil time	=	$-\ 1^h 35^m 50^s.6$
		$+24^h$
Washington civil time of U.T. at place	=	$22^h 24^m 9^s.4$
$\Delta\lambda$	=	$-\ 3^h 0^m 47^s.0$
Local civil time	=	$19^h 23^m 22^s.4$
Difference between local and standard time	=	$+\ 9^m 2^s.8$
Pacific standard time of U.T.	=	$19^h 32^m 25^s.2 - 12^h$
	=	$7^h 32^m 25^s.2$ p.m.

Example 2: What is the hour angle of Betelgeux at $11^h 32^m 25^s.2$ p.m. Pacific standard time on Feb. 27, 1927, at a place whose longitude is $8^h 9^m 2^s.8$?

From example 1, the Pacific standard time of upper transit is $7^h32^m25^s.2$. The mean time interval since upper transit is therefore $4^h0^m0^s$; sidereal time gains on mean time at the rate of $9^s.856$ per hour. The hour angle of the star is

$$4^h0^m0^s + 4.0 \times 9.856 = 4^h0^m39^s.4$$

Solutions similar to those of the preceding examples are expedited by using tables for the conversion of mean solar into sidereal time interval or *vice versa,* which tables are given in the American Ephemeris and in the Nautical Almanac.

323a. Determination of Latitude.—To determine latitude by an observation on a star at upper transit, the star's declination and right ascension are found in an ephemeris, and the approximate standard time of upper transit at the given place is determined as illustrated by example 1, Art. 323. Before this time, the transit is set up and the estimated altitude of the star is laid off on the vertical circle. (The latitude will be roughly known, the declination is known, and hence the altitude can be estimated with sufficient precision to bring the star within the field of view.) The telescope is pointed approximately along the meridian, and the instrument is revolved about the vertical axis back and forth through a small angle until the star is sighted. The star is followed with the horizontal cross-hair until the maximum altitude is reached and then the vertical angle is read. The latitude is determined as described in Art. 308.

In this way several stars whose times of upper transit vary by short intervals can be observed and the latitude can be computed by taking the mean of the values thus found.

323b. Determination of Time.—To determine time by observing the upper transit of any star, the direction of the meridian and the longitude of the place being known, the standard or local civil time of upper transit is calculated as in the preceding example. The star's declination is found from the ephemeris, and its altitude is roughly calculated, the latitude of the place being at least approximately known. Before the estimated time of upper transit, the instrument is set up and a sight is taken along the meridian. The estimated altitude of the star is laid off on the vertical circle, and the course of the star is followed until it crosses the vertical hair. At this instant, time is observed. The difference between this time and the calculated time is the error of the timepiece. Time determinations should be made on stars near the celestial equator.

For more accurate results a succession of observations such as the observation just described may be made on stars whose calculated times of upper transit differ from each other by only a few minutes. Most instrumental errors will be eliminated if the instrument is reversed

between two successive observations. The average clock error thus determined is considered to be the error of the timepiece.

It is evident that time and latitude observations may be made simultaneously if, in addition to the observations just described, the vertical angle to each star as it crosses the meridian is measured.

323c. Determination of Longitude.—To determine longitude by observing the upper transit of any star, the direction of the meridian and the standard time of the place being known, the standard time of upper transit is calculated for a longitude estimated to be that of the place. The star is found as described in Art. 323*a*, and the standard time of its upper transit is observed. The interval between the calculated time and the observed time of upper transit, changed to sidereal time, is the difference between the estimated longitude and the true longitude of the place.

323d. Determination of Azimuth.—To determine azimuth by observation on any star other than a circumpolar star, the same general procedure is followed as for solar observations of this character described in Arts. 309 and 312. To determine the approximate position of a given star at a given time so that the star may be brought within the field of view, the right ascension and declination are found from an ephemeris, and the hour angle of the star is calculated as illustrated in example 2, Art. 323. The approximate altitude is then calculated by Eq. (20), Art. 296*e*, and the approximate azimuth is computed by Eq. (21), Art. 296*f*. With these two approximate values, the star is readily brought within the field of view if the direction of the meridian is roughly known. The intersection of the cross-hairs is then sighted at the star, the time is observed, and the horizontal and vertical circles are read.

If a star chart is available the position of a given star may be readily determined with the eye, and it may be sighted through the telescope at once without first laying off the approximate azimuth and altitude.

Where the azimuth is to be determined with a fair degree of precision, usually a procedure similar to that described for the sun in Art. 312 is followed, and a star is chosen which is in a favorable position (see Art. 309).

This method would be employed only when satisfactory observations could not be taken on Polaris, either on account of clouds or on account of the star's being too near the horizon.

324. Problems.

1. In connection with a solar observation, sights are taken to an index mark in the direction of the sun. The vertical angles to the mark are $-3°17'00''$ with telescope direct and $-3°18'30''$ with telescope reversed. With the telescope still in the reversed position, a sight is

taken to the sun for which the observed vertical angle is $+36°02'30''$. Correct the observed angle for index error of the vertical circle.

2. Find the apparent declination of the sun for the instant of 9^h0^m a.m. Central standard time on February 15 of the current year, using an ephemeris giving values of 0^h Greenwich civil time.

3. Find the apparent declination of the sun for the instant of local apparent noon at a place whose longitude is $5^h52^m54^s.1$W for the date of July 21 of the current year, using an ephemeris giving values for Greenwich apparent noon.

4. Find the apparent declination of the sun at 3^h18^m p.m. Pacific standard time on November 4 of the current year, using an ephemeris giving values for Greenwich apparent noon, and taking into consideration the equation of time. Determine the effect of neglecting the equation of time.

5. The observed altitude of the lower limb of the sun as it crosses the meridian at a given place is $55°31'30''$. The observation is made at $11^h34^m20^s$ a.m. Eastern standard time on May 16 of the current year. Calculate the latitude of the place.

6. The observed altitude of the upper limb of the sun as it crosses the meridian at a given place is $52°13'$. The longitude of the place is 7^h32^mW. The date is March 4 of the current year. What is the latitude of the place?

7. At a place whose latitude is $41°58'10''$N the observed altitude of the sun's center at 3^h12^m p.m. Central standard time on October 21 of the current year is $20°04'30''$. The horizontal angle (measured on the azimuth circle) from a reference line to the sun is $81°32'20''$. The temperature is 40°F. What is the azimuth (measured from south) of the sun at the given instant and what is the azimuth of the reference line? Compute the azimuth of the sun by Eq. (13′), Art. 296c.

8. On August 1 of the current year the observed altitude of the sun at a given place is $30°51'45''$ at $7^h42^m20^s$ a.m. local apparent time. The latitude of the place is $37°18'20''$N and the longitude is $102°17'30''$W. The temperature is 75°F. The horizontal angle (measured clockwise) from reference line to sun is $89°39'15''$. What is the azimuth of the sun measured from north, and what is the azimuth of the reference line? Compute the azimuth of the sun by Eq. (13), Art. 296.

9. Compute the changes in azimuth of sun due to a $01'$ change in latitude, in declination, and in altitude for latitude 50°, declination 0°, and altitudes of 15° and 30°. Use Eq. (13), Art. 296, as a basis for calculations. Compare results with corresponding quantities for latitude 40° given in table of Art. 309.

10. Same as problem 9 but for latitude 30°.

11. When an azimuth observation is taken on the sun, the vertical axis of the transit is inclined $01'$ to the true vertical, the inclination being in a plane normal to the sight plane when the line of sight is directed towards the sun. The altitude of the sun is 60°. Due to this inclination, what error will be introduced in the horizontal angle from reference line to sun?

12. At Orono, Maine, on December 5 of the current year the sun's center is observed to cross the meridian at a watch time of $11^h24^m21^s$ a.m. The longitude of the place is $4^h34^m40^s.3$W. What is the watch correction to give local mean time? What is the watch correction to give Eastern standard time?

13. At Denver, Colorado, on May 5 of the current year the sun's eastern limb crosses the meridian at a watch time of $11^h57^m48^s$ a.m. The longitude of the place is $6^h59^m47^s.7$W. What is the watch correction to give local mean time? What is the watch correction to give Mountain standard time?

14. At a given place the center of the true sun crosses the meridian at $11^h41^m37^s$ a.m. Pacific standard time on September 15 of the current year. What is the longitude of the place?

15. By a series of observations the true altitude of the sun's center at a given station is 24°28′44″ at the instant of $4^h13^m12^s$ p.m. Mountain standard time on March 14 of the current year. The clockwise horizontal angle from reference line to sun is 312°16′37″. The latitude of the place is 39°01′42″N. By Eqs. (15′) and (16′), Art. 296c, compute the azimuth and hour angle of the sun at the given instant. Determine the longitude of the place and the azimuth of the reference line reckoned from south.

16. Compute the declination settings for a solar attachment at local apparent times of 1^h, 2^h, 3^h, and 4^h after noon on October 13 of the current year for a place whose latitude is 59°06′N and whose longitude is 118°36′45″W.

17. On September 7 of a given year, upper culmination of Polaris at the meridian of Greenwich occurs at $2^h35^m29^s$ Greenwich civil time. What is the Eastern standard time of upper culmination on September 10 of the same year at a place whose longitude is 78°30′15″W?

18. On January 12 of the current year the observed altitude of Polaris at upper culmination at a given place is 44°36′25″. The temperature is 15°F. What is the latitude of the place?

19. From Table IV find the Central standard time of upper culmination of Polaris on December 7, 1933, at Des Moines, Iowa, (longitude $6^h14^m30^s.6$W).

20. The altitude of Polaris is observed 20^m after the time of upper culmination and found to be 48°32′20″. The polar distance is 1°04′35″. What is the latitude of the place?

21. Compute the azimuth and hour angle of Polaris when at elongation, the polar distance being 1°04′35″ and the latitude of the place being 43°00′49″N.

22. What is the time of western elongation of Polaris at a given place when upper culmination occurs at $2^h15^m20^s$ p.m. Eastern standard time and the latitude is 42°22′47″.6N?

23. By Table IV find the Pacific standard time of eastern elongation of Polaris on August 18 of the current year for latitude 37°52′24″N and longitude $8^h9^m3^s$W.

24. By Table V find the azimuth of Polaris when at elongation on August 18 of the current year for a place whose latitude is 37°52′24″N.

25. By Table V*a*, determine the azimuth correction to be applied to an observation on Polaris 15^m after elongation to reduce to elongation, the azimuth at elongation being 1°34′12″. Compute the corresponding perpendicular offset at the reference monument beneath the star when the monument is 600 ft. from the station occupied.

26. At a given place upper culmination of Polaris occurs at $3^h15^m20^s$ p.m. Central standard time on a given date. On the same date an azimuth observation is made at $7^h0^m20^s$ p.m. The latitude of the place is 41°15′30″N and the polar distance of the star is 1°04′12″. Compute the hour angle and azimuth of the star. Check the azimuth by Table VII.

27. By use of the American Ephemeris compute the Central standard time of upper transit of α Canis Minoris on January 8 of the current year at a place whose longitude is $6^h15^m12^s$W.

28. What is the hour angle of α Leonis at $2^h0^m0^s$ a.m. Eastern standard time on April 30 of the current year at a place whose longitude is $4^h49^m8^s$W?

References

1. Cary, E. R., "Geodetic Surveying," John Wiley and Sons, New York, 1916.
2. Hosmer, G. L., "Geodesy," Second Edition, John Wiley and Sons, New York, 1930.
3. Nassau, J. J., "Text Book of Practical Astronomy," McGraw-Hill Book Co., New York, 1932.
4. "Manual of Instructions for the Survey of the Public Lands of the United States," General Land Office, Government Printing Office, Washington, D. C., 1930.

SECTION III
PRACTICE OF SURVEYING

CHAPTER XIX

LAND SURVEYING—RURAL AND URBAN

325. General.—Land surveying deals with the laying off or measurement of the lengths and directions of lines forming the boundaries of real or landed property. Land surveys are made for one or more of the following purposes:

1. To secure the necessary data for writing the legal description and for finding the area of a piece of land, the boundaries of the property being defined by visible objects.
2. To reestablish the boundaries of a tract for which a survey has previously been made and for which the description as defined by the previous survey is known.
3. To subdivide a tract into two or more smaller units in accordance with a definite plan which predetermines the size, shape, and location of the units.

The functions of the land surveyor are to carry out field surveys as suggested above, to calculate dimensions and areas, to prepare maps showing the lengths and directions of boundary lines and areas of lands, and to write descriptions by means of which lands may be legally conveyed, by deed, from one party to another.

In the present chapter, land surveying practices as applied to both rural and urban properties are described and some of the legal aspects of land surveying are discussed. In the succeeding chapter the United States system of subdividing the public lands is outlined. Methods of calculating and subdividing areas are discussed in Chap. XVI.

326. Kinds of Land Surveys.—In accordance with the purposes listed in Art. 325, surveys may be divided into the following classes:

1. *Original surveys*, made for the purpose of measuring the unknown lengths and directions of boundaries already established and in evidence. Surveys of this character are usually of rural lands. For example, Adams may purchase from Brown a certain parcel of pasture land bounded or defined by features or objects,

such as fences, roads, trees, etc. In order that the deed may contain a definite description of the tract, a survey is necessary.

2. *Resurveys,* run for the purpose of relocating the boundaries of a tract for which a survey has previously been made. The surveyor is guided by a description of the property based upon the original survey, and by evidence on the ground. The description may be in the form of the original survey notes, an old deed, or a map or plat on which are recorded the measured lengths and bearings of sides and other pertinent data. When, without further division, land is transferred by deed from one party to another, a resurvey is often made.

3. *Subdivision surveys,* run for the purpose of subdividing land into more or less regular tracts according to a prearranged plan. The division of the public lands of the United States into townships, sections, and quarter sections is an example of the subdivision of rural lands into large units. The laying out of blocks and lots in a city addition or subdivision is an example of the subdivision of urban lands.

327. Instruments and Methods.—Nearly all land surveys are run with the transit and tape, by methods described in Chaps. VI, XII, and XIII. Distances are ordinarily measured to feet and decimals, and angles are measured to minutes or fraction thereof. On the U. S. public-land surveys all distances are in Gunter's (66-ft.) chains, as prescribed by law, measurements being taken with a tape graduated to read chains and links (1 chain = 100 links). The directions of lines are usually referred to the true meridian, and angular measurements are transformed to bearings.

Formerly the surveyor's compass and 66-ft. link chain were extensively used, particularly in rural surveying, and the lengths and directions of lines contained in many old deeds are given as Gunter's chains and magnetic bearings. In retracing old surveys of this character, due allowance must be made for change in magnetic bearing since the time of the original survey. Also, due allowance must be made for the inferior instruments of that time, and for the fact that great precision was not regarded as necessary since the land values were low. Further, the United States public lands were surveyed under contract, at the low price of a few dollars per mile. Many of the lines and corners established many years ago are not where they theoretically should be; nevertheless these boundaries legally remain fixed as they were originally established.

Whenever possible, the field procedure is such that the lengths of boundary lines and the angles between boundaries are obtained by direct measurement. The land survey is therefore in general a

traverse, the transit stations being at corners of the property and the traverse lines coinciding with property lines. Where obstacles render direct measurement of boundaries impossible, traverse stations are established and a traverse is run as near the property lines as is practicable. By use of the data thus collected, supplemented by measurements from traverse lines to property corners, the lengths and directions of the property lines are computed. Distances are usually measured horizontally. On urban surveys where the desired precision is high, the distances are frequently measured on the slope and are then reduced to the horizontal. On rural surveys where slopes are steep, measurements are often made on the slope, vertical angles being observed with a clinometer. Where the boundary is irregular or curved, the traverse is established in a convenient location and offsets are taken from the traverse line to points on the boundary, after which the length of the boundary is calculated.

328. Monuments and Reference Marks.—It is customary to mark the corners of landed property by visible monuments. In the case of original surveys, many of the corners were natural objects such as trees and stones already marking the borders of a tract before the survey was actually made. In general, however, the corner monuments are established by the surveyor either to mark the intersections of boundaries already in existence or to define new boundaries. Unfortunately in the great majority of cases monuments thus established are of temporary character, often consisting of wooden stakes driven in the ground, the stakes soon rotting away. This is particularly true in the case of city lots, and resurveys made necessary by the obliteration of such temporary markers constitute a large part of the practice of the surveyor whose work is confined to the surveying of urban lands.

Examples of markers of a more permanent character are an iron pipe or bar driven in the ground; a concrete or stone monument with drill hole, cross or metal plug marking the exact corner; a stone with identifying mark placed below the surface; charcoal placed below the surface; a mound of stones; and a metal marker set in concrete below the surface, reached through a covered shaft leading to the surface. Monuments for city lots are usually set nearly flush with the ground. The boundaries of many of the old English, Dutch, and Spanish grants are marked by large stone monuments, often projecting several feet above ground. Subsurface stones are commonly used for corner monuments in localities where roads follow section lines. On many old government surveys, through wooded country where stones were not available, corners were established by building up a mound of earth over a quart of charcoal or a charred stake, or by building a mound about a tree at which the corner

fell. The U. S. General Land Office has more recently adopted as the standard for the monumenting of the public-land surveys a post made of iron pipe filled with concrete, the lower end of the pipe being split and spread to form a base, and a brass cap with identifying marks being riveted to the upper end.

When there is any possibility of a corner monument's becoming displaced, the corner should be *referenced* or connected to nearby objects of more or less permanent character in such manner that it may be readily replaced in case of loss (Art. 232). Trees, stones, and buildings are examples of objects which are often utilized as reference marks. In many of our larger cities systems of permanent monuments are established, and to these all surveys are referred. On public-land surveys, the bearing and distance from a corner to a tree are taken where possible, the tree being blazed and so marked as to identify the section on which the tree stands, the mark terminating with the letters "B.T." signifying bearing tree. The General Land Office specifies that every corner established in the public-land surveys shall be referenced by one or more objects of any of the following classes: (*a*) "bearing trees, or other natural objects . . . ; permanent improvements; and memorials; (*b*) mound of stone; and (*c*) pits."

Where a corner falls in such location as to make it impossible or impracticable to establish a monument in its true position, it is customary to set a point on one or more of the boundary lines leading to the corner. A point thus established is called a *witness corner.* Everything that has been said concerning monuments at the true corners also applies to witness corners. Witness corners are necessary where the true corner falls in a road, stream, lake, or marsh, within a building, or upon a precipitous slope. Under certain circumstances, as when boundaries are in roads, it is impossible to place the witness corner on any of the property lines approaching the true corner, in which case the witness corner is established in any convenient location.

The field notes should give detailed information concerning the character, size, and location of all monuments and reference marks, and the data should be recorded in such manner that there will be no possibility of misinterpretation. As far as possible, all points established in the field should be so marked as clearly to indicate the object which they represent.

329. Boundary Records.—There are three general sources from which the descriptions of the boundaries of real property may be found, *viz.*, deeds, official plots or maps, and notes of original surveys.

Throughout the United States, records of the transfer of land from one person to another are kept either in the office of the city or town clerk or more usually in the county registry of deeds, exact copies of all deeds of transfer being filed in deed books. These files are free to be examined by anyone and are a frequent source of information for the land surveyor in search of boundary descriptions when it is inconvenient or impossible to secure permission of the owner to examine the original deed.

In connection with the register an alphabetic index is kept, usually by years, one part of the index giving the names of *grantors* or persons selling property, and the other part giving the names of *grantees* or persons buying property during the given year. It is therefore a simple matter to find a given deed if either or both parties to the transfer and the approximate date of transfer are known. The preceding transfer of the same property is usually indicated by a note on the margin of the deed.

As the United States public lands are subdivided, official plats are prepared showing the dimensions of subdivisions and the character of monuments marking the corners. The notes of the U. S. surveys, together with the official plats prepared from the notes, are kept until all public lands within the state are surveyed, when all records are given to the state. An exception is Oklahoma, for which U. S. survey records are filed with the Commissioner of the General Land Office at Washington, D. C. Surveys have been completed in all but the Rocky Mountain and Pacific Coast states and the Territory of Alaska (Art. 346). States in possession of records have them on file at the state capitol. Information concerning these records may usually be secured from the Secretary of State. Photographic copies of the official plats are obtainable for a small sum.

In most cases the deeds of transfer of city lots give only the lot and block number and the name of the addition or subdivision. The official map of the subdivision on which are shown the dimensions of all lots and the character and location of permanent monuments is on file either in the office of the city or town clerk of the municipality or in the county registry of deeds. Copies of the official plats giving dimensions and ownership of all lots are also on file in the offices of city and county assessors.

330. Description of Rural Lands.—Nearly all of the original land grants in the older portions of the United States were of irregular shape, the boundaries often following stream and ridge lines. In the process of subdivision, units were taken here and there without much regard for regularity, and it was thought sufficient if lands were specified as bounded by natural or artificial features of the terrain and

if the names of adjacent property owners were given. Thus a description of a tract as recorded in a deed reads:

"Bounded on the north by Bog Brook, bounded on the northeast by the irregular line formed by the southwesterly border of Cedar Swamp of land now or formerly belonging to Benjamin Clark, bounded on the east by a stone wall and land now or formerly belonging to Ezra Pennell, bounded on the south and southeast by the turnpike road from Brunswick to Bath, and bounded on the west by the irregular line formed by the easterly fringe of trees of the wood lot now or formerly belonging to Moses Purington."

330a. Metes and Bounds.—As the country developed, land became more valuable, many boundaries such as those listed in the preceding description ceased to exist, and land litigations were numerous. It then became the general practice to determine the lengths and directions of the boundaries of land by measurements with the link chain and surveyor's compass and permanently to fix the locations of corners by monuments. The lengths were ordinarily given in rods or chains and the directions were expressed as bearings, usually referred to the magnetic meridian. At the present time, surveys of this character are usually made with the transit and tape, distances being recorded in feet or chains, and directions being given in true bearings computed from angular measurements. In describing a tract surveyed in this manner the lengths and bearings of the several courses are given in order, and the objects marking the corners are made known; in the event of any boundary's being coincident with some prominent feature of the terrain it is so stated; and the calculated area of the tract is given. When the bearings and lengths of the several sides are thus given, a tract is said to be described by *metes and bounds*. Within the limits of the precision of the original survey it is possible to relocate the boundaries of a tract if its description by metes and bounds is available, provided at least one of the original corners can be identified and the true direction of one of the boundaries can be determined.

The following is illustrative of a description by metes and bounds typical of rural lands in the Eastern States and of isolated grants in the Western States where the subdivision of lands has been outside the rectangular system of the public-land surveys:

"Beginning at an iron pipe sunk in the ground thirty-three and no tenths (33.0) feet west of the center line of the concrete highway extending from Fryeburg to North Conway and twelve hundred and seventy-three and four-tenths (1,273.4) feet south of the north face of the south abutment of the bridge over Jordon Brook in the town of Redstone, New Hampshire, at the southeast corner of land now or formerly belonging to

William Bancroft, and running by fence and land of said Bancroft south 89°26′ west a distance of fourteen hundred and seventy-nine and four-tenths (1,479.4) feet to a cross cut in the surface of granite ledge; thence turning and running along a stone wall and land now or formerly belonging to Joseph A. Winship south 28°17′ east for a distance of one thousand and fifty-two and no tenths (1,052.0) feet to a drill hole in the top of a six-inch by six-inch stone monument sunk in the ground, and thence turning and running along said wall and the land of said Winship south 29°37′ west a distance of one thousand and fifteen and three-tenths (1,015.3) feet to a point in the thread of the channel of Spring Brook, from which point a nail in a rock maple tree ten inches in diameter bears north 43°37′ east a distance of thirty-seven and seven-tenths (37.7) feet; and thence turning and following the thread of said channel at land now or formerly belonging to Alfred S. Weeks in a general easterly direction a distance of fifteen hundred and sixty-nine (1,569 ±) feet more or less to a point on the thread of the channel of said brook and distant thirty-three and no tenths (33.0) feet west of the center line of aforesaid road; thence turning and running along the west boundary of said road north 2°15′ west a distance of nineteen hundred and eighty-nine and two-tenths (1,989.2) feet to the point of beginning; all bearings being referred to the true meridian, and the parcel of land containing a calculated area of fifty-nine and sixty-seven hundredths (59.67) acres more or less."

330b. Subdivisions of Public Lands.—A third type of description is employed for lands which have been divided in accordance with the rectangular system of the General Land Office. This method of subdividing the public domain is described in detail in Chap. XX.

The records and plats of the U. S. surveys are a part of the permanent public records and are accessible to anyone desiring to consult them. In conveying by deed a U. S. subdivision or fraction thereof, no doubt can at any time exist as to the tract involved if it is described by stating its sectional subdivision, section number, township, range, and name of the principal meridian on which the initial point is located (Figs. 347*b* and 347*c*). Following is an example of the legal description of a 40-acre tract comprising a full quarter-quarter section:

"The north-east quarter of the south-west quarter of section ten (10), Township four (4) South, Range six (6) East, of the Initial Point of the Mount Diablo Meridian, containing forty (40) acres, more or less, according to the United States Survey."

For reasons which are discussed in Chap. XX, certain subdivisions called *fractional sections* do not contain the nominal acreage of that subdivision. Following is a description of a fractional half of a fractional quarter-section:

"The west fractional half of the northwest fractional quarter of Section seven (7), Township four (4) North, Range twelve (12) East of the Initial Point of the . . . Meridian, comprising lots one and two and containing sixty-three and seventy-nine hundredths (63.79) acres, according to the official plat of the United States Survey."

331. Original Survey of Rural Land.—The need for an original survey usually arises when one person desires to transfer to another a tract of land which has not been previously surveyed but which is defined by certain natural or artificial features of the terrain.

After having been shown the boundaries of the land, the surveyor establishes monuments of one kind or another at the corners, and proceeds to run a closed transit traverse about the property, measuring the lengths of lines and the angles between intersecting lines. Where boundaries are not straight, offsets from transit line to curved boundary are measured at known intervals, and where obstructions make direct measurement along boundaries impossible, the traverse is run as close to the boundary as convenient and measurements are taken from transit stations to corners of the tract, so that the length and bearing of the boundary lines may be calculated. Angular measurements may be taken by any of the methods described in Chap. XIII, but most often the interior angles are observed. Preferably the corners should be referenced to permanent objects. Also the direction of the true meridian should be determined, this usually being done by a solar observation (see Art. 309).

The information thus obtained is recorded in the surveyor's notebook, the angles and distances of the main traverse being tabulated, and the remaining data being recorded in the form of a sketch. The bearings of the sides are then calculated, properly with respect to the true rather than the magnetic meridian.

A description of the tract, similar in arrangement to that given in Art. 330*a*, is prepared. Usually a plat is drawn, the boundaries being plotted by one or another of the methods described in Chap. XV, and details being shown as suggested in Art. 40, p. 44. The area is computed as described in Chap. XVI. In the process of computation, the error of closure of the traverse is determined and thus a check on the reliability of the survey is obtained. A copy of the description and a tracing of the plat are submitted to the person for whom the survey is made.

332. Resurvey of Rural Land.—The resurvey of lands is attended with greater difficulty than is usually appreciated by those inexperienced in work of this character. Particularly is this true in the older sections of the United States where the early surveys were not of the rectangular system and were not under the control of the U. S.

General Land Office. The proper relocation of old lines calls for greater ingenuity and broader experience on the part of the surveyor than any other kind of surveying.

The purpose of a resurvey is to reestablish boundaries in their original positions. The information that the surveyor has to guide him is a description, contained in the deed or obtained from old records, perhaps similar to that given in Art. 330*a*. Descriptions of adjacent property may also be obtained.

If the description of the property were without error and one or more of the original corners were in evidence, and further, if the resurvey could be run without error, the problem would be as simple as running the original survey. When the lengths and directions given in the description had been laid off, the surveyor could say with assurance that the reestablished corners were in their original position. The facts are, however, that the original survey did contain errors and probably rather large ones if it were made during the era of the compass and link chain. Further complications may be added by directions in the description being given by magnetic bearings and the declination at the time of the original survey being unknown, or by no statement having been made as to whether the bearings of the original survey were referred to the magnetic or to the true meridian. Not infrequently large mistakes are made in transposing from one record to another or are present in the measurements of the original survey. The loss of corners, the lack of reference measurements, the removal or alteration of physical boundaries, the conflicting testimony of persons having knowledge concerning the position of boundaries, and numerous other factors may add to the uncertainties of the problem.

332a. As a first step the surveyor usually calculates the latitudes and departures of the several courses as given in the description, determines the error of closure, and plots the boundaries of the tract to scale. If original bearings are magnetic, the declination at the time of the original survey is sought, and true bearings are calculated.

If true bearings cannot be found in this manner, and one or more boundaries can be positively identified, observations are made to determine the true bearings of these known lines, and by a comparison of true and original magnetic bearings the declination at the time of the original survey is estimated and the true bearings of other lines are computed.

If a single boundary line can be identified by means of described corners which are found in place or by reliable reference marks, a comparison is obtained between the length of the chain or tape used on the original survey and that to be used on the resurvey, and the proportionate lengths of other sides of the tract are computed.

With these calculated directions and proportionate lengths, the surveyor, starting from a known corner, reruns the courses, and at each estimated position of a corner seeks evidence of the position of the original monument.

Thus if a stake had originally been set at the corner, careful slicing of the top soil with a shovel might reveal rotted wood, a hole in the ground, or even discolored earth, which might be considered rather positive evidence of the old location. Ties to bearing trees or other objects to which reference measurements were taken would also prove useful in finding the probable position of an obliterated monument. At any point where the surveyor finds what he regards as positive evidence as to the original location of the corner and this location does not agree with the relocation measurements derived from the description of the property, a monument is set at the original location and new measurements of angles and distances are made to refer to the mark thus established.

At any point where physical evidence as to the original location of the corner is entirely lacking, the corner is temporarily located by measurements derived from the description of the property. The survey is then continued until positive evidence of the location of a succeeding corner is found, or until the traverse is brought to a closure at the initial point.

In the former case, the temporary monuments established between two corners which are located with certainty are regarded as correctly located and are replaced by more permanent markers if the points established by the angles and distances of the resurvey fall at the true position of corresponding corners as indicated by visible evidence; if the points do not so fall, the error is determined and the positions of intermediate temporary corners are adjusted by proportionate measurements, the discrepancy between the original survey and the resurvey being assumed to have gradually accumulated.

If no further evidence of the original location of corners is found, the survey is run to the point of beginning, and the linear error of closure is measured. The total error in latitudes and in departures is then computed; the corrections in latitudes and in departures of the preceding lines are computed as described in Art. 263; and these corrections are applied by moving the preceding temporary monuments. Finally the lengths and bearings of the adjusted courses are measured in the field.

332b. Where only a single corner can be found, the process of reestablishing boundaries is not so simple, particularly when the bearings of the original survey were observed with a compass and were referred to the magnetic meridian without the declination being given. Usually an estimate of the amount of the magnetic declina-

tion at the time of the original survey can be obtained by consulting old records, but often the date of the survey from which the description is derived is unknown and cannot be closely determined.

By means of the estimated declination, the magnetic bearings are changed to true bearings, the latitudes and departures are calculated, and the error of closure of the original survey is determined. If this is reasonably small (say not greater than $1/300$ if the old survey was run with a compass), it indicates that there are no mistakes in the lengths and bearings given in the description. About the only course then open to the surveyor is to rerun the survey in accordance with the old description, first establishing the direction of the true meridian, and to consider the corners as being relocated to the best of his ability if the error of closure of the resurvey is no larger than that of the original survey. This error of closure is distributed proportionally among the several courses as described in the preceding article.

It is evident that there might be a large error in the length of the chain or tape used in making the original survey and still the traverse would close. Inasmuch as there is no way of making a comparison between the original and resurvey lengths, distances laid off during the resurvey may be considerably different from corresponding ground distances measured during the original survey. Hence while the resurveyed tract may have the same shape as the original tract, and its boundaries may maintain the same direction as the corresponding boundaries of the original, yet the actual area of the resurveyed tract may be considerably different from that of the original. Also, by a similar course of reasoning, it is evident that any error in the estimated declination will result in a resurvey figure which, while it may close, will be composed of lines each of which will make a constant angle with the corresponding boundary of the original tract. The surveyor should realize that, for a case such as this, a small error of closure of the resurvey is not conclusive evidence of the closeness with which corners are reestablished with respect to their original positions.

332c. When a tract is to be resurveyed, a description of the tract being accessible but all evidence of the location of original corners having been lost, the surveyor will find it expedient to search the records for descriptions of adjoining property and by means of these descriptions to reestablish by measurement as many corners of the tract in question as seems feasible. The locations of the corners thus determined are, of course, not without error. Often one corner may be reestablished by measurements from several different sources, each of which results in a different location. In cases of this kind the surveyor is called upon to exercise his judgment as to the most probable location of the original corner.

Sometimes it is possible to locate the position of an obliterated corner through evidences of previously existing lines such as fences and roads. Thus, if the surveyor has reason to believe that a fence once stood on the line, he may be able to find evidences of rotted posts in the ground. Differences in the ground surface, or even differences in vegetation along a definite line, are valuable clues.

Having thus tentatively fixed the location of one or more corners, the surveyor attempts to reconcile these locations with the description of the given tract, the resurvey being conducted somewhat as just described, and readjustments of the position of the tentatively located corners being made to conform to the judgment of the surveyor in light of the information he obtains as the survey progresses. Occasionally he may find it desirable to consult old settlers who were familiar with the original boundaries, but while such persons are usually very positive in their opinions, the information secured from such sources is seldom of much value and is frequently very misleading.

332d. When a resurvey has been completed, it is the duty of the surveyor to render a report to his client stating exactly what he found and what course of procedure he employed in attempting to reestablish missing corners. The report should be accompanied by a plat showing the observed lengths and directions of the sides of the tract and other data similar to that shown on the plat of an original survey (see Art. 331). In addition, it should indicate which are original monuments and which are monuments established at the time of the resurvey. Mistakes in the original description should be pointed out, but the surveyor should clearly understand that it is his function to reestablish boundaries of a given tract in as nearly as possible their original position.

333. Surveys for Subdivision of Rural Lands.—A subdivision survey implies a survey which is conducted for the purpose of subdividing into two or more tracts an area whose boundaries are already established, all in accordance with some prearranged plan. In the United States, surveys of this kind made in connection with rural lands may be classified as follows:

1. *U. S. Surveys of Public Lands.*—The public lands are divided into townships, sections, and quarter sections by United States land surveyors, in a manner prescribed by law. The work of surveying the public lands is carried out under the direction of the General Land Office. The system is described in Chap. XX.

2. *Surveys for Fractional Subdivision of Sections.*—Surveys of this class are conducted for the purpose of further subdividing lands by an extension of the U. S. system. In general the U. S. surveys

establish the boundaries of sections and establish quarter-section corners on section lines; and any further subdivision is made after the lands have passed into the hands of private individuals, the work being carried out by surveyors in private practice. Subdivisions of this kind are described in detail in Arts. 357 and 358.

3. *Surveys for Irregular Subdivisions.*—Surveys of this class are conducted for a variety of purposes. The following examples serve to illustrate the procedure for certain cases:

Example 1: A railroad is to traverse the land belonging to Black and the railroad company desires to secure title to a right-of-way of a definite width on either side of the center line which has already been surveyed and marked with stakes. A description of Black's tract has been secured.

The right-of-way surveyor reruns the boundaries of Black's tract that are intersected by the railroad line, establishes the directions of the right-of-way boundaries parallel with the center line, and sets monuments at the intersection of these boundaries with those of Black's tract. He then makes a survey of the tract thus defined, securing sufficient data so that the lengths and directions of the boundaries of the right-of-way tract now within Black's tract are obtained. He also ties right-of-way corners which he has established to the nearest old corners of Black's property.

With these data the area of the right-of-way tract is calculated, and a description of the tract is prepared as for an original survey (see Art. 330*a*). The point of beginning is referred to one of the old monuments marking the original tract, and not to the center line of the railroad.

Example 2: It is stipulated in the will of Green that his New England farm is to be divided equally among his three sons, each to have an equal frontage on the highway which forms one of the boundaries of the tract. The farm is of irregular shape and has not been surveyed for many years.

In a case of this kind, the surveyor first makes a resurvey of the entire tract, angles being measured probably to minutes, and distances being measured probably to tenths of feet. From the data thus obtained the area is computed. In connection with the resurvey, subdivision corners are established on the highway.

The simplest division is one for which the subdividing lines are straight, each cutting off the required area from the given tract. With the area of the entire tract known, the area each son is to receive is calculated, and the length and bearing of each of the lines rendering the subdivision is calculated as described in Art. 283, p. 411.

Finally, from each of the two subdivision corners already established on the highway frontage, the surveyor lays off the calculated direction of the subdividing line through that point, and establishes the remaining unknown corner at the point where this subdividing line intersects the opposite boundary. The distances from this latter corner to adjacent corners are measured, and the survey is considered as checked if these measured distances agree closely with the computed lengths of the same courses. A plat is drawn as for an original survey, the lengths and

bearings of all lines and the area of each subdivision being shown; and a description of each of the three subdivisions is prepared.

334. Surveys for Subdivision of Urban Lands.—As a city or town develops, unimproved lands are subdivided into lots which are placed on sale as residential or business property. In most instances such extensions are the result of the activities of real-estate operators who acquire a tract of rural land of considerable area, develop a plan of subdivision which is approved by the authorities of the municipality to which the tract is to be attached, and cause surveys to be made for the purpose of establishing the boundaries of individual lots. A tract thus divided according to an acceptable plan is known as an *addition* or *subdivision.*

For large and important developments the work of originating the general plan is often carried out by persons specializing in city planning and landscape architecture, under whose direction the surveyor works. Such developments require a great deal of care and skill on the part of the designer, and usually extensive surveys (particularly in hilly sections) are carried out before the actual plan of subdivision can be decided upon. Problems of this character can be adequately discussed only in treatises on city planning, to which the reader is referred, but it is perhaps appropriate to state that the preliminary studies should consider the probable future character of the district, the probable location of business sections, the possible magnitude, direction, and character of future traffic, the topography of the land, the location, width, grade, and character of paving of streets, the size and shape of lots and blocks, the location and size of storm and sanitary sewers, and the disposition of electric and telephone wires and cables.

For the ordinary real-estate development the owner usually calls for the services of an engineer or surveyor who has had experience in such work. The surveyor confers with the owner and they discuss a general plan. The surveyor makes a resurvey of the entire property, and if the character of the topography is irregular, he usually makes certain preliminary surveys for the purposes of finding the location and elevation of the governing features of the terrain. In some cases a complete topographic survey may be made. With the general plan fixed, and having studied the results of the field investigation and having considered the items listed in the previous paragraph, the surveyor works out a detailed plan on paper, showing on the drawing the names of all streets and the numbers of all blocks and lots, the dimensions of all lots, the width of streets, the length and bearing of all street tangents, and the radius and length of all street curves. He also prepares a report which, in addition to a

discussion of the plan of subdivision, may consider the cost of subdividing, including not only the establishing of boundaries but also the work of grading, paving, constructing sewers, and landscaping.

This detailed plan, when approved by the owner, is submitted to the governing body in the municipality. If it meets with the requirements of this body, it is approved.

Upon the authority of the owner, the surveyor then proceeds to execute the necessary subdivision surveys, including the laying out of roads, walks, blocks, and lots. Often the lot and block corners are marked with permanent monuments, but in many cases, contrary to what may be considered good practice, the lot corners are marked by wooden stakes. When the surveys are completed, the map of the subdivision is revised to show minor changes made during the survey, together with location and character of permanent monuments. A tracing is submitted to the municipality which tracing, when duly signed by those in authority, becomes the official map of the subdivision. It then becomes a part of the public records and is usually filed in the Registry of Deeds of the county in which the municipality lies. Upon this approval, if the subdivision is outside the corporate limits of the municipality, they are extended to include it.

335. Description of Urban Lands.—The manner of legally describing the boundaries of a tract of land falling within the corporate limits of a city or town depends upon conditions attached to the surveys by which the boundaries of the tract were first established, as indicated by the following classifications:

1. *By Lot and Block.*—When the boundaries of the tract exactly conform with a lot which is a part of a subdivision or addition for which there is recorded an official map, a statement giving the lot and block numbers and the name and date of filing of the official map constitutes a definite and legal description of the tract. Most city property is described in this way. Following is a description of this character occurring in a deed:

"Lot 15 in Block No. 5 as said lots and blocks are delineated and so designated upon that certain map entitled *Map of Thousand Oaks, Alameda County, California,* filed August 23, 1909, in Liber 25 of Maps, page 2, in the office of the County Recorder of the said County of Alameda."

2. *By Metes, Bounds, and Lots.*—When the boundaries of a given tract within a subdivision for which there is a recorded plot do not exactly conform to boundaries shown on the official plot, the tract is described by metes and bounds, with the point of beginning referred to a corner shown on the official plat. The numbers of lots of which

the tract is composed are also given. The following is an example of a description of this kind:

"Beginning at the intersection of the Northern line of Escondido Avenue, with the Eastern boundary line of Lot No. 16, hereinafter referred to; running thence Northerly along said Eastern boundary line of Lot 16, and the Eastern boundary line of Lot 17, eighty-nine (89) feet; thence at right angles Westerly, fifty-one (51) feet; thence South 12°6′ East, seventy-five (75) feet to the Northern line of Escondido Avenue; thence Easterly along said line of Escondido Avenue, fifty-three and $^{13}/_{100}$ (53.13) feet, more or less, to the point of beginning.

"Being a portion of Lots 16 and 17, in Block No. 5, as said lots and block are delineated and so designated upon that certain Map entitled *Map of Thousand Oaks, Alameda County, California*, filed August 23, 1909, in Liber 25 of Maps, page 2, in the office of the County Recorder of the said County of Alameda."

3. *By Metes and Bounds to City Monuments.*—A number of the larger and older cities of the United States have, by precise surveys, established an elaborate system of reference monuments and have determined their coordinates with respect to some arbitrarily assumed initial point. When the tract cannot be defined by descriptions such as the preceding, the point of beginning may be definitely fixed by stating its direction and distance from an official reference monument and by describing the character of the monument by which the point of beginning is marked. The boundaries of the tract may then be described by metes and bounds.

The location of corners may also be defined by rectangular coordinates referred to the origin or initial point of the system.

When the tract is within a city or town not so monumented, the point of beginning of the boundary description may be referred by direction and distance to the intersection of the center lines of streets. It is not good practice to refer to the intersection of sidewalk or curb lines, for these are apt to be changed from time to time. In sections of the country within the rectangular system of U. S. surveys, the point of beginning of a boundary description may properly be referred to section lines and corners.

336. City Surveying.—It has been stated (Art. 8) that the term *city surveying* is sometimes applied to the surveying operations within a municipality with regard to mapping its area, laying out new streets and lots, and constructing streets, sewers, buildings, etc. While the principles of city surveying are not different from those of ordinary surveying, there are some differences in the details of the methods employed. Some features which are pertinent to urban surveying are as follows:

1. Measurements are made with a greater degree of refinement than for land of less value.
2. Some cities maintain a standard of length with which tapes may be compared.
3. Usually the control of the survey for the map of a city is by triangulation (Chap. XXVIII) rather than by traversing, which would be employed for an equal area outside of the city.
4. A system of reference points and bench marks is established at points a few blocks apart (usually at street intersections) and this system is preferably tied in with the United States precise surveys. Points are located either in the street, at the curb, or on the sidewalk, one such point being sufficient for each chosen intersection. (For a description of monuments and reference marks, see Art. 328.) In subsequent surveys, it is good practice to tie in to more than one of these established points, as monuments may have been moved.
5. The established points are well referenced (see Art. 232) to more or less permanent objects such as building corners, curb or walk lines, centers of street intersections, and manhole covers. In undeveloped districts, these points are referenced to stakes.
6. Maps showing the location of proposed sewers, street extensions, and other improvements usually show, to scale and in figures, the exact dimensions of adjacent lots and of all other lots that will be benefited by, or assessed for, the proposed improvement.
7. Sometimes separate maps are made of surface and underground utilities such as car lines, sewers, water lines, gas lines, electric power and telephone lines and conduits, tunnels, etc., both for convenient reference and in order to avoid interference in the location of new projects.
8. In connection with lawsuits regarding automobile or train collisions, falls, injuries during construction, and other accidents, the surveyor is sometimes called on to prepare a large-scale map for exhibit in the courtroom. While the map for this purpose should be extremely simple in character, it should include all details that might have a bearing upon the accident. Some of these details may be the grade and crown of the roadway; height of curb; depressions; location (at time of accident) of obstructions to traffic or to view such as trees, poles, signs, and parked automobiles; sources of light (if at night); and location of points (in plan and elevation) from which it is stated that the accident has been witnessed. Colors are sometimes employed to make the various features more intelligible to the layman.

The subdivision of urban lands is discussed in Art. 334, and typical descriptions of urban lands are given in Art. 335. The usual methods of keeping records are described in Art. 329.

The operations of setting grades for buildings, sewers, pipe lines, pavements, and railroad tracks are described in Art. 141, and the building-site survey in Art. 473. Some details of running lines and

locating details, pertinent to urban surveys, are given in Arts. 202, 203 and 233*a*.

Further details with regard to the width of streets, sizes of blocks and lots, location of utilities, etc. are to be found in texts on city planning, highway engineering, and sanitary engineering (see Refs., p. 523).

337. Cadastral Surveying.—Cadastral surveying, as defined in Art. 8, includes (1) the establishment over the area of the city of a system of reference marks for control and for use in future surveys, (2) the accurate location on the ground of property lines, and (3) the securing of necessary data for an intermediate- or large-scale map.

The surveying methods (including the control) for intermediate- and large-scale maps are described in Chap. XXV. An excellent system of locating all points and property lines is by reference to a rectangular system of coordinates similar to that described in the examples of Arts. 452*a* and 454*a*. Such a system is well adapted to the use of sectional sheets for the map. Further, if the rectangular coordinates of the survey points are recorded, the work of restoring lost and obliterated corners is facilitated.

338. Legal Terms.—Following are definitions, quoted from Bouvier's "Law Dictionary," (Ref. 1, p. 523) of a few of the more common legal terms having to do with the conveyance of landed property.

Adverse Possession.—The enjoyment of land, under such circumstances as indicate that such enjoyment has been commenced and continued under an assertion of right on the part of the possessor, is adverse possession.

When such possession has been actual, and has been adverse for twenty years, the law raises the presumption of a grant.

Where one enters into possession of real property by permission of the owner, without any tendency whatever being created, possession being given as a mere matter of favor, he can never acquire title by adverse possession, no matter how long continued against the true owner thereof.

The adverse possession must be "actual, continued, visible, notorious, distinct, and hostile."

The title by adverse possession for such a period as is required by statute to bar an action, is a fee-simple title, and is as effective as any otherwise acquired.

Alluvium.—That increase of earth on a bank of a river, or on the shore of the sea, by the force of the water, as by a current or by waves, or from the recession of water in a navigable lake, which is so gradual that no one can judge how much is added at each moment of time, is known as alluvium. The proprietor of the bank which is increased by alluvium

is entitled to the addition, this being regarded as the equivalent for the loss he may sustain from the encroachment of the waters upon his land.

Avulsion.—The removal of a considerable quantity of soil from the land of one man and its deposit upon or annexation to the land of another, suddenly and by the perceptible action of water, is avulsion. In such case the property belongs to the first owner. Avulsion by the Missouri River, the middle of whose channel forms the boundary line between the states of Missouri and Nebraska, works no change in such boundary, but leaves it in the center line of the old channel.

Color of Title.—Color of title, for the purposes of adverse possession under the statute of limitations, is that which has the semblance or appearance of title, legal or equitable, but which in fact is no title.

A writing which upon its face professes to pass title but which does not in fact do so, either from a want of title in the person making it or from the defective conveyance used, is also known as color of title. The term is also applied to a title that is imperfect, but not so obviously that it would be apparent to one not skilled in the law.

Fee.—The word fee signifies that the land or other subject of property belongs to its owner and is transmissible, in the case of an individual, to those whom the law appoints to succeed him under the appellation of heirs.

Fee Simple.—An estate of inheritance is a fee simple. The word "simple" adds no meaning to the word fee standing by itself. But it excludes all qualifications or restrictions as to the persons who may inherit it as heirs.

High-water Mark.—The high-water mark is wherever the presence of the water is so common as to mark on the soil a character, in respect to vegetation, distinct from that of the banks; it does not include low lands which, though subject to periodical overflow, are valuable for agricultural purposes.

That part of the shore of the sea to which the waves ordinarily reach when the tide is at its highest is also known as the high-water mark.

Low-water Mark.—Low-water mark is that part of the shore of the sea to which the waters recede when the tide is lowest, *i.e.*, the line to which the ebb tide usually recedes; or it is the ordinary low-water mark unaffected by drought. It has been said to be the point to which a river recedes at its lowest stage.

Parol.—Parol is a term which is used to distinguish contracts which are made verbally, or in writing not under seal, which are called parol contracts, as distinguished from those which are under seal, which bear the name of deeds or specialties.

Patent.—A patent is the title deed by which a government, either state or federal, conveys its lands.

Reliction.—An increase of the land by the retreat or recession of the sea or a river is known as reliction.

339. Legal Interpretation of Deed Description.—As indicated in the preceding pages, the descriptions of the boundaries of a tract

include the objects which fix the corners, the lengths and directions of lines between the corners, and the area of the tract. It frequently happens that a deed description will contain errors of measurement or calculation, or mistakes of record, thus introducing inconsistencies which can not be harmonized completely when retracement becomes necessary.

In such cases, where uncertainty has arisen as to the location of property lines, it is a universal principle of law that the endeavor is to make the deed effectual rather than void, and to execute the intentions of the contracting parties. The following general rules are pursuant of this principle.

1. *Monuments.*—It is presumed that the visible objects which marked the corners when a conveyance of ownership was made, indicated best the intentions of the parties concerned; hence it is agreed that a corner is established by an existing material object or by conclusive evidence as to the previous location of the object. A corner thus established will prevail against all other conflicting evidence, whatsoever may be its character. The kinds of evidence which are valid in relocating obliterated corners are stated in Art. 365, p. 555.

2. *Calls for Distance and Direction.*—In the case of discord between the described courses and the calculated area, deed-description calls for distances or directions of courses will prevail against the call for area of a tract, again on the assumption that the boundary lines are more visible and actual evidence of the intentions of the parties than is the calculated area of the tract.

3. *Mistakes.*—It is a well-established principle that a deed description which taken as a whole plainly indicates the intentions of the parties concerned will not be invalidated by evident mistakes or omissions. For example, such obvious mistakes as the omission of a full tape length in a dimension, or the transposition of the words *Northeast* for *Northwest* will have no effect on the validity of a description, provided it is otherwise complete and consistent, or provided its intention is plainly manifest.

4. *Purchaser Is Favored.*—In the case of a description which is capable of two or more interpretations, that one will prevail which favors the purchaser.

5. *Ownership of Highways.*—Land described as being bounded by a street or highway conveys ownership to the center of the street or highway. Any variation from this interpretation must be explicitly stated in the description.

6. *Original Government Surveys Are Presumed Correct.*—Errors found in original surveys will in no way affect the location of the

boundaries established under those surveys. In other words, no errors can be attributed to original government surveys, and the boundaries remain fixed as originally established.

340. Riparian Rights.—An owner of property which borders on a body of water is a riparian proprietor and has riparian rights which in many cases are extremely valuable. Because of the value of these rights, and because of the difficulties arising from the irregularity of such boundaries, it is important that the surveyor should be familiar with the general principles relating thereto.

Many of these principles receive different interpretations in the various states, and each state is sovereign in the matter of these interpretations, hence the surveyor should know the statutes and precedents established in his particular state. For example, as regards the ownership of the bed of a navigable river, the two states of Iowa and Illinois bordering on the same river have very different laws. Clark (Ref. 3, p. 523) states: "It is a rule of property in Illinois, that the fee of the riparian owner of land in that state bordering on the Mississippi River extends to the middle line of the main channel of the river," whereas, the Iowa courts hold "that the bed of the Mississippi River and the banks to the high-water mark belong to the state, and that the title of the riparian proprietor extends only to that line."

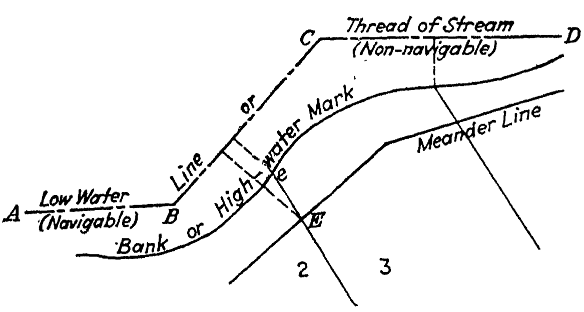


FIG. 340*a*.

340a. Meander Lines.—In public-land surveys of the United States where regular corners fall in water, traverses called *meander lines* are run roughly following the bank of stream or shore of lake. It is a well-established principle that government patents of land bordering on meandered streams or lakes convey ownership, not to the meander line, but to the thread of a non-navigable stream, or to the bank of a navigable stream, or to the shore of a lake. Hence, meander lines are not property lines and riparian rights are limited in no way by the meander lines. Of course, if it is specified in

the deed as such, a meander line may be a property line, but not otherwise.

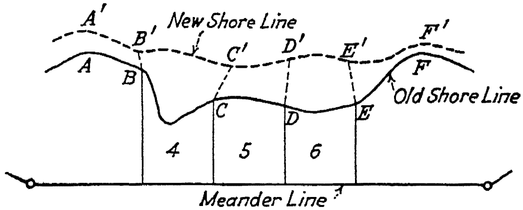


FIG. 340*b*.

340b. Establishment of Property Lines of Riparian Owners.—In establishing the property lines of riparian owners many dissimilar

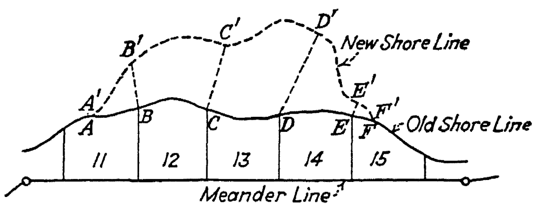


FIG. 340*c*.

and complex situations are encountered, but the principles which usually apply are stated below under five general cases.

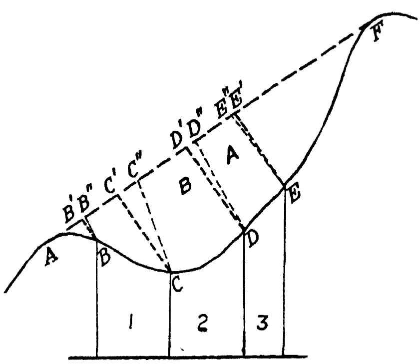


FIG. 340*d*.

1. *Origin of Dividing Lines.*—There are two opposing lines of decisions; under one it is held that a dividing line has its origin at the high-water line of a river, or at the shore line of a lake, and not at the meander line. Thus in Fig. 340*a*, the dividing line between lots 2 and 3 would be made perpendicular to the thread of the stream,

beginning at e (on the high-water line) and not at E (on the meander line). Under the other line of decisions, the reverse is held.

2. *Alluvium and Reliction.*—The direction of the property lines dividing areas created by alluvium or by reliction is determined by the proportional lengths of the old and of the new shore lines. The extremities of these lines are fixed either by definite bends, as A, F, A', and F' (Fig. 340*b*) or by the intersections of the old and new lines, A, F (Fig. 340*c*). The general rule is to measure along the old shore line between the old extremities (as A and F); measure along

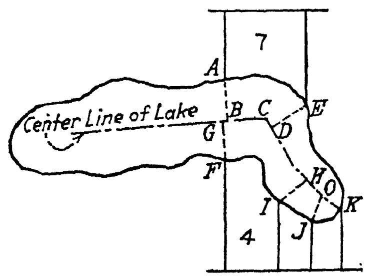


Fig. 340*e*.

the new shore line between the new extremities (as A' and F'); divide the new line ($A'F'$) into parts proportional in length to those of the old line (AF). Thus for lots 4 and 12 by proportion $\frac{B'C'}{BC} = \frac{A'F'}{AF}$. The area $BB'C'C$ represents the area added by alluvium to lots 4 and 12.

3. *Bays or Coves.*—Property lines fixing riparian rights in bays or coves sometimes are established by lines beginning at the extremities of the property lines on shore, and having a direction perpendicular to a line connecting the adjacent headlands of the bay or cove. Thus the lines BB', CC', etc., for lots 1, 2, and 3 (Fig. 340*d*) are established perpendicular to the line AF, which connects the two headlands A and F.

Other court decisions have fixed the lines according to the following rule: Divide the straight line joining the headlands (AF in Fig. 340*d*) into parts proportional to the lengths of the shore line held by each owner; the property line of inundated land is determined by joining the extremities of the property lines on shore and the corresponding points of subdivision on the line between headlands. These are shown in the figure by lines BB'', CC'', etc.

4. *Streams and Rivers.*—The lines fixing the riparian rights of owners of property bordering on streams and rivers are established

by lines perpendicular to the thread of the stream if non-navigable, or to the low-water line (sometimes to the middle of the channel) if navigable. Thus the lines for lots 2 and 3 of Fig. 340*a*, are established perpendicular to the line *ABCD*.

5. *Lakes.*—In the case of lakes, the riparian property lines are established perpendicular to the center line of the lake; or in the case of a circular shore line, by lines to the center of the lake. Thus in Fig. 340*e*, the lines for lot 7 are established by the boundary *ABCDE*, and for lot 4 by the boundary *FGCHI*. Where the shore line is circular at the end of the lake, the land lines terminating at *J* and *K* are drawn to *O*, the center of the circular shore line.

341. Adverse Possession.—The many legal aspects of adverse possession cannot be treated in this brief article, but it is desirable to direct the attention of the surveyor to the important fact that property lines are sometimes fixed by possession and use of the land as against original survey boundaries. The conditions and the period of time necessary to gain title are fixed by statute in the various states.

According to the definition given in Art. 338, adverse possession, to become effective, must be plainly evident to the owner and hostile to his interests. Such possession may be evidenced by fencing, cultivation, erection of buildings, etc., and must be held without the permission of and to the exclusion of the owner.

Right to title by adverse possession may be acquired by individuals, corporations, and even by the state. But the statute does not run *against* the state; *i.e.*, property in a street or highway cannot be acquired by adverse possession.

Under this principle, if a person should use the land up to a fence, and should recognize it as a boundary line, to the exclusion of the owner, for the statutory period, the fence then becomes the legal property line even though it may be shown later that it is not on the true and original line. However, if the possession of the land has not been held adversely, *i.e.*, to the exclusion of the owner, and if the fence has merely served the convenience of the persons concerned, both parties recognizing that it was probably not on the true line, title cannot be claimed.

It is therefore clear that the application of the principle of adverse possession is entirely a matter of intention and belief. If land is held openly and notoriously with the intent to acquire title, or with the belief that the occupation is proper and right, then title will be granted if and when the statutory requirements are fulfilled. But if by parol agreement or by actions it is manifest that the parties concerned had no intention to occupy beyond the true line, at the same

time knowing that the location of the true line was uncertain, then title cannot be gained adversely.

Adverse possession under "color of title" will "ripen into title" under the statute of limitations in some jurisdictions in half the time required without color of title; for example, if title may be gained without color of title in 20 years, it may be gained in 10 years with it.

342. Legal Authority of the Surveyor.—Often resurveys are run to settle controversies between adjoining property owners. The surveyor should understand that while he may act as an arbiter in such cases, it is not within his power to fix boundaries without the mutual consent and authority of all interested parties. In the event of a dispute involving court action he may present evidence and argument as to the proper location of a boundary, but he has no authority to establish such a boundary against the wishes of either party concerned. A competent surveyor by his wise counsel will prevent most litigation; but if he cannot bring his clients to an agreement, the boundaries in dispute become valid and defined only by a decision of the court. In boundary disputes the surveyor is therefore an expert witness but not a judge.

343. Liability of the Surveyor.—It has been held in court decisions that a surveyor is a member of a learned profession and may be held liable for incompetent services rendered. This principle applies both to a county surveyor and to any person engaged in private practice. Thus Clark (Ref. 3, p. 523), quoting from court decisions, states: "If a surveyor is notified of the nature of a building to be erected on a lot, he may be held liable for all damages resulting from an erroneous survey; and he may not plead in his defense that the survey was not guaranteed." Similarly, it has been held that in any case where the surveyor is cognizant of the purpose for which the survey is made, he is liable for damages resulting from incompetent work.

The general principle invoked in such cases is that the surveyor is bound to exhibit that degree of prudence, judgment, and skill which may reasonably be expected of a member of his profession. Thus in the following quotations from Clark (Ref. 3, p. 523) a Connecticut court says "the gist of the plaintiff's cause of action was the negligence of the defendant in his employment as a civil engineer. Having accepted that service from the plaintiff, the defendant . . . was bound to exercise that degree of care which a skilled civil engineer would have exercised under similar circumstances." Also, a Kansas court declares, "reasonable care and skill is the measure of the obligation created by the implied contract of a surgeon, lawyer, or any other professional practitioner." But Ruling Case Law says, "yet a

person undertaking to make a survey does not insure the correctness of his work, nor is absolute correctness the test of the amount of skill the law requires. Reasonable care, honesty, and a reasonable amount of skill are all he is bound to bring to the discharge of his duties."

References

1. Bouvier's "Law Dictionary," Rawle's Third Edition.
2. Cautley, R. W., "Descriptions of Land," The Macmillan Company, New York, 1913.
3. Clark, F. E., "Law of Surveying and Boundaries," Bobbs-Merrill Co., Indianapolis, 1922.
4. Hodgman, F., "Land Surveying," The F. Hodgman Co., Climax, Michigan, 1904.
5. Holt, A. H., "The Surveyor and His Legal Equipment," *Proc. Am. Soc. Civ. Engr.*, Vol. 59, No. 7, September 1933, pp. 1105–1119, and succeeding discussion.
6. Robinson, C. M., "City Planning, with Special Reference to the Planning of Streets and Lots," G. P. Putnam's Sons, New York, 1916.
7. Skelton, Ray H., "Legal Elements of Boundaries and Adjacent Properties," Bobbs-Merrill Co., Indianapolis, 1931.
8. Williams, B. F., "Relocating Land Surveys in Texas," *Trans. Am. Soc. Civ. Engr.*, Vol. 96, pp. 735–743, 1932.
9. "Municipal Field Engineering," *Proc. 38th Convention Am. Soc. Mun. Engr.*, 1932.

CHAPTER XX

UNITED STATES PUBLIC-LAND SURVEYS

344. General.—This chapter deals with the methods of subdividing the Public Lands of the United States in accordance with regulations imposed by law and carried out by the Field Surveying Service of the General Land Office. The methods described are those now in force, but with minor modifications these methods have in principle been followed in all the surveys of the public domain since the rectangular system of subdivision was inaugurated. Under this system the public lands of 29 states and the Territory of Alaska have been or are in progress of being surveyed (Art. 346). In general, these methods of subdividing land do not apply in the thirteen original states and in Kentucky, Tennessee, and Texas; further, since the progress of the public-land surveys has been from east to west, the details of the surveys in states east of the Mississippi River differ somewhat from those of present practice.

The laws regulating the subdivision of public lands and the surveying methods employed are fully described in the "Manual of Instructions for the Survey of the Public Lands of the United States" (1930), published by the General Land Office,[1] from which the material for this chapter is freely drawn.

345. Laws Relating to Public-land Surveys.—Beginning with an ordinance passed by the Continental Congress in May, 1785, which provided for townships 6 miles square, each containing 36 sections 1 mile square, laws regulating the surveying, marking, and disposal of the public lands of the United States have from time to time been enacted by Congress. Following are the provisions of the Public Land Laws in which the surveyor is principally interested:

1. All responsibility for the surveying and sale of the public lands of the United States is placed in the hands of the Commissioner of the General Land Office, who under the direction of the Secretary of the Interior is authorized to carry into execution every part of the Public Land Laws not otherwise specially provided for.
2. When the surveys and records of a state are completed, all the field notes, maps, and records pertaining to land titles are delivered to the Secretary of State of that state.

[1] Superintendent of Documents, Washington, D. C., $1.50.

3. Any agent of the United States, acting upon the authority of the Commissioner of the General Land Office, has free access to public records delivered to any State, but no transfer of such records is made to any State until the State has enacted legislation providing for the safekeeping of such records and for the allowance of free access thereto by authorities of the United States.

4. It is required that all surveys and resurveys of public lands under the supervision of the Commissioner of the General Land Office are to be made by surveyors selected by the General Land Office. (Prior to 1910 surveys were made by contract.) The field work is now performed by a permanent corps of engineers under civil-service regulations.

5. It is provided that resurveys may be made by the Government under certain conditions.

6. Boundaries of public lands, when established by duly authorized surveyors and when approved by the Commissioner, are unchangeable.

7. The original corners established by the surveyors stand as the true corners they were intended to represent, whether in the place shown by the field notes or not.

8. The unit of length is the 66-ft. or Gunter's chain divided into 100 links.

9. Quarter-quarter-section corners not established by the original surveys are to be on the line joining the section and quarter-section corners and midway between them, except in the northern and western half miles of the township.

10. The center lines of sections are to be straight between opposite quarter-section corners.

11. In a fractional section where no opposite quarter-section corner has been or can be established, the center line of such section is to be run from the proper quarter-section corner as nearly in a cardinal direction as due parallelism with section lines will permit to the meander line, reservation, or other boundary of such fractional section.

12. Lost or obliterated corners of the approved surveys are to be restored to their original location, if possible.

346. Historical Notes.—The first surveys of the public lands of the United States, made under the ordinance of May 1785, divided lands north of the Ohio River. Only the exterior lines of the townships were run, but section corners were established at intervals of 1 mile on the township lines, and the plats were marked into subdivisions 1 mile square. These surveys were made under the direction of the Geographer of the United States.

The act of Congress of May 1796 provided for a surveyor general and directed the survey of lands northwest of the Ohio River and above the mouth of the Kentucky River. Under this law it was provided that the sections be numbered according to the plan in operation at the present time. The townships surveyed were divided into sections, "by running through the same, each way, parallel lines at the end of every 2 miles" and by marking corners at intervals of 1 mile on each of such lines.

In 1800 an act of Congress provided for the subdivision of lands into half sections by running parallel lines through the townships from east to west and from north to south at intervals of 1 mile, and the marking of corners at distances of ½ mile on lines running from east to west and 1 mile on lines running from south to north. This act also required that excesses or deficiencies in measurement should be placed in the sections or half sections in the most northerly or westerly half miles of each township.

In 1805 an act of Congress directed that the public lands should be divided into quarter sections, and provided that all corners marked in the public surveys should be established as the proper corners which they were intended to designate, and that corners of half and quarter sections should be placed as nearly as possible equidistant from the two adjacent section corners on the same line.

The General Land Office was established in 1812 as a branch of the Treasury Department, and the office of Commissioner of the General Land Office was created.

In 1820 an act of Congress provided for the sale of public lands in half-quarter sections and required that the line of division of the quarter section should in every case run north and south.

In 1832 an act of Congress directed the subdivision of the public lands into quarter-quarter sections and required that the line of division of the half-quarter section should in every case run east and west. This act also provided that fractional sections be subdivided in accordance with regulations prescribed by the Secretary of the Treasury.

In 1849 the Department of the Interior was created and the control of the General Land Office was transferred from the Department of the Treasury to the Department of the Interior, where it still remains.

By act of Congress in 1909 it was provided that resurveys may be made at the discretion of the Secretary of the Interior, if such resurveys are essential to mark properly the boundaries of the public lands previously surveyed but remaining undisposed of, provided such resurvey shall not be so executed as to impair the rights of entrymen or owners of lands affected. By act of Congress in 1918, resurveys may be made of public lands which are in private ownership upon application of the owners of three fourths of the privately owned lands in any township covered by public-land surveys, when more than fifty per cent of the area of such township is privately owned, provided there be deposited a sum equal to the estimated cost of the resurveys. Any portion of the deposit which may remain after the work is completed is repaid pro rata to the persons making the deposit.

In 1925, the office of surveyor general of the several districts was abolished, and all activities were transferred to the Field Surveying Service, under the jurisdiction of the U. S. Supervisor of Surveys.

Under the regulations imposed by Congress surveys of the public lands have been completed, or practically so, in the States of Alabama, Arkansas, Florida, Illinois, Indiana, Iowa, Kansas, Louisiana, Michi-

gan, Minnesota, Mississippi, Missouri, Nebraska, North Dakota, Oklahoma, Ohio, South Dakota, and Wisconsin. The survey records and plats have been transferred to the respective States except those for lands in Oklahoma which are on file in the General Land Office, Washington, D. C.

Surveys of the public lands are still in progress (1930) in the States of Arizona, California, Colorado, Idaho, Montana, Nevada, New Mexico, Oregon, Utah, Washington, and Wyoming, and in the Territory of Alaska.

It must be kept in mind that the early surveys were made under a contract system, with crude instruments and often under unfavorable field conditions; hence, often the lines and corners will be found to depart from their theoretical positions. However, the original corners legally stand as the true corners, and the surveyor must be guided by them in making resurveys or subdivisions, regardless of irregularities in the original survey.

347. General Scheme of Subdivision.—The preceding article makes it evident that the regulations for the subdivision of public lands have been altered from time to time in the years since 1785, when the rectangular system was inaugurated; hence, the methods employed in surveying various regions of the United States show marked differences, depending upon the dates when the surveys were made. In general principle, however, the system has remained unchanged, the primary unit being the township, bounded by meridional and latitudinal lines, as nearly as may be 6 miles square, divided into 36 secondary units, called *sections*, as nearly as may be 1 mile square. Obviously, since the meridians converge, it is impossible to lay out a square township by such lines, and having a township which is not square, it is impossible that each of the 36 sections contained therein should be 1 mile square even though all measurements are without error.

Since the time of the earliest surveys, townships and sections have been located with respect to principal axes passing through an origin or an *initial point*, as it is called, the lines which form the axes consisting of a true meridian, called the *principal meridian*, and a true parallel of latitude, called the *base line*.

The principal meridian is given a name to which all subdivisions are referred. Thus the principal meridian which governs the rectangular surveys (wholly or in part) of the states of Ohio and Indiana is called the First Principal Meridian; its longitude is 84°48′50″ and the latitude of the base line is 41°00′00″. The extent of the surveys which are referred to a given initial point may be found by consulting a map, published by the General Land Office, entitled

"United States, Showing Principal Meridians, Base Lines, and Areas Governed Thereby."

Secondary axes are established at intervals of 24 miles north or south of the base line and at intervals of 24 miles east or west of the principal meridian, thus dividing the tract being surveyed into quadrangles bounded by true meridians 24 miles long and by true parallels, the south boundary of each quadrangle being 24 miles long and the north boundary being 24 miles long less the convergency of the meridians in that distance. These secondary parallels are called *standard parallels* or *correction lines* and each is continuous throughout its length. The secondary meridians are called *guide meridians*, and each is broken at the base line and at each standard parallel.

A typical system of principal and secondary axes is shown in Fig. 347*a*. Both base line and standard parallels, being everywhere perpendicular to the direction of the meridian, are laid out on the ground as curved lines, the rate of curvature depending upon the latitude. The principal meridian and guide meridians, being true north and south lines, are laid out as straight lines but converge toward the north, the rate of convergency depending upon the latitude.

Standard parallels are counted north or south of the base line. Thus the *second standard parallel south* indicates a parallel 48 miles south of the base line. Guide meridians are counted east or west of the principal meridian. Thus the *third guide meridian west* is 72 miles west of the principal meridian.

347a. The division of the quadrangles created by the standard parallels and guide meridians into townships is accomplished by laying off true meridional lines called *range lines* at intervals of 6 miles along each standard parallel, the range line extending north 24 miles to the next standard parallel; and by joining the township corners established at intervals of 6 miles on the range lines, guide meridians, and principal meridian with latitudinal lines called *township lines*.

A row of townships extending north and south is called a *range;* and a row extending east and west is called a *tier*. Ranges are counted east or west of the principal meridian, and tiers are counted north or south of the base line. Usually for purposes of description the word "tier" is omitted and "township" is substituted therefor. The plan of subdivision is illustrated by Fig. 347*b*. A township is designated by the number of its tier and range, and the name of the principal meridian.

Thus, T7S, R7W is read *Township seven south, Range seven west,* and designates a township in the seventh tier south of the base line and the

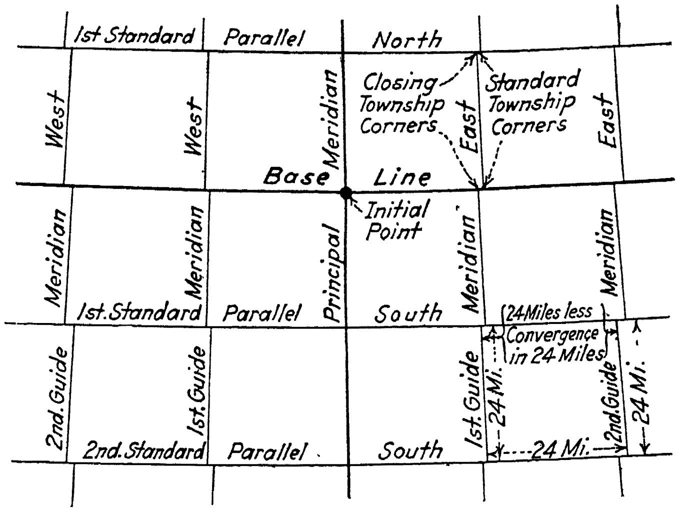


FIG. 347*a*.—Guide meridians and standard parallels.

seventh range west of the principal meridian. If the township were among those referred to the initial point of the Third Principal Meridian,

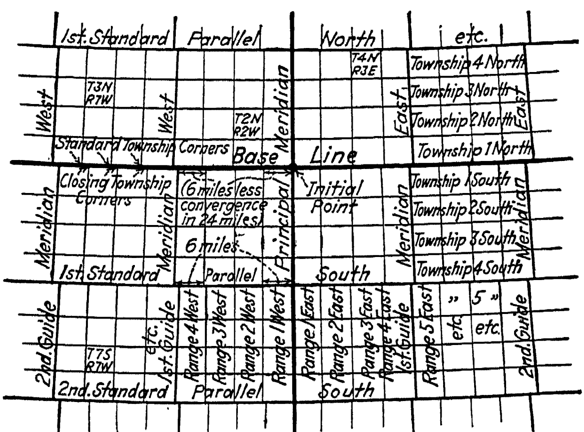


FIG. 347*b*.—Township and range lines.

for example, the location of the township would be completely described by the designation T7S, R7W of the Third Principal Meridian.

347b. The division of townships into sections is performed by establishing, at intervals of 1 mile, meridional lines[1] parallel with the east boundary of the township and by joining the section corners established at intervals of a mile with straight latitudinal lines. These lines, called *section lines,* divide each township into 36 sections. The plan of designating the sections in each township, illustrated by Fig. 347*c*, consists in numbering them consecutively from east to west and from west to east, beginning with No. 1 in the northeast corner and ending with No. 36 in the southeast corner. Thus Section 16 is a section whose center is 3½ miles north and 3½ miles west of the southeast corner of a township.

A section is legally described by giving its number, the tier and range of the township in which the section is located, and the name of the principal meridian to which the surveys in the given locality are referred. For example, the location of a section is defined by the description *Section* 16, T7S, R7W, *of the Third Principal Meridian.*

On account of the convergency of the range lines (true meridians) forming the east and west boundaries of townships, the north and south boundaries are in general less than 6 miles in length, exceptions being the south boundaries of townships which lie just north of a standard parallel or correction line. Since the meridional section lines are run parallel to the east boundary of the township, it follows that, if the surveys be without error, all sections except those adjacent to the west boundary will be 1 mile square, but that those adjacent to the west boundary will have a latitudinal dimension less than 1 mile by an amount equal to the convergency of the range lines within the distance from the section to the nearest correction line to the south.

347c. The principal meridian, base line, standard parallels, and guide meridians are called *standard lines.* Corners called *standard corners* are established on the base line and standard parallels at intervals of 40 chains; these standard corners govern the meridional subdivision of the land lying between each standard parallel and the next standard parallel to the north. Other corners called *correction corners* or *closing corners* are later established on the base line and standard parallels during the process of subdivision; these corners fall at the intersection of the base line or standard parallel either with the meridional lines projected from the standard township corners of the next standard parallel to the south (see Fig. 347*b*) or

[1] Strictly speaking, these lines are not meridional, but they are parallel to the east boundary of the township, which is a meridional line.

with the intermediate section and quarter-section lines. Standard parallels are also called *correction lines.*

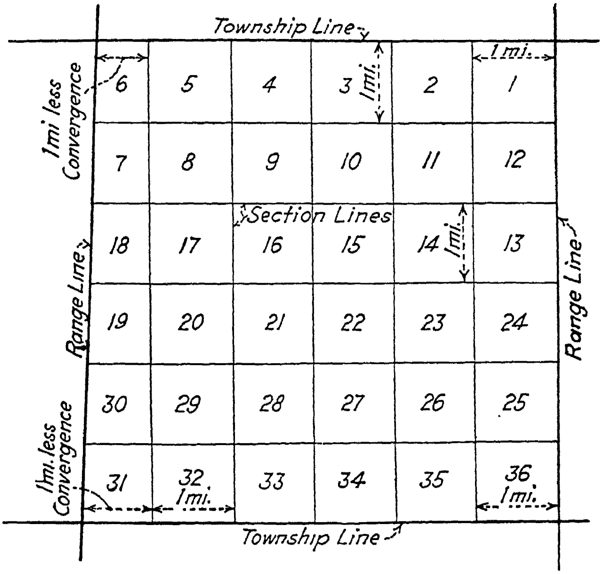


FIG. 347c.—Numbering of sections.

347d. It should be understood that the plan of subdivision just described is the one which is carried out when conditions allow. There are, of course, always present the errors of measurement, so that the actual lengths and directions established in the field do not entirely agree with the theoretical values. But in addition, conditions met in the field often make it inexpedient or impossible to establish the lines of the survey in exact accordance with the specified plan. Thus there are numerous instances of standard parallels and guide meridians having been originally established at intervals of 30 and 36 miles, under old regulations, and regions having been partially surveyed. Later, under present regulations, meridians have been established between the old guide meridians, and recent subdivisions are, hence, referred to standard lines many of which are less than 24 miles apart. Also the presence of large bodies of water, mountain ranges, Indian reservations, etc., may greatly modify the method of division, many townships and sections being made fractional.

348. Principal Meridian.—The principal meridian is established as a true meridian through the initial point, either north or south, or

in both directions, as conditions require. Permanent quarter-section and section corners are established alternately at intervals of 40 chains (½ mile) and regular township corners are placed at intervals of 480 chains (6 miles).

Independent linear measurements are taken by two sets of chainmen, or when this is not possible, by the duplication of each measurement by one set of chainmen. When the discrepancy between two sets of measurements taken in the prescribed manner exceeds 20 links per mile, it is required that the line be remeasured to reduce the difference. The corners are set at the mean distances. When successive independent tests of the alinement, as determined by astronomical observations, indicate that the line has departed from the true meridian by more than 03′, it is required that the necessary correction be made to reduce the deviation in azimuth.

349. Base Line.—From the initial point the base line is extended east and west on a true parallel of latitude, standard quarter-section and section corners being established alternately at intervals of 40 chains (½ mile) and standard township corners being placed at intervals of 480 chains (6 miles). The manner of taking the linear measurements of the base line and the required accuracy of both linear measurements and alinement is the same as for the survey of the principal meridian. Any one of the three methods described in Art. 353, for laying out the true latitude curve, may be used.

350. Standard Parallels.—At intervals of 24 miles north and south of the base line, true parallels of latitude called *standard parallels* or *correction lines* are run east and west from the principal meridian, these lines being established in a manner identical with that prescribed for the survey of the base line.

It is specified that two independent sets of linear measurements should be taken unless subdivisional closings are possible, in which case the subdivisional closing will furnish a satisfactory verification of the length of the lines. The maximum allowable discrepancy between measurements is 20 links per mile, and the maximum allowable error in direction is 03′, the necessary correction being made to reduce the error in azimuth when it exceeds this value.

351. Guide Meridians.—These lines are extended north from the base line and standard parallels at intervals of 24 miles east and west of the principal meridian. Each meridian is established as a true meridian in a manner identical with that employed in laying off the principal meridian. The guide meridians terminate at the points of their intersection with the standard parallels, and hence are broken lines, each segment being theoretically 24 miles long. Errors

of measurement are placed in the most northerly half mile of each 24-mile segment. At the true point of intersection of the guide meridian and standard parallel, a closing township corner (correction corner) is established by retracing the standard parallel between the first standard corners to the east and to the west of the point for the closing corner; and the distance from the closing corner to the nearest standard corner on the standard parallel is measured.

352. Convergency of Meridians.—In Fig. 352*a*, let ACP and BDP represent two meridians, P being the north pole of the earth, O the center of the earth, and AB an arc of the equator intercepted by the two meridians; and let CD be the arc of a parallel of latitude at a latitude $\phi = COA = DOB$ at which latitude it is desired to determine the angular and linear convergency of the meridians. Consider the earth as a perfect sphere.

The difference in longitude between the two meridians is

$$\lambda = \frac{CD}{CO'} \text{ or } CD = CO' \cdot \lambda$$

The latitude of the arc CD is

$$\phi = DOB = DEO'$$

Then,

$$\sin \phi = \frac{DO'}{DE} \text{ or } DE = \frac{DO'}{\sin \phi}$$

With a negligible error the angle of convergency is

$$\theta = \frac{CD}{DE}$$

Substituting the values for CD and DE obtained above

$$\theta = \lambda \sin \phi \tag{1}$$

Let the distance between two meridians measured along a parallel be $d = CD$, and let the radius of the earth at the parallel be R. Then from the figure

$$\lambda = \frac{CD}{CO'} = \frac{d}{R \cos \phi}$$

Substituting this value in (1), there results

$$\theta = \frac{d \sin \phi}{R \cos \phi} = \frac{d \tan \phi}{R}, \text{ where } \theta \text{ is in radians.} \tag{2}$$

If d is in miles and R = 20,890,000 ft., the approximate mean radius of the earth, then θ in seconds is from (2)

$$\theta'' = 52.13d \tan \phi \tag{3}$$

In Fig. 352*b* let l be the length of the meridian between two parallels and let θ be the mean angle of convergency of two meridians whose mean latitude is ϕ and whose mean distance apart measured on a parallel is d. Also let the linear convergency of the two meridians, measured along a parallel, be c. Then, with small approximation, $\theta = \frac{c}{l}$.

Substituting this value in (2) and solving for c, there results

$$c = \frac{dl \tan \phi}{R} \tag{4}$$

which gives with sufficient accuracy for land surveying the linear convergency between two meridians. If d and l are in miles and R is the mean radius of the earth, then c in feet is given approximately by the expression

$$c_f = \frac{4}{3} dl \tan \phi \tag{5}$$

which is derived from Eq. (4) and which gives the linear convergency with sufficient accuracy for land surveying. The convergency in 66-ft. chains, where d and l are miles, is then

$$c_c = 0.0202 dl \tan \phi \tag{6}$$

Example 1: Find the angular convergency of two guide meridians 24 miles apart at latitude 43°20′. By Eq. (3),

$$\theta'' = 52.13 \times 24 \tan \phi = 1{,}182''$$
$$\theta = 19'42''$$

Example 2: Find the convergency in chains of two guide meridians 24 miles apart and 24 miles long at a mean latitude of 43°20′. By Eq. (6),

$$c_c = 0.0202 \times 24 \times 24 \tan \phi$$
$$c_c = 10.95 \text{ chains}$$

It will be noted in example 1 that the convergency of the two guide meridians is nearly a third of a degree. In example 2 the linear convergency is nearly 11 chains in the 24 miles; this would represent the jog at the correction line in the first guide meridian east or west, or one half of the jog in the second guide meridian. The north boundary of a township 24 miles north of a correction line at the given latitude is approximately 2¾ chains less than 6 miles.

In Table XI is given the angular convergency of meridians 6 miles apart, for each degree of latitude, together with the linear convergency of these meridians in a length of 6 miles. The linear convergency represents the correction to be applied to the north boundary of a regular township in computing the error of closure

about the township. This value likewise represents double the amount of the offset from the tangent to the parallel at a distance of 6 miles from the point of tangency.

Table XI also gives for the various latitudes, the difference in longitude for 6 miles in both angle and time, and the difference in latitude for both 1 and 6 miles in angular measure.

352a. In the subdivision of townships into sections, the establishment of meridional section lines parallel to the east boundary of the township necessitates a correction in azimuth on account of the angular convergency of the meridians. Meridional section lines to the west of the east boundary being run north, are deflected to the left or west of the true meridian by an angle equal to the convergency in the distance from the section line to the east boundary. Hence, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$ of the angles of convergency given in Table XI represent, respectively, the deflections from the true meridian for section lines respectively 1, 2, 3, 4, and 5 miles west of the east boundary of the township.

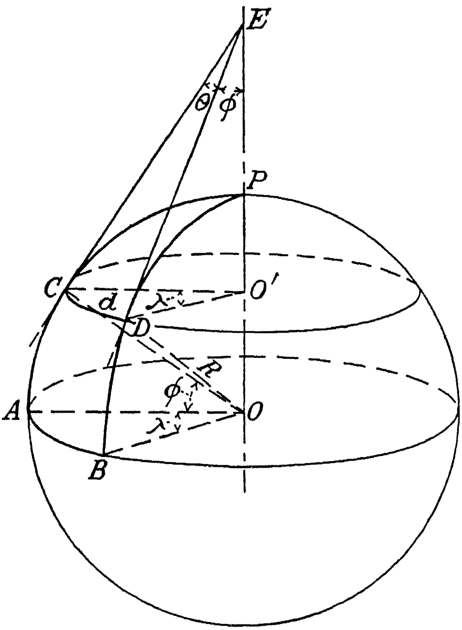


FIG. 352*a*.—Convergency of meridians.

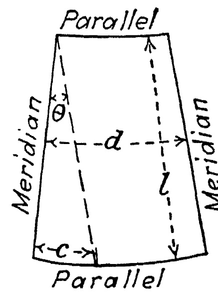


FIG. 352*b*.

353. To Lay Off a Parallel of Latitude.—Since the base line, standard parallels, and latitudinal township lines are true parallels of latitude, they are curved lines when established on the surface of the earth, as is evident from the fact that meridians converge and that a parallel of latitude is a line whose direction at any point is perpendicular to the direction of the meridian at that point. Its

projection on the surface of the earth is the base element of a cone whose vertex is at the earth's center, and the radius of whose base is $R \cos \phi$, in which R is the earth's radius and ϕ is the latitude. It is defined by a plane at right angles to the earth's polar axis cutting the earth's surface on a circle whose radius decreases as the latitude increases. It is true that the rate of curvature within the latitudes of the United States is so small that two points, say a quarter of a mile apart, on the same parallel of latitude will, for all practical purposes, define the direction of the curve at either point, but the projection of a line so defined in either direction would describe a great circle of the earth, gradually departing southerly from the true parallel.

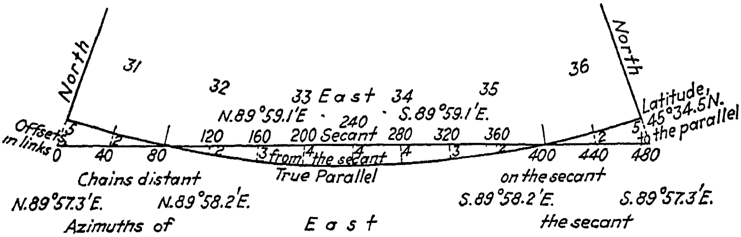


FIG. 353.

The great circle tangent to the parallel at any point along the parallel is called the *tangent to the parallel*, and it coincides with the true latitude curve only at the point of origin.

Though the tangent is a straight line, its bearing is not constant, but varies with the distance from the point of tangency, its deflection from true east or true west being equal to the angle of convergency of the meridians within the distance from the point of tangency to the given point. Hence the angles of convergency given in Table XI also represent the deviation in azimuth of the tangent from the parallel in a distance of 6 miles, and $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$ of the tabulated angles represent the changes in azimuth in distances respectively 1, 2, 3, 4, and 5 miles from the point of tangency.

By means of a sketch it can be readily shown that, within the limits of precision necessary in land surveying, the offset from tangent to parallel at any distance from the point of tangency is one half of the linear convergency of the meridians within the same distance. Hence values one half as great as the values of the linear convergency in 6 miles given in Table XI represent the offsets from tangent to parallel, measured along the meridian, at a distance of 6 miles from the point of tangency. With small error a parallel of latitude may, within the limits of distance here considered, be assumed to behave as a parabola. Hence, the offset from tangent to curve at any point,

for all practical purposes, may be said to vary as the square of the distance from the point of tangency, and the offsets at $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, etc., miles from the point of tangency would bear to the offset at 6 miles respectively the ratios $\frac{1}{144}$, $\frac{1}{36}$, $\frac{9}{144}$, $\frac{1}{9}$, etc.

There are three general methods of establishing a true parallel of latitude which may be employed independently to arrive at the same result: (1) the solar method, (2) the tangent method, and (3) the secant method.

353a. Solar Method.—By this method a solar attachment to the engineer's transit is employed as described in Art. 314*a*, p. 473. If the instrument is in good adjustment the true meridian may be established with sufficient accuracy at each transit station, and the true parallel may be established by turning an angle of 90° in either direction from the meridian. If sights taken with the telescope in the latter position are not longer than 20 to 40 chains, the line thus defined will not depart appreciably from the true parallel.

353b. Tangent Method.—This method consists in determining the true meridian at the point of tangency, from which the tangent to the parallel is established by laying off an angle of 90°. The tangent is extended in a straight line for a distance of 6 miles, and as each 40 chains is laid off along the tangent, the corresponding section or quarter-section corner is established on the parallel by laying off along the meridian the appropriate offset from tangent to parallel.

At the end of 6 miles a new tangent is laid off, and the process just described is repeated. The values of the offsets may be found from Table XI, as suggested in Art. 353.

353c. Secant Method.—This is a modification of the tangent method, in which the secant is a straight line 6 miles in length forming the arc of a great circle, which cuts the true parallel at the end of the first and fifth miles from the point of beginning, as illustrated by Fig. 353. From the figure it is clear that the secant is parallel with a tangent to the parallel at the end of the third mile (240 chains); hence, the offset south from the third-mile point on the secant line to the corner on the true parallel is the same as the offset from the tangent to the parallel in a distance of two miles. Also, it is evident that the offset south of the point of beginning to the initial point on the secant, and the offset north of the secant to the true parallel at the end of the sixth mile, is equal to the difference between the tangent offset in a distance of 3 miles and the tangent offset in a distance of 2 miles.

If the secant is laid off toward the east, the direction of the secant from the point of beginning to the end of the third mile is north of true east, and beyond the end of the third mile is south of true east, the variation from true east increasing directly with the distance in either direction from the third-mile point. At the third-mile point the secant bears true east; at the initial point the secant bears north of east by an amount equal to the angular convergency of meridians 3 miles apart; and at

the end of the sixth mile the secant bears south of east by the same amount. In Table XII are given, for various latitudes, the azimuths measured in either direction from true north, of the secant at intervals of 1 mile. In Table XIII are tabulated the offsets from the secant to the parallel at intervals of ½ mile.

The procedure employed in establishing a true parallel by this method is as follows:

The initial point on the secant is located by measuring south of the beginning corner a distance equal to the secant offset for 0 mi. given in Table XIII. The transit is set up at this point and the direction of the secant line is established by laying off the azimuth from true north given in Table XII in the column headed 0 mi. The secant is then projected in a straight line for 6 miles and as each 40 chains is laid off along the secant, the proper offset is taken to establish the corresponding section or quarter-section corner on the true parallel.

At the end of 6 miles, if it is not convenient to determine the true meridian, the succeeding secant line may be established by laying off, at the sixth-mile point, a deflection angle from the prolongation of the preceding secant to the succeeding secant, the angle being equal to the convergency of meridians 6 miles apart. Values of these deflection angles are given in the last column of Table XII. When the direction of the new secant has been thus defined, the process of measurement to establish corners on the true parallel is continued as before.

The secant method is recommended by the General Land Office for its simplicity of execution and proximity to the true latitude curve, as all measurements and all cutting (to clear the line) by this method are substantially on the true parallel.

354. Township Exteriors.—The exact procedure employed in establishing township boundaries depends upon factors so variable that a complete discussion of the subject will not be attempted here. When practicable to do so, however, the township exteriors are surveyed successively through a 24-mile quadrangle in ranges, beginning each range with the township on the south. The range lines or meridional boundaries of the townships take precedence in the order of survey and are run from south to north on true meridians, quarter-section and section corners being established alternately at intervals of 40 chains. At the end of 6 miles a temporary township corner is set, pending latitudinal measurements necessary to close the township exterior and to calculate the error of closure.

Each township line forming the north or south boundary of a township is run as a random line, as described in Arts. 353*a* to 353*c*, from the old towards the new meridional boundary, and if the error of closure is within the permissible value the line is corrected back on

a true parallel joining the two township corners. On the true parallel are established quarter-section and section corners, alternately at intervals of 40 chains, measurements being made from the boundary last run. The fractional measurement is placed in the most westerly half mile.

Where both meridional boundaries of a township are new lines, or where both have been established by a previous survey, the random latitudinal boundary is run from east to west, but in other particulars the procedure is as outlined above.

A range line is terminated at its intersection with a standard parallel, the excess or deficiency in the measured distance between standard parallels being placed in the most northerly half mile. At the point of intersection between the range line and standard parallel, a closing township corner (correction corner) is established. In order to determine the alinement of the line closed upon, the standard parallel is retraced between the two standard corners adjacent to the closing corner. The distance from the closing corner to the nearest standard corner is measured in order that the error of closure may be calculated.

Following the ideal procedure outlined above, when a full 24-mile quadrangle is to be divided into townships, the survey is usually begun at the southeast corner of the southwest township of the quadrangle (see Fig. 347*b*). The range line is run 6 miles north, and the latitudinal boundary connecting the regular township corner previously established on the guide meridian or principal meridian, with the 6-mile point on the range line, is established as described in the preceding paragraphs. The range line is then continued another 6 miles and a second latitudinal boundary is established in the same manner as the first, connecting the second regular township corner north of the standard parallel on the guide or principal meridian, with the 12-mile point on the range line. Again the process is repeated, and then the range line is extended north of the 18-mile point to the closing township corner on the standard parallel. The most westerly range of townships is thus surveyed.

In a similar manner the boundaries of the townships forming the next range to the east are established.

Finally, the third range line, started at the southwest corner of the southeast township, is laid off as the others, but at the 6-, 12-, and 18-mile points latitudinal lines are run to the west to connect with corresponding township corners, and also to the east to connect with the first, second, and third regular township corners north of the standard parallel on the guide meridian.

355. Limits of Error.—The maximum allowable error of closure prescribed for the United States rectangular surveys is $1/452$ provided the error of closure in neither latitude nor departure exceeds $1/640$.

Where a survey qualifies under the latter limit, the former is bound to be satisfied. It is equivalent to a systematic error of 12½ links, in either latitude or departure, per mile of perimeter. On this basis both the latitudes and the departures for the exterior lines of a normal township should close within 3 chains; of a normal range or tier of sections within 1¾ chains; of a normal section within ½ chain. The above general requirement is applied as a test of the accuracy of the angular and linear measurements incidental to all classes of lines embraced in the division of the public lands. When-

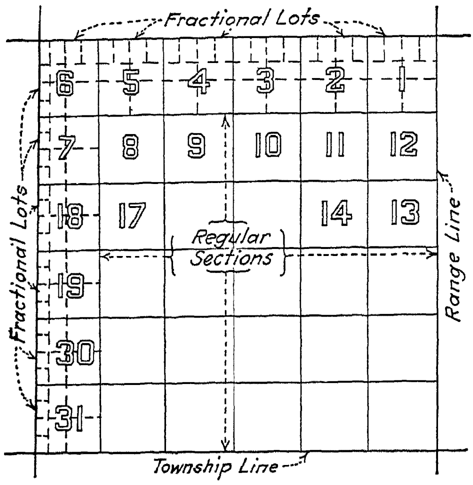


FIG. 355.—Township subdivision.

ever a closure is effected, the latitudes, departures, and error of closure of the lines composing the figure (quadrangle, township, section, meander, etc.) must be calculated, and corrective steps must be taken whenever the test discloses an error in excess of the allowable value.

In addition to the above requirement, township exteriors must be so established that the rectangular limits of township subdivisions, as discussed in the following article, are not exceeded.

Normally the boundaries of a township are considered to be established within satisfactory governing limits from which to control the subdivisional surveys when the calculated position of the section lines may be theoretically projected from the township boundaries without invading the danger zone in respect to the rectangular limits.

355a. Rectangular Limits.—Before considering further the methods employed in the subdivision of townships, the legal requirement

relative to the rectangular surveys of the public lands should be stated. Of the 36 sections in each normal township (Fig. 355), 25 are returned as containing 640 acres each; 10 adjacent to the north and west boundaries (comprising sections 1–5, 7, 18, 19, 30, and 31) each contain regular aliquot parts totaling 480 acres and 4 additional fractional lots each containing 40 acres plus or minus definite differences to be determined in the survey; and one section (section 6) in the northwest corner contains regular aliquot parts totaling 360 acres with 7 additional fractional lots each containing 40 acres plus or minus certain definite differences to be determined in the survey. The above-mentioned aliquot parts of 640 acres, called the *regular* subdivisions of a section, are the quarter section, the half-quarter or eighth section, and the quarter-quarter or sixteenth section, the last containing 40 acres and being the legal minimum for purposes of disposal under the general land laws.

With regard to the allowable limits of precision, the "Manual of Surveying Instructions" of the General Land Office is quoted as follows:

"In the administration of the surveying laws it has been necessary to establish a definite relation between rectangularity * * * , as contemplated by law, and the resulting unit of subdivision consequent upon the practical application of surveying theory to the marking out of the lines on the earth's surface, wherein the ideal section is allowed to give way to one which may be termed 'regular.' Such relation, as applied to the boundaries of a section, has been placed at the following limits: (a) For alinement, not to exceed 21′ from cardinal in any part; (b) for measurement, the distance between regular corners to be normal according to the plan of the survey, with certain allowable adjustments not to exceed 25 links in 40 chains; and (c) for closure, not to exceed 50 links in either latitude or departure.

"Township exteriors, or portions thereof, will be considered defective when they do not qualify within the above limits. It is also necessary, in order to subdivide a township regularly, to consider a fourth limit, as follows:

"(d) For position, the corresponding section corners upon the opposite boundaries of the township to be so located that they may be connected by true lines which will not deviate more than 21′ from cardinal."

356. Subdivision of Townships.—The procedure to be employed in the subdivision of a township into sections depends upon the regularity of the established boundaries of the township. If these boundaries are within the governing limits previously mentioned, the subdivision may proceed in a normal order, the south and east boundaries of the township being the governing lines. When the

township exteriors are irregular the variations in the procedure of subdivision are too numerous to allow of description here.

Following the normal plan for subdividing townships with regular boundaries, the subdivisional survey is begun on the south boundary of the township at the section corner between sections 35 and 36 (see Fig. 356) and the line between sections 35 and 36 is run in a northerly direction parallel to the east boundary of the township, the quarter-section corner between 35 and 36 being set at 40 chains, and the section corner common to sections 25, 26, 35, and 36 being

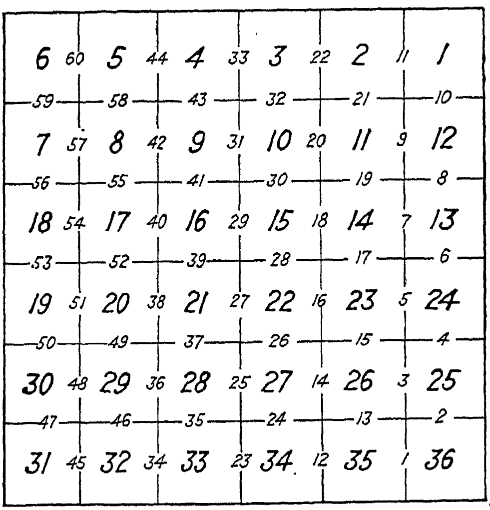


FIG. 356.—Order of establishing section lines.

set at 80 chains. From this latter corner a random line is run eastward on a course calculated to be parallel with the south boundary of section 36, a temporary quarter-section corner being set at 40 chains. If this random line intersects the east boundary of the township exactly at the corner of sections 25 and 36, it is blazed and established as the true line, and if the linear error of closure is within the allowable limits, the temporary quarter-section corner is made permanent by shifting it to a position midway between adjacent section corners, as determined by field measurements.

If the point of intersection between the random line and east bounaary falls to the north or to the south of the section corner on the township boundary, as will generally be the case, the falling is measured and from the data thus obtained the bearing of the true return course is calculated and the true line joining the section corners is blazed and established, the

quarter-section corner common to sections 25 and 36 being placed midway between section corners, as described above.

This process is repeated for the successive meridional and latitudinal lines in the eastern range of sections until the north boundary of section 12 is established, the order in which the lines are surveyed being as indicated by the numbers on the section lines in Fig. 356.

When the northern boundary of the township is not a base line or standard parallel, the line between sections 1 and 2 is run north as a random line parallel to the east boundary, the distance to its point of intersection with the northern boundary of the township being measured. If the random intersects the northern boundary at the corner of sections 1 and 2 and the linear error of closure of the tier of sections is within the allowable, the random is blazed back and established as the true line, the fractional measurement being thrown into that portion of the line between the quarter-section corner and the north boundary of the township.

If, as is usually the case, the random intersects the north boundary to the east or to the west of the corner of sections 1 and 2, the falling is measured, the bearing of the true return course is calculated, and the true line joining the section corners is established, the permanent quarter-section corner common to sections 1 and 2 being placed a full 40 chains from the south boundary of these sections. In this way the excess or deficiency in linear measurement is, as before, placed in that portion of the line between the permanent quarter-section corner and the north boundary of the township.

When the north boundary of the township is a base line or standard parallel the line between sections 1 and 2 is run as a true line parallel to the east boundary of the township, a permanent quarter-section corner being set at 40 chains, a closing section corner being established at the point of intersection of the section line and base line or standard parallel, and the distance from this closing corner to the nearest standard corner being measured.

The successive ranges of sections from east to west are surveyed in a manner identical with the procedure described in the preceding paragraphs for the most easterly range until the two most westerly ranges are reached.

Meridional section lines are thus established by running north parallel to the east boundary of the township, and random latitudinal lines are laid off parallel to the south boundary of the sections of which they are the north boundaries; all meridional section lines are a full 80 chains in length, except the most northerly ones, where all errors in linear measurement are thrown in the most northerly half mile, and the permanent

quarter-section corners on true latitudinal section lines are placed equidistant from the adjoining section corners.

The west and north boundaries of section 32 are established as for corresponding sections to the east. A random line parallel to the south boundary of the township is then run west from the corner of sections 29, 30, 31, and 32, the point of intersection between the random line and the west boundary of the township being determined. The falling of the intersection from the true corner is then measured, the course of the true line is calculated, and the true line is blazed and established, the permanent quarter-section corner being placed on the true line at a full 40 chains from the corner of sections 29, 30, 31 and 32. Thus the deficiency due to convergency of the meridians and the excess or deficiency due to errors in linear measurements are thrown in the most westerly half mile.

The survey of the other sections comprising the two most westerly ranges is continued in similar manner, the order in which the lines are surveyed being indicated by the numbers shown in Fig. 356.

357. Subdivision of Sections.—While acts of Congress contain the fundamental provisions for the subdivision of sections into quarter sections and quarter-quarter sections, the sections are only in rare instances subdivided in the field by United States surveyors. However, certain lines of subdivision are shown upon the official plats, and the surveyor in private practice who may be employed by the entrymen or landowners to establish the lines of subdivision is compelled to correlate conditions found on the ground with those shown on the approved plat. The function of the United States surveyor is to so establish the official monuments that the officially surveyed lines may be identified and the subdivision of the section may be controlled as contemplated by law. There the duties of the United States surveyor cease, and those of the surveyor in private practice begin. In the work of subdividing sections into the parts shown on the official plat the local surveyor can not properly serve his client unless he is familiar with the land laws regarding the subdivision of sections, nor in the event of the loss of original monuments can the surveyor expect legally to restore the same unless he understands the principles employed in the execution of the original survey.

357a. Subdivision by Protraction.—Upon the official government township plats the interior boundaries of quarter sections are shown as straight dotted lines connecting opposite quarter-section corners. The sections adjacent to the north and west boundaries of a normal township, except section 6, are further subdivided into parts containing two regular half-quarter sections and four lots, the latter containing the fractional areas resulting from the plan of subdivision

of the normal township. Figure 357*a* is illustrative of the plan of the normal subdivision of sections. The regular half-quarter sections are protracted by laying off a full 20 chains from the line joining opposite quarter-section corners. The lines subdividing the fractional half-quarter sections into the fractional lots are protracted from mid-points of the opposite boundaries of the fractional quarter sections.

In section 6 the two interior quarter-quarter-section corners on the boundaries of the fractional northwest quarter are similarly fixed,

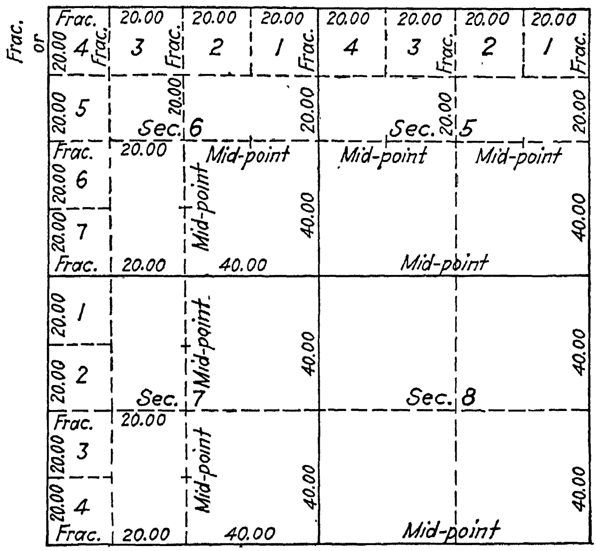


FIG. 357*a*.—Subdivision of sections.

one at a point 20 chains north and the other at a point 20 chains west of the center of the section, from which points lines are protracted to corresponding points on the west and north boundaries of the section. Hence the subdivision of the northwest quarter of section 6 results in one regular quarter-quarter section and three lots.

In all sections bordering on the north boundary the fractional lots are numbered in succession beginning with No. 1 at the east. In all sections bordering on the west boundary the fractional lots are numbered in succession beginning with No. 1 at the north, except section 6 which, being common to both north and west boundaries, has its fractional lots numbered in progression beginning with No. 1 in the northeast corner and ending with No. 7 in the southwest corner, all as illustrated by Fig. 357*a*.

Figure 357*b* illustrates a typical plat of section 6 on which the protracted areas are shown. Figure 357*c* is a similar section giving the calculated dimensions of the protracted areas.

Fractional Lots.—In addition to sections made fractional by reason of their being adjacent to the north and west boundaries of a township, they are also frequently made fractional on account of meanderable bodies of water, mining claims, and other segregated areas being within their limits. Such sections are subdivided by protraction into such regular and fractional parts as are necessary for the entry of the undisposed public lands and to describe these lands separately from the segregated areas.

Figures 357*d* and 357*e* illustrate two sections made fractional by meanderable bodies of water. The practice is to number the lots in each section in sectional tiers beginning with No. 1 as the most easterly lot in the most northerly tier containing fractional sections, and to number the lots progressively toward the west in that tier, then toward the east in the tier to the south, and so on, tier by tier. This system of lot numbering is shown in both of the figures. A lot extending north and south through two or more tiers is numbered in the tier containing its greater area.

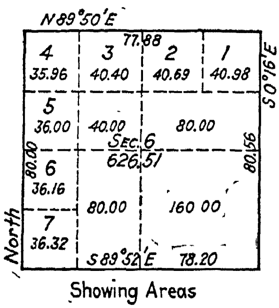


FIG. 357*b*.—Subdivisional areas of section 6.

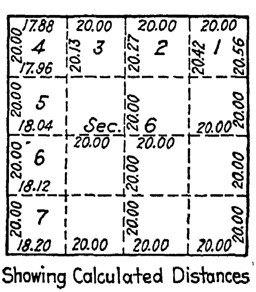


FIG. 357*c*.—Subdivisional dimensions of section 6.

357b. Subdivision by Survey.—The rules for the subdivision of sections given in the following paragraphs are based upon the general land laws. When an entryman has acquired title to a certain legal subdivision he becomes the owner of the identical ground area represented by the same subdivision on the official plat. Preliminary to subdivision it is necessary to identify the actual boundaries of the section, as it cannot be legally subdivided until the section and exterior quarter-section corners have either been found or have been restored to their original positions, and the resulting courses and distances have been redetermined in the field. Having located the

opposite quarter-section corners, the legal center of the section, or interior quarter-section corner may be placed. If the boundaries of quarter-quarter sections or fractional lots are to be established on the ground, it is necessary to measure the lengths of the boundaries of the quarter section and to fix thereon the quarter-quarter-section corners at distances in proportion to those given upon the official plat, when the legal center of the quarter section may be located.

Subdivision of Sections into Quarter Sections.—According to law, the procedure to be followed in the subdivision of a section into quarter sections is to run straight lines between the established opposite quarter-section corners. The point of intersection of lines thus run is the quarter-section corner common to each of the four quarter sections into which the section is divided. It is called the *interior quarter-section corner* and is the legal center of the section.

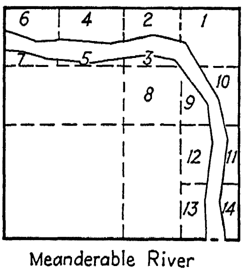


FIG. 357*d*.—Fractional lots.

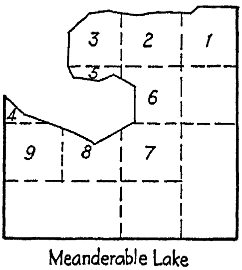


FIG. 357*e*.—Fractional lots.

Subdivision of Fractional Sections.—Where opposite corresponding quarter-section corners of a section have not been or cannot be fixed, as is frequently the case when sections are made fractional by streams, lakes, etc., the lines of sectional subdivisions are run on courses mean between those of adjoining established section lines, or are run on courses parallel to the east, south, west, or north boundary of the section where there is no opposite section line.

358. Subdivision of Quarter Sections into Quarter-quarter Sections.—Preliminary to the subdivision of regular quarter sections, the quarter-quarter-section corners are established at points midway between the section and exterior quarter-section corners and between the exterior quarter-section corners and the center of the section. The quarter-quarter-section corners having thus been established, the center lines of the quarter section are run as straight lines between opposite corresponding quarter-quarter-section corners on the boundaries of the quarter section. The intersection of these lines is common to the four quarter-quarter sections into which the

quarter section is divided. It is called the *interior quarter-quarter-section corner*, and it marks the legal center of the quarter section.

Irregular Quarter Sections.—This case arises when the quarter section is adjacent to the north or west boundary of a regular township and also when the quarter section adjoins any irregular boundary of an irregular township. The procedure is the same as that outlined in the preceding paragraph, except that the quarter-quarter-section corners on the boundaries of the quarter section which are normal to the township exterior are placed at 20 chains, proportionate measurement, counting from the regular quarter-section corner.

Fractional Quarter Sections.—The subdivisional lines of fractional quarter sections are run from properly established quarter-quarter- or sixteenth-section corners, with courses governed by the conditions represented upon the official plat, to the lake, water course, or reservation which renders such tracts fractional.

359. Meandering.—In the process of surveying the public lands, all navigable bodies of water and other important rivers and lakes below the line of mean high water are segregated from the lands which are open to private ownership. In the process of subdivision the regular section lines are run to an intersection with the mean high-water mark of such a body of water where corners, called *meander corners*, are established. The traverse which is run between meander corners, approximately following the margin of a permanent body of water, is called a *meander line*, and the process of establishing such lines is called *meandering*. The mean high-water mark is taken as the line along which vegetation ceases. The fact that an irregular line must be run in tracing the boundary of a reservation does not entitle such a line to be called a meander line except where it follows closely the shore of a lake or the bank of a stream.

Meander lines are not in any sense boundaries but are lines which are run for the purpose of approximately locating the water boundaries, and while the official plats show fractional lots as bounded in part by meander lines, it is an established principle that ownership does not stop at such boundaries (see Art. 340*a*). A Supreme Court decision reads as follows:

> "Meander lines are run in surveying fractional portions of the public lands bordering on navigable rivers, not as boundaries of the tract, but for the purpose of defining the sinuosities of the banks of the stream and as the means of ascertaining the quantity of land in the fraction subject to sale, which is to be paid for by the purchaser. In preparing the official plat from the field notes, the meander line is represented as the border line of the stream, and shows to a demonstration that the water-course, and not the meander line as actually run on the land, is the boundary."

In running a meander line, the surveyor begins at a meander corner and follows the bank or shore line, as closely as convenience permits, to the next meander corner, the traverse being a succession of straight lines. The true length and bearing of each of the courses of the meander line is observed with precision, but for convenience in platting and computing areas the intermediate courses are laid off to the exact quarter degree and each intermediate transit station is placed a whole number of chains, or at least a multiple of 10 links, from the preceding station. Inasmuch as meander lines are not true boundaries, this procedure defines the sinuosities of the mean high-water line with sufficient accuracy. When a meander line is "closed" on a second meander corner, the latitudes and departures of the courses bounding the fractional lot are computed and the error of closure is determined. If this exceeds the allowable value, the line is rerun until an error in bearing or distance is discovered which will bring the closure within the specified limits (maximum error in either latitude or departure is $1/640$).

Rivers.—Proceeding downstream, the bank on the left hand is termed the left bank and that on the right hand the right bank. Navigable rivers and bayous as well as all rivers not embraced in the class denominated "navigable," the right-angle width of which is 3 chains and upwards, are meandered on both banks, at the ordinary mean high-water mark, by taking the general courses and distances of their sinuosities.

Lakes.—Regulations provide for the meandering of all lakes having an area of 25 acres or greater, the procedure being the same as for the meandering of streams. In case the lake lies entirely within a section there will be obviously no regular meander corners, and a *special meander corner* is established at the intersection of the shore of the lake with a line run from one of the quarter-section corners on a theoretical course to connect with the opposite quarter-section corner, the distance from the quarter-section corner to the special meander corner being measured. The lake is then meandered by a line beginning and ending at the special meander corner. If a meanderable lake is found to lie entirely within a quarter-section, an *auxiliary meander corner* is placed at any convenient place on its margin, and this is connected by traverse with one of the regular corners established on the boundary of the section.

Islands.—In the progress of the regular surveys, every island of any meanderable body of water, except those islands which have formed in navigable streams since the admission of a state to the union, is located with respect to regular corners on section boundaries, and is meandered and shown upon the official plat. Also in the survey of lands fronting on any non-navigable body of water, any island opposite such lands is subject to survey.

360. Marking Lines between Corners.—As a final step in the survey of the public lands, it is the aim permanently to fix the posi-

tion of the legal lines of subdivision with reference to objects on the surface of the earth. This is accomplished (1) by setting monuments, of a character later to be defined, at the regular corners, (2) by finding the location of the officially surveyed lines with respect to natural features of the terrain, and (3) by indicating the position of the regular lines through living timber by *blazing* and by *hack marks*.

The latter method of fixing the position of the regular subdivisional lines is required by law just as definitely as is the establishment of monuments at the corners, and all legal lines of the public-land surveys through timber are marked in this manner, those trees which are on the line, called *line trees*, being marked with two horizontal notches, called *hack marks*, on each side of the tree facing the line, and an appropriate number of trees on either side of the line and within 50 links thereof being marked by flat axe marks, called *blazes*, a single blaze on each of two sides quartering towards the line.

361. Corners.—In the subdivision of the public lands as described in the preceding articles, it is required that the United States surveyors shall permanently mark the position of the township, section, exterior quarter-section, and meander corners, and also such quarter-quarter-section corners as it is necessary to establish in connection with the subdivision of fractional sections. For this purpose are employed monuments of a character specified by regulations of the General Land Office.

The location of every such corner monument is, in accordance with definite rule, referred to such nearby objects as are available and suitable for this purpose; and where the corner itself can not be marked in the ordinary manner an appropriate witness corner is established (Art. 361*b*).

At the appropriate place in the field notes of the survey a record of each established monument is introduced, this record including the character and dimensions of the monument itself, the manner in which it is placed, the significance of its position, its markings, and the nature of the objects to which reference measurements are taken, together with these measurements.

361a. Corner Material.—At the present time the regulations of the General Land Office specify as the official corner monument, which is to be used when circumstances will permit, a post made from commercial wrought-iron pipe, 1 to 3 in. in diameter and 36 in. long, filled with concrete; the bottom end of the pipe is split in halves for 4 or 5 in., the halves being spread outward to form a base, and to the top of the pipe is riveted a brass cap on which the corner markings are stamped with steel dies. The posts are set in the ground for about three fourths of their length. The 3-in. posts are ordinarily

employed for township corners, 2-in. posts for section corners, and 1-in. posts for quarter-section and meander corners and all other permanent points. Witness corners are of the same size as the corresponding regular corners.

Where the procedure is duly authorized, durable native stone may be substituted for the model iron post described above, providing the stone is at least 20 in. long and at least 6 in. in its least lateral dimension. Stone may not be used as a monument for a corner whose position is among large quantities of loose rock. The required corner markings are cut with a chisel, and the stone is ordinarily set with about three fourths of its length in the ground.

Where the ground is underlaid with rock close to the surface, making it impossible to complete the excavations for monuments to the regular depth, the monument is placed as deep as practicable and it is supported above the natural ground surface by a mound of stone. Where the solid rock is at the surface, the exact corner point is marked by a cross cut in the rock; and if practicable to do so, the corner monument is established in its proper position and is supported by a mound of stones.

Where the corner point falls within the trunk of a living tree which is too large to be readily removed, the tree becomes the corner monument and, as such, is scribed with the proper marks of identification.

361b. Witness Corners.—Where a true corner point falls within an unmeandered stream or lake or within a marsh or in an inaccessible place, a witness corner is established in a convenient location nearby, preferably upon one of the surveyed lines leading to the position of the regular corner. Also where the true point falls within the traveled limits of a road, a cross-marked stone is deposited below the road surface, and a witness corner is placed in a suitable location outside the roadway.

The witness corner is placed on any one of the surveyed lines leading to a corner, if a suitable place within a distance of 10 chains is available, but if there is no secure place to be found on a surveyed line within the stated limiting distance, the witness corner may be located in any direction within a distance of 5 chains.

362. Marking Corners.—Although to treat completely the system of marking employed by the General Land Office on corner monuments established in the survey of the public lands is beyond the scope of this text, a brief description of the general features of the system is here given. For further details the reader is referred to the "Manual of Surveying Instructions" of the General Land Office.

All classes of monuments are marked in accordance with a system which has been designed to provide a ready identification of the posi-

tion and character of the monument on which the markings appear. Iron posts and tree corners are marked with capital letters which are themselves keys to the character of the monument and with arabic figures giving the section and township and range numbers of the adjacent subdivisions and the year in which the survey was made. Certain additional marks in the shape of *notches* and *grooves* are placed on the vertical edges or faces of stone monuments, the numbers of marks in the case of an exterior corner being equal to the distance

SC
T25N
R17E|R18E
S36|S31
1916

(*a*) Standard township corner.

T27N|R17W
S31|S32
T26N R17W
S6
1916

(*b*) Interior section corner.

FIG. 362.—Typical markings on iron monuments.

in miles along the township or range line from the monument to the adjoining township corner, and in the case of interior corners, being equal to the distance in miles along section lines from the monument to the adjoining township boundary, thus furnishing a means of determining the numbers of the adjoining sections.

A witness corner and its accessories are constructed and marked similarly to a regular corner for which it stands, with the additional letters "WC" to signify *witness corner*.

Following is an index of the ordinary markings common to all classes of corners:

Mark	Meaning	Mark	Meaning
AMC	Auxiliary meander corner	S	South
BO	Bearing object	SC	Standard corner
BT	Bearing tree	SMC	Special meander corner
C	Center	T	Township
CC	Closing corner	W	West
E	East	WC	Witness corner
MC	Meander corner	WP	Witness point
N	North	¼	Quarter section
R	Range	1/16	Quarter-quarter, or sixteenth section
S	Section		

All standard township, section, and quarter-section corners on base line and standard parallels are marked "SC." All closing township and section corners on these lines are marked "CC."

362a. Markings on Iron Monuments.—Following are the descriptions of the markings on the caps of certain of the iron post monuments. These markings are made to read from the south side of the monument, and the year number of the date of establishing the corner is placed below the markings.

1. *Standard Township Corner.*—The township number (as T25N) on north half, and the ranges and sections of the two adjoining subdivisions to the northeast and northwest (as R18E, S31, and R17E, S36) in the appropriate quadrants (Fig. 362*a*).
2. *Corners Common to Four Townships.*—Township numbers (as T23N, T22N) on north and south halves; range numbers (as R18E, R17E) on east and west halves; section numbers (as S31, S6, S1, S36) in the four quadrants.
3. *Closing Section Corners.*—Township and range on the half from which the closing line approaches the monument; section numbers in proper quadrants; also, if known at the time, the township, range, and section on the side of the correction line opposite the closing section line.
4. *Corners Common to Four Sections on Township Exterior.*—Township (or range) common to the adjoining townships (as T25N); ranges (or townships) upon opposite sides of the exterior (as R17E, R18E); section numbers in appropriate quadrants.
5. *Interior Section Corners Common to Four Sections.*—Township and range in northern half; sections in appropriate quadrants.
6. *Standard Quarter-section Corners.*—On north half marked "¼" followed by section number (as ¼S36).
7. *Quarter-section Corners.*—On a meridional line, "¼" on north and sections on east and west halves; on a latitudinal line, "¼" on west half and sections on north and south halves.

362b. Markings on Stone Monuments.—The letters and figures on stone monuments are cut on the exposed faces of the stone, and not on the top. In addition, grooves are cut in the faces of certain monuments, and notches are cut in the vertical edges of certain others. Grooves are employed when the faces are oriented to the cardinal directions, and notches are employed when the vertical edges are turned to the cardinal directions.

362c. Markings on Tree Monuments.—The system of marking tree monuments is practically the same as that employed in marking the caps of the iron monuments, already described in some detail. If *side of tree* be substituted for *quadrant of cap*, the markings given in Art. 362*a* are applicable to corresponding tree monuments. The appropriate marks are made on the trunk of the tree just above the root crown, and the series of marks on a particular side of a tree are scribed to read downward in a vertical line. The scribe marks are usually made in a vertical blaze. The marks thus made will remain

long after the blaze is covered with new growth, and will in fact be destroyed only with the wood itself.

363. Corner Accessories.—When a corner is referred by direction and distance to some other more or less permanent object, and the operation becomes a matter of record, it is possible to relocate the former with respect to the latter. In land surveying a recorded measurement of this kind from a monument to some fixed natural or artificial object in the immediate vicinity is often called a *connection*, and the object thus located is called a *corner accessory*. It is specified that the United States surveyors in the survey of the public lands shall employ at least one accessory for every corner established, the character of the accessories to fall within the following groups: (*a*) bearing trees, or other natural objects such as notable cliffs and boulders; permanent improvements; and memorials; (*b*) mounds of stone; and (*c*) pits.

The marks upon a bearing tree are made on the side nearest the corner, in the manner already described for tree-corner monuments. The mark includes the section number in which the tree stands and is terminated by the letters "BT."

When a bearing object is of rock formation, the point to which measurements are taken is indicated by a cross, and it is marked with the letters "BO" and the section number, all marks being cut with a chisel.

Where it is impossible to make a single connection to a bearing tree or other bearing object, and where a mound of stone or pits are impracticable, such articles as glassware, stoneware, a cross-marked stone, a charred stake, a quart of charcoal, or pieces of metal are deposited at the base of the monument.

Where native stone is at hand and bearing trees or other suitable bearing objects are not available, a mound of stones, of sufficient size to be conspicuous, may be employed as an accessory.

Where accessories such as those mentioned in the preceding paragraphs are not available, pits may be used if conditions are favorable to their permanence. Where the ground is covered with sod, the soil is firm, and the slope is not steep, the pits will gradually fill with a material different in color or in texture from the original soil, and often a new species of vegetation springs up. This sometimes makes it possible to identify the location of pits after the lapse of many years.

364. Field Notes.—The field notes taken in connection with the survey of public lands are required to be in narrative form and are designed to furnish not only a record of the exact surveying procedure followed in the field, but also to provide a report showing the character of the land, soil, and timber traversed by the line of subdivision, and to give a detailed schedule of the topographic features adjacent to the lines, together with reference measurements showing

the position of the lines with respect to natural objects, to improvements, and to the lines of other surveys. In this way the notes serve three purposes: (1) The field procedure is made a matter of official record; (2) the general characteristics of the territory served by the subdivision surveys are secured; and (3) the reference measurements to objects along the surveyed lines furnish evidence by which the established points and lines become practically unchangeable.

365. Restoration of Lost Corners.—While it has been the aim of the General Land Office in the subdivision of the public lands so to monument the established corners that there should always be physical evidence of their position, yet it is a matter of common experience that many corner marks become erased with the progress of time. It is one of the very important duties of the local or county surveyor, in the relocation of property lines or in the further subdivision of lands, as well as of the United States surveyor in the extension of public-land surveys, to examine such evidence as is available and to identify the official corners if they exist. Should a search of this kind result in failure, then it is the duty of the surveyor to employ a process of field measurement which will result in the lost corner being restored to its most probable original position. As here employed the term *corner* is used to designate a point established by a survey, while the term *monument* is used to indicate the object placed to mark the corner point upon the surface of the earth.

A corner is said to *exist* when its position within very narrow limits can be determined beyond all reasonable doubt, either by means of the original monument, by means of the accessories to which connections were made at the time of the original survey, by the expert testimony of surveyors who may have identified the original corner and recorded connections to other accessories, or even by land owners who have indisputable knowledge of the exact location of the original monument. If the monument of an existing corner cannot be found, the corner is said to be *obliterated*, but it is not *lost*.

In the absence of an original monument, a line tree or a definite connection to natural objects or to improvements, which can be identified. may each fix a point of the original survey for both latitude and departure. The mean position of a blazed line, when identified as the original line, may sometimes help to fix a meridional line for departure, or a latitudinal line for latitude. Other calls of the original field notes in relation to various items of topography may assist materially in the recovery of the locus of the original survey. Such evidence may be developed in infinite variety.

When the original position of a corner cannot be determined beyond reasonable doubt, either from traces of the original monument or from other reliable evidence relating to the position of the original monument, the corner is said to be *lost;* and it is restored to its original position, as nearly as possible, by processes of surveying which involve the retracement of lines leading to the corner. Restoration of a corner does not insure that it is placed exactly in its original location, and when a corner is restored the record of the survey should so state.

365a. Proportionate Measurement.—In connection with all resurveys of this character, since the purpose is to restore corners to their original location, it is essential that the laying off of a given distance at the time of the resurvey should render the same absolute distance between two points on the ground as was the case during the original survey when corresponding measurement was effected. For reasons which have been discussed in earlier chapters (Art. 87, etc.), the measurement of a given known line at the time of a resurvey will not in general agree with the length of the line as recorded in the original survey. Thus where linear measurements are necessary to the restoration of a lost corner, the principle of *proportionate measurements* (see Art. 332*a*) must be employed; this consists first in comparing the resurvey measurement with the original measurement between two existing corners between which lies a lost corner, and then in laying off a distance from one of the existing corners to the lost corner which bears the same proportion to the corresponding original measurement, that the resurvey distance between the two existing monuments bears to the original recorded length between these points. In this way there is established by resurvey the same relation between the several parts of the line that originally existed.

The term *single proportionate measurement* is applied to a new measurement made on a single line to determine the position thereon for restoring a lost corner, for example, a quarter-section corner on line between two original section corners. The term *double proportionate measurement* is employed to signify new measurements between four original corners on intersecting meridional and latitudinal lines for the purpose of fixing by relation to both lines the position of a lost corner, for example, a corner common to four sections.

365b. Field Process.—In the succeeding paragraphs are given the field processes to be followed in a few of the simpler cases of the restoration of lost corners. In any event the restorative process must be in harmony with the methods employed in the original establishment of the lines involved, and the preponderant lines must be given the greater weight in determining whether a corner should be relo-

cated by single or double proportionate measurement or by some other method. Thus, standard parallels are given precedence over township exteriors, the latter are given precedence over subdivisional lines, and quarter-section corners are relocated after adjoining section corners have been restored.

1. *Township Corner Common to Four Townships.*—When all the connecting lines have been established in the field, retracement is made between the nearest existing corners on the meridional line north and south of the lost corner, and a temporary stake is set at the proportionate distance for the lost corner; this defines the latitude of the lost corner. Similarly measurement is made between the nearest existing corners on the latitudinal line through the point, and at the proper proportionate distance a second temporary stake is set; this marks the departure of the lost corner. The position of the lost corner is then found at the intersection of an east and west line through the first stake and a north and south line through the second, and the corner is thus relocated by double proportionate measurement.

2. *Section Corner Common to Four Sections in Interior of Township.*—Where all lines have been run, the section corner common to four sections in the interior of a township is restored in the manner described in (1).

3. *Regular Corner on Range Line but Not at Corner of Township.*—The range line is straight between township corners. Two original corners on the 6-mile segment of the range line, one north and one south of the point sought, are identified and a line is run between them. The lost corner is relocated by a single proportionate measurement along this line. This procedure applies either to section or quarter-section corners.

4. *Regular Corner on Township Line but Not at Corner of Township.*—The township line was originally run as a parallel of latitude for 6 miles. A parallel is rerun between the nearest existing corners to the east and west of the point sought, and the corner is relocated by proportionate measurements along this line.

5. *Standard Corner.*—The standard corner includes any township, section, quarter-section, or meander corner, established on a base line or standard parallel at the time the line was originally run. The corner is relocated by the process explained in (4), that is, by single proportionate measurement along the parallel reestablished between the nearest existing standard corners on opposite sides of the point sought.

6. *Quarter-section Corner on Either Meridional or Latitudinal Section Line but Not on Range or Township Line.*—The corner is relocated by single proportionate measurement along the straight line joining the adjacent section corners of the same section. If these section corners can not be identified, they must be restored, as previously explained, before the quarter-section corner can be reestablished.

7. *Quarter-section Corner at Center of Section.*—The corner is relocated at the intersection of meridional and latitudinal lines between opposite quarter-section corners on the boundaries of the section.

8. *Closing Corner on Standard Parallel.*—The parallel is reestablished between the nearest existing corners on opposite sides of the corner sought. The lost corner is relocated by single proportionate measurement along the parallel from the nearest *standard* corners on opposite sides of the point sought.

9. *Quarter-quarter-section Corner on Section and Quarter-section Lines.*—The corner is relocated by single proportionate measurement between quarter-section and section corners on opposite sides of the point sought.

10. *Quarter-quarter-section Corner at Center of Quarter Section.*—The corner is relocated at the intersection of the meridional and latitudinal lines adjoining opposite quarter-quarter-section corners on the exterior of the quarter section.

366. Problems.

1. Find the angle of convergency between two meridians 6 miles apart whose mean latitude is 32°20′. Calculate the linear convergency, measured along a parallel of latitude, in a distance of 6 miles.

2. Find the angle of convergency between two meridians whose distance apart is 24 miles and whose mean latitude is 45°. Compute the linear convergency in a distance of 24 miles.

3. Find the length of 1° longitude at a latitude of 40°06′20″.

4. Calculate the offsets between the tangent and the parallel at intervals of ½ mile over a distance of 6 miles when the latitude is 40°06′20″.

5. Calculate the azimuth of the secant and the offsets from the secant to the parallel at intervals of ½ mile over a distance of 6 miles when the latitude is 40°06′20″.

6. Show the dimensions and areas of the protracted subdivisions of Section 2, as required by law to be shown by official plat, when the north, east, south, and west boundaries are respectively 80.24, 80.16, 79.92, and 80.20 chains.

7. Show the dimensions and areas of the protracted subdivisions of Section 7, as required by law to be shown on the official plat, when the north, east, south, and west boundaries are respectively 76.84, 80.00, 76.64, and 80.00 chains.

8. Show the dimensions and areas of the protracted subdivisions of Section 6, as required by law to be shown on the official plat, when the north, east, south, and west boundaries are respectively 76.36, 80.44, 76.60, and 80.00 chains.

9. A meanderable river follows a winding course from southwest to northeast across a section, the position of the regular southwest corner falling in the water. Draw a sketch assuming the described conditions and indicate thereon the positions of meander and witness corners and meander lines, and indicate the numbering of the fractional lots.

10. An interior section corner is lost and it is to be restored by a resurvey. The nearest corners which can be identified are regular section corners 1 mile north, 2 miles east, 3 miles south, and 1 mile west of the point sought. The records show the corresponding original measured

distances to be 80.40, 160.56, 240.00, and 78.32 chains. The resurvey measurement between the nearest existing monuments on the meridional line through the lost corner is 320.16 chains, and that along the latitudinal line between the nearest existing corners is 238.48 chains. Calculate the proportionate measurements to be used in the relocation of the lost corner, and state the procedure to be employed in its reestablishment.

11. A section corner on a range line is lost, and it is to be restored by a resurvey. One mile to the south the township corner is identified and 2½ miles to the north the quarter-section corner is found. According to the records the corresponding distances measured at the time of the original survey were 80.00 and 200.00 chains. The resurvey distance between the existing corners is 279.64 chains. State procedure to be followed in restoring the lost corner, and calculate the proportionate measurements to be employed.

CHAPTER XXI

ROUTE SURVEYING

367. General.—Surveys made for the purpose of locating and building railways, highways, canals, power transmission lines, and other utilities which are constructed across country for purposes of transportation or communication are called *route surveys*, and the work of conducting such surveys is called *route surveying*. Surveys of this character are necessary for the purposes of selecting the general route to be followed and of fixing the grades, alinement, and other details of the selected route in order that the project may be faithfully constructed in accordance with a definite plan.

Obviously the character of the enterprise has its influence upon the route selected. The economic location of a highway between two towns, for example, might be quite different from that of a power transmission line between the same terminals, owing to the wide variations in the purpose, character, and design of the two projects. It is clear that the location of any route involves a study to determine the manner in which certain definite requirements of the enterprise may be met at the least expense, including not only the cost of construction but also the cost of maintenance and operation. It is therefore a problem in engineering economics in which the conditions are few or many, simple or complicated, depending upon the character and magnitude of the undertaking and upon the nature of the territory through which the route must pass. While a discussion of these economic questions is beyond the scope of this text, it is desired to draw attention to the fact that all conditions imposed by a given problem must receive full consideration before a route is definitely selected.

The details of the surveying methods employed naturally vary somewhat with the character of the project, but certain general field methods are generally applicable.

368. Field Procedure.—Ordinarily, it is impossible to complete in one operation the work of surveying for the location of a route and for the construction of a utility between two given terminals; but the country through which the route is to pass must be gone over and thoroughly examined a number of times, and a series of surveys must be run. First, a study is made to determine the route

to be selected. In some cases, as for a railroad, this may involve extensive exploration in the field; in other cases the conditions allow no alternative for the route, as, for example, the placement of a sanitary sewer in a certain street. The location for a storm sewer, on the other hand, may involve careful study of several streets to make possible a selection of the best route. Any such study, whether made by means of some sort of a survey in the field, or by means of maps, profiles, or other records in the office, may be called a *reconnaissance*.

Following the reconnaissance, a transit line is run over the general route selected. This may mark exactly the line upon which construction is to be undertaken, in which case it is called the *location* or the *located line*. As it is not always possible to stake out the location immediately following the reconnaissance, it may be necessary to run a transit line called the *preliminary line* or *preliminary location* over the approximate route, and from the information thus secured, to stake out the location.

After the location has been run, so-called *construction surveys* may be necessary for staking out special structures, or to measure the amount of work actually performed in construction.

RAILROAD SURVEYS

369. Reconnaissance.—The purpose of the reconnaissance is the selection of the general route. It consists in an extensive study of the whole area that might possibly be used for the location, in order that no possible route may be overlooked or disregarded. As a result of the reconnaissance, the most of this area will be discarded from further investigation, and only one or two narrow strips of territory will be subjected to the more detailed and accurate study which is to follow. Consequently no possible route that is missed during the reconnaissance will be discovered by the later work, and very probably no amount of care and refinement in the later work will compensate for the loss of a better route which may have been overlooked during the reconnaissance. The importance of studying the whole area for all possible routes cannot be too strongly emphasized.

369a. Use of Maps.—The locating engineer on reconnaissance must secure a mental picture of the topography of the whole area under investigation. Maps are of the greatest assistance, and all available maps should be studied. If contour maps of the region can be obtained the problem of the reconnaissance becomes relatively simple.

The United States Geological Survey has published contour maps of a considerable part of the territory of the United States. The charts of the United States Coast and Geodetic Survey are useful for areas along the coast and along navigable streams. Both surveys publish the results of their precise triangulation, traverse, and level work, giving accurately the positions of established points; this information is useful in checking surveys and in referring one survey to another. Other bureaus of the federal government publish maps, most of which can be secured from the Superintendent of Documents, Washington, D. C. Some States publish maps valuable as a guide to location work. Of the maps thus published those prepared by the State Geological Surveys are the most useful.

If such data are not available, it may be found desirable roughly to map to small scale all or a part of the area under consideration. The reconnaissance methods developed by the army engineers are applicable to this problem (see Books 5, 7 and 8, p. 583).

369b. Reconnaissance Methods.—A study is first made of the main streams, their position, size, direction, and approximate velocity and slope. This is followed by a similar investigation of the smaller streams. Then the divides or watershed-limit lines between the different drainage basins are examined and the elevations and positions of low points on the ridges, known as "passes" or "saddles," are determined. When these items have been fully investigated, a general knowledge of the country will have been obtained.

Approximate elevations and approximate distances are necessary to give some idea of the grades that may be secured and the probable necessary length of line. Such elevations and distances may be obtained from maps that are taken into the field. Where maps are not available the distances can be found by some form of range finder, by pacing, or by timing, and the elevations can be found by aneroid barometer or by clinometer.

As a clear idea of the topography of the country takes shape in the mind of the locating engineer, he realizes that there are certain *controlling points*, that is, points decided upon definitely as those through which the line will run. Typical controlling points are important towns, passes, or bridge sites.

Suppose that between the ends of the line three or four points are found to be controlling points. To that extent the route has been fixed. Let A and B be the towns at the ends of the line, and let C be the first controlling point, going from A toward B. Then the points A and C being definitely fixed, the country between them is again gone over and studied, and the route between them is more or less definitely selected. Similarly for the part of the line from C to the next controlling point D; and so on.

It may not be possible, on reconnaissance, to eliminate from consideration all routes but one, and to say definitely that one general route is surely superior to all others. But rarely will more than two routes need further study than that here outlined.

370. Preliminary Survey.—After the reconnaissance has fixed one or two general routes for further study, a narrow strip of country along each possible route is carefully surveyed and mapped, the strip being of sufficient width to ensure the final location falling within the limits of the survey. The area outside the strip thus surveyed is disregarded as offering no route as favorable to economic construction as the route or routes selected for preliminary survey.

The preliminary survey may be run by use of (1) transit, tape, and level; (2) transit and stadia; or (3) the plane table.

370a. Transit-tape-level Method.—Formerly this method was employed practically to the exclusion of all others. It is especially adapted to lines through wooded country, but for lines through open country it has been largely supplanted by the transit-stadia method described in the succeeding article.

The survey corps consists of a transit party, a level party, and a topography party. In the transit party, the chief of party usually runs the transit and, following the instructions of the locating engineer, he runs a continuous traverse at random approximately along the middle of the strip through which it appears that the final line will lie, observing angles in the traverse line, directing the chainmen, and recording angular and linear measurements. Usually the traverse is run by the method of deflection angles described in Art. 226, and the notes are kept up the page in the form shown in Fig. 226*b*, p. 311. The chainage is carried forward from the point of beginning, stakes marked with the station number being set at all full 100-ft. stations and at any plus stations that are established. Hubs are set at all angles in the traverse and at all other transit stations. To expedite the progress of the survey a rear flagman is usually employed. A stakeman drives each stake after it has been marked by the head chainman. In wooded country the line is cleared by axemen, usually under the direction of the head chainman, the number depending upon the density of the forest growth, but being rarely more than five. Roads, streams, land lines, etc., intersected by the traverse line are shown by sketch, and the plus to such features is determined. The results of each day's work are usually plotted at the close of the day.

The level party, consisting of levelman and rodman, follows the transit party, taking profile levels along the traverse in the usual manner (see Arts. 132 and 133), ground elevations being determined

at all stakes set by the transit party, at changes in slope, and at roads and streams. At the same time, bench marks of more or less permanent character are established along the line at intervals of a mile or less; and every opportunity is taken to check the line of levels by observations on existing bench marks, on bodies of still water, etc. Figure 133, p. 184, illustrates the usual form of notes. Usually the elevations of turning points and bench marks are calculated in the field as the work progresses. Bench marks commonly chosen are tree roots and large stones. The bench marks are plainly marked with number and usually with elevation. The preliminary profile is usually brought up to date at the close of each day's work.

The topography party, consisting of the topographer, the topographer's rodman, and frequently a tapeman, follows the level party, the elevations of traverse stations having been previously obtained from the levelman. By use of the hand level or clinometer, sometimes by use of the engineer's level in flat country, elevations are determined along lines at right angles to the traverse, for a distance on either side of the traverse sufficient to construct a map wide enough to include the position of the located line, all as described in Art. 135, p. 186. Article 466, p. 683, describes the use of the hand level more in detail, and Fig. 466*a* shows a form of notes when contours are located directly with the hand level. The transverse lines are normally taken at each 100-ft. station along the traverse, but in very irregular country they may be as close together as 25 ft. and in very smooth country they may be as far apart as 500 ft. In addition to the determination of elevations at critical points along the transverse lines, the topography party also locates such details as buildings, roads, property lines, streams, etc., and notes the character of cultivation, quality of the land, and any other features which may have a bearing upon the location of the railroad.

370b. Transit-stadia Method.—This method is particularly adapted to railroad preliminary surveys through open country where clear sights may be obtained without cutting. Where the topography is not badly broken and the country is open, the transit-stadia method is, under normal conditions, the most satisfactory method in use today. As compared with the method of the preceding article, the transit-stadia survey as ordinarily performed requires fewer men and is considerably more rapid in the field.

The usual procedure is to run the traverse and to take side shots at the same time, as described in Art. 248*b*, p. 341. Thus horizontal and vertical control are established and details are observed in one operation. The party usually consists of a chief of party, a transitman, two or more rodmen, and usually a recorder, though sometimes

the chief of party acts as the recorder. Hubs are established at transit stations, but no intermediate stakes are set. The details of conducting a survey of this character are more fully described in Art. 463*a*, p. 679.

When the survey is extensive, vertical control is established by direct leveling with the engineer's level, unless the survey can be tied to bench marks of known elevation at appropriate intervals. Also on long lines, distances between transit stations are sometimes found by direct measurement with the tape. Under these circumstances the stadia is employed merely for the location of topographic details.

370c. Plane-table Method.—This method is occasionally employed for preliminary surveys, but is not adapted to use in wooded country, nor is it conveniently employed in extremes of weather. For very short lines the field procedure is much the same as with the transit-stadia method, except that the map of the strip of country is constructed in the field as the work progresses. The use of the plane table for such work is described in Arts. 413, 414, and 465. On extensive surveys the plane table is often employed as an auxiliary for the mapping of topographic details, the main traverse being run with transit and tape, and elevations of traverse stations being obtained by direct leveling, as described in Art. 370*a*.

371. Preliminary Profile and Map.—A profile of the ground along the traverse line is prepared as described in Art. 149, p. 206, the usual horizontal scale being 1 in. = 400 ft. and the most common vertical scale being 1 in. = 20 ft.

From the notes of the preliminary survey there is also prepared a map showing the topographic features and other details of the strip of country along the selected route. Contours are normally shown at 5-ft. intervals, but the interval may be 2 ft. or even 1 ft. in level country and may be 10 ft. or greater in rough country. (For methods of plotting see Chap. XV and for contour-map construction see Art. 434, p. 636.)

Both the map and profile are employed by the locating engineer as a guide during the progress of the preliminary survey, and for this reason each day's work is ordinarily plotted before the next day's work in the field is begun. Often a rough preliminary profile is kept in the field and plotted as the levels progress.

372. Location Survey; Paper Location.—Upon the basis of the preliminary-line map and further detailed study of the ground surface, the position of the located center line is determined. Generally a location, called a *paper location,* is drawn upon the map at this stage of the work, before the line is staked out in the field; but

sometimes the line may be run on the ground directly, the map being used in the field to assist in placing the line.

If a complete paper location is made, it is possible to construct from the contours a profile of the paper location, to fix a grade line on that profile, and to make an estimate of the cost of construction.

The field work of the location survey is described in Art. 376.

In running the location survey, curves are introduced at the places where the direction of the line changes; such curves are fully staked

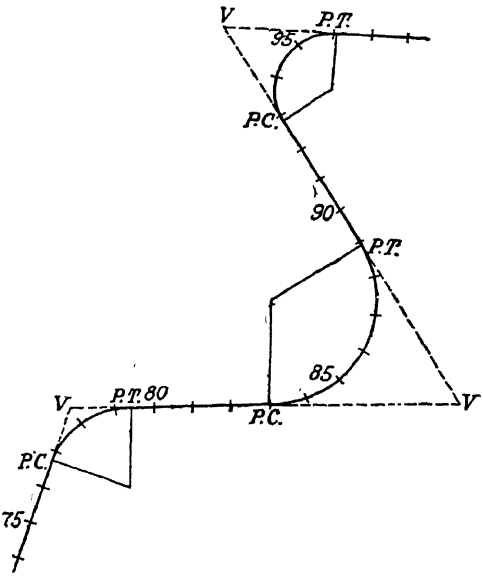


FIG. 373*a*.—Route stationing.

out, with stakes not more than 100 ft. apart and sometimes with stakes 25 to 50 ft. apart.

373. Circular Curves.—Most of the curves used in railroad location are circular curves, that is, arcs of circles connected by straight lines tangent to them, and therefore called *tangents*.[1] The point where the curve begins is commonly called the *point of curve*, written P.C.; similarly the point where the curve ends is called the *point of tangent*, written P.T.; and the point where two tangents produced intersect is called the *point of intersection* or the *vertex*, written P.I. or V.

Stationing progresses from the point of beginning, measurements

[1] For the completed line of railroad, the transition from tangent to circular curve and from circular curve to tangent is accomplished gradually by means of a section in the form of a *spiral* (see Art. 375).

being made from point to point around the curves as indicated in Fig. 373*a*. The 100-ft. distances from station to station on a curve are normally measured as *chord* distances of 100 ft. each, but these 100-ft. distances are sometimes considered as being distances along the *arcs*, the proper corresponding chord lengths being calculated for measurement in the field. Such will be the case along curved property boundaries and where curves of very short radius are used, say anything less than 100 or 200 ft.; also in highway surveys, discussed in Art. 381.

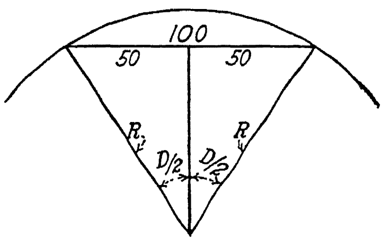


FIG. 373*b*.—Degree of curve.

The sharpness of the curvature may be expressed in either of two ways:

1. By stating the length of the radius.
2. By stating the angle subtended at the center by a chord of 100 ft. This angle is called the "degree of curve," usually denoted by the letter D. Then, as illustrated by Fig. 373*b*, if the radius be R,

$$\frac{50}{R} = \sin \tfrac{1}{2}D; \text{ or } R = \frac{50}{\sin \tfrac{1}{2}D} \qquad (1)$$

From this relation the radius may be found if the degree of curve is known or *vice versa*.

In highway practice, the degree of curve is defined as the central angle subtended by an *arc* 100 ft. in length (see Art. 381).

On a heavy-traffic, high-speed railroad, every effort is usually made to have no curve sharper than 5 or 6 degrees. On lines of light traffic and relatively low speeds, 20° curves or sharper are used.

373a. Geometry of the Circular Curve.—In discussing circular curves, the following geometrical facts are employed:

1. An inscribed angle is measured by one-half its intercepted arc, and inscribed angles having the same or equal intercepted arcs are equal. Thus, in Fig. 373*c*, the angle ACB, at any point C on the circumference, subtending an arc AB, is one-half the angle AOB at the center O, subtending the same arc AB; and the angles at the points C and C' are equal.

2. An angle formed by a tangent and a chord is measured by one-half its intercepted arc. Thus in Fig. 373*c*, the angle at the point A between AD, the tangent to the curve at that point, and the chord AB, is one-half the angle at the center O subtending the same arc AB. This is a special case of the proposition above, when the point C moves to A.

These two propositions are of great importance in locating curves, and are applicable also to certain other surveying problems.

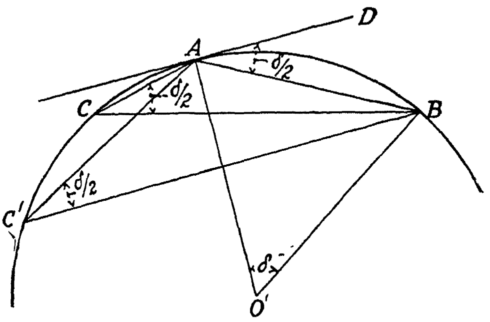


FIG. 373*c*.

3. The two tangent distances to a circular curve, from the point of intersection of the tangents to the points of tangency, are equal in length.

4. Two angles are equal if their sides are perpendicular each to each, in the same order.

373b. Curve Formulas.—Figure 373*d* represents a circular curve joining two straight lines or tangents. In the field the intersection angle I between the two tangents is measured. The radius of the curve in any particular case is selected to fit the topography and the operating conditions on the line when constructed. The line OV bisects the angles at V and at O, bisects the chord AB and the arc ADB, and is perpendicular to the chord AB at F. From the figure $\angle AOB = I$ and $\angle AOV = \angle VOB = \frac{1}{2}I$.

The chord $AB = C$ from the beginning to the end of the curve is called the *long chord*. The distance $AV = BV = T$ from vertex to P.C. or P.T. is called the *tangent distance*. The distance $DF = M$ from the mid-point of the arc to the mid-point of the chord is called the *middle ordinate*. The distance $DV = E$ from the mid-point of the arc to the vertex is called the *external distance*.

Given the radius of the curve $OA = OB = R$, and the intersection angle I, then in the triangle OAV,

$$\frac{T}{R} = \tan \tfrac{1}{2}I$$

$$T = R \tan \tfrac{1}{2}I = \text{tangent distance} \qquad (2)$$

$$E = R \sec \tfrac{1}{2}I - R = R(\sec \tfrac{1}{2}I - 1) = R \operatorname{exsec} \tfrac{1}{2}I = \text{external distance} \quad (3)$$

In the triangle AOF,

$$\frac{C}{2} = R \sin \tfrac{1}{2}I$$

$$C = 2R \sin \tfrac{1}{2}I = \text{long chord} \quad (4)$$

$$M = R - R \cos \tfrac{1}{2}I = R(1 - \cos \tfrac{1}{2}I) = R \operatorname{vers} \tfrac{1}{2}I = \text{middle ordinate} \quad (5)$$

In the triangle AVF, in which $\angle VAF = \tfrac{1}{2}I$,

$$\frac{C}{2} = AV \cos \tfrac{1}{2}I = T \cos \tfrac{1}{2}I$$

$$C = 2T \cos \tfrac{1}{2}I \quad (6)$$

In the triangle ADF, in which $\angle DAF = \tfrac{1}{4}I$

$$\frac{M}{\tfrac{1}{2}C} = \tan \tfrac{1}{4}I$$

$$M = \tfrac{1}{2}C \tan \tfrac{1}{4}I \quad (7)$$

373c. Length of Curve. *Arc Lengths.*—The length of the circumference of a circle is $2\pi R$. This is the arc length for a full angle or

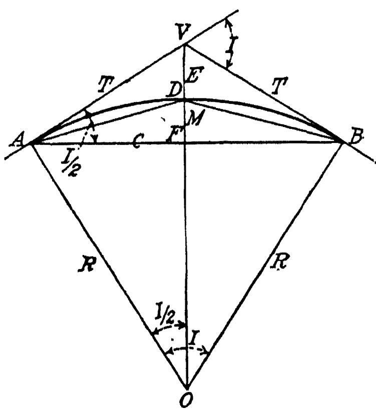


FIG. 373*d*.

360°. As the arc length corresponding to any one given radius varies in direct proportion to the central angle subtended by the arc, the length of arc for any central angle I is

$$\text{Arc} = \left(\frac{I^\circ}{360^\circ}\right) 2\pi R \quad (8)$$

in which the angle I° is expressed in degrees. This solution is simplified by the use of a table of arc lengths for various angles and for unit radius.

Chords.—As the distances measured in the field with the tape must of necessity be measured along straight lines, and as the 100-ft. tape is most commonly used, the *length of curve* in railroad practice is considered to be the sum of the lengths of a number of chords, normally each 100 ft. long. Thus if the central angle I of the curve AD (Fig. 373*e*) is equal to three times the degree of curve D, as shown, then there are three 100-ft. chords between A and D, and the length of curve on this basis is

$$L = 100\frac{I}{D} = 300 \text{ ft.}$$

This length is less than the corresponding arc length AD.

373d. Curves by Deflection Angles.—Curves are usually staked out by the use of deflection angles turned from the tangent to the curve at the transit station, and by chords measured from station to station along the curve. The method is illustrated in Fig. 373*f*, where ABC represents the curve, AX is a tangent to the curve at the point A, and angles XAB and XAC are the deflection angles from the tangent to the chords AB and AC.

Assume the transit to be set up at the point A. Given: R, δ, θ. Required to locate the points B and C.

$$\angle XAB = \frac{\delta}{2} \tag{9}$$

$$AB = 2R\sin\frac{\delta}{2} \tag{10}$$

Turning the angle $\frac{\delta}{2}$ at A and measuring the distance AB, the point B is located in the field.

$$BC = 2\,R\,\sin\frac{\theta}{2}$$

$$\angle BAC = \frac{\theta}{2}$$

$$\angle XAC = \frac{\delta + \theta}{2} \tag{11}$$

At the point A the deflection angle XAC is set off, the distance BC is measured from B, and the forward end of the tape at C is lined in with the transit at A, the line of sight of which has been pointed along the line AC. Thus the point C is located.

Should the chord lengths be given instead of the central angles, then the angles are calculated by means of the formula

$C_1 = 2R \sin \frac{\delta}{2}$, $C_2 = 2R \sin \frac{\theta}{2}$, in which the radius R and chord lengths C_1 and C_2 are known.

If B is at a full station and the distance BC is the 100-ft. distance to the next full station at C, then $\theta = D =$ degree of curve, and $\angle BAC = \frac{D}{2}$.

373e. Laying Off a Curve by Deflection Angles.—The first step in the field location of a curve is to mark on the ground the P.C. and

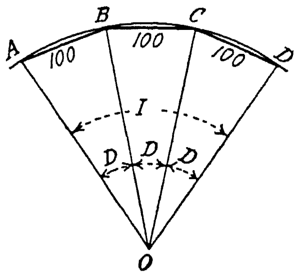


FIG. 373*e*.

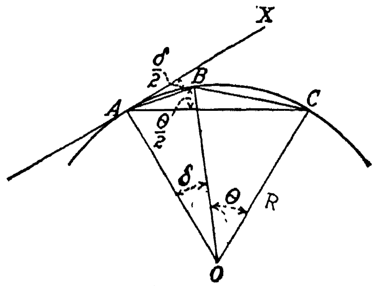


FIG. 373*f*.

the P.T. The deflection angle for each full station on the curve is calculated. If intermediate or plus stations are to be located also, deflection angles are calculated for these points. The transit is placed at the P.C., a sight is taken along the tangent, and each point on the curve is located by deflection angle and distance, the angle being turned from the tangent at the P.C. (except as noted below), and the distance being measured from each full station to points not more than 100 ft. ahead. If full stations only are to be located, the distance to each of such stations is measured from the previous station. The procedure is illustrated by the following example.

Example: Assume that curve AB (Fig. 373*g*) has been located and stations set as far as B, the P.T. of the curve. The directions of the tangents BV and VM have been fixed by hubs, but distances along these tangents have not been measured. The degree of the curve CD is to be 12°00′. It is desired to stake out the curve CD.

The tangents BV and VM are run to an intersection at V, the transit is set at V, and the angle I is read and found to be 104°36′. The degree of curve having been assumed for the curve CD, the radius is determined and the equal tangent distances CV and VD are calculated, as follows:

$$R = \frac{50}{\sin \frac{1}{2}D} = 478.34 \text{ ft.}$$

$$T = R \tan \tfrac{1}{2}I = 478.34 \tan 52°18' = 618.90 \text{ ft.}$$

By measurement from V, hubs are set at C and D. Chaining is then carried forward from B, and the station and plus of C, the P.C. of the curve CD, is found to be 89 + 85.0.

Station 90, the first full station on the curve, is 0.15 station beyond C, that is, the central angle subtended by the arc from C = 89 + 85.0 to station 90 is 0.15 the degree of curve.

The central angle from C to station 90 is 12°00′ × 0.15 = 1°48′. The exact distance along the chord C to 90 calculated by the formula $C = 2R \sin \frac{1°48'}{2}$ is 15.03 ft. In curves such as this, with relatively long radius, this chord would usually be assumed as proportional to the central angle, in this case 15.00 ft. long.

The length of curve P.C. to P.T. is

$$L = 100\frac{I}{D} = 100\frac{104.6}{12.00} = 871.7 \text{ ft.}$$

$$\begin{aligned} \text{Station at P.C.} &= 89 + 85.0 \\ L &= 8 + 71.7 \\ \text{Station at P.T.} &= 98 + 56.7 \end{aligned}$$

The deflection angle for station 90 is $\frac{1°48'}{2} = 0°54'$.

The angle at P.C. between station 90 and station 91 is $\frac{12°00'}{2}$ = 6°00′. Therefore the deflection angle from tangent to station 91 (angle V-P.C.-91) is 6°54′.

Similarly the angle 91-P.C.-92 is 6°00′ and the angle V-P.C.-92 is 12°54′. By the same process the remaining deflection angles from the tangent to full stations on the curve are computed and tabulated up the page as shown below.

Station	Point	Deflection angle	Curve or bearing
100			
99	....	......	N73°10′W
98 + 56.7	P.T. ⊙	52°18′	
98	....	48°54′	
97	....	42°54′	D = 12°
96	....	36°54′	I = 104°36′
95	....	30°54′	T = 618.90
94	⊙	24°54′	R = 478.34
93	....	18°54′	L = 871.7
92	....	12°54′	
91	....	6°54′	
90	....	0°54′	
89 + 85.0	P.C. ⊙	0°00′	12°L
89			
88			

Consider the angle 98-P.C.-P.T. The central angle 98-0-P.T. from 98 to P.T. is subtended by an arc which is 0.567 stations long; hence the angle 98-0-P.T. is $12°00' \times 0.567 = 6°48'$. One-half of this is 3°24', the angle 98-P.T.-V. This added to the deflection angle for station 98

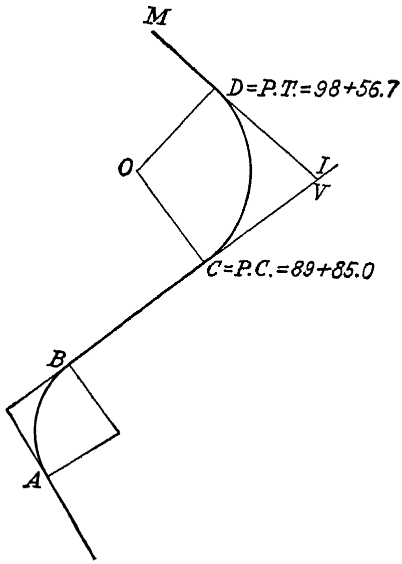


FIG. 373*g*.

(48°54') gives the total deflection angle 52°18'. It may be noted that the total deflection angle should equal $\frac{I}{2}$, and in this example since the angle $52°18' = \frac{104°36'}{2}$, there is given a check on the calculation of all the deflection angles.

The tabulation illustrates the usual form of field notes, the direction of the curve with respect to the tangent being designated by the letter R or L for right or left, following the degree of curve; thus 12°L indicates a 12° curve to the left of the tangent at the P.C.

373f. Transit Set-ups on the Curve.—Often it is impracticable or impossible, on account of obstacles, great length of curve, etc., to run all of a given curve with the transit at the P.C., and one or more set-ups are required on the curve between P.C. and P.T.

Figure 373*h* illustrates the case where the transit is set up at some intermediate point *A*. The curve is begun at P.C. and located as far as *A*, where a hub is set. The transit is then set at *A*. A back-

sight[1] is taken on the last previous station at which the transit was set up, in this case the P.C., and the angle $\frac{\alpha}{2}$ (half the central angle subtended by the chord sighted over) is turned off as shown. The line of sight is then directed along the tangent at A. Deflections to points beyond A are turned as in the case previously explained. It is to be observed that the angle $\frac{\alpha}{2}$ between the tangent at A and the chord A-P.C. is equal to the deflection angle at the P.C.

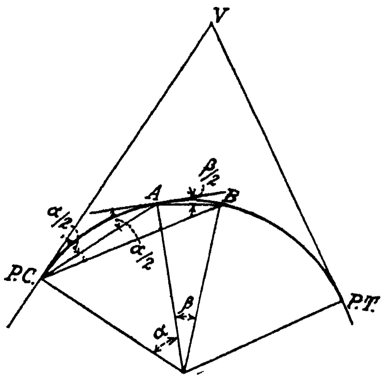


FIG. 373*h*.

for the point A. Therefore, the vernier setting to locate point B from the transit station A, is the same as would have been used had the transit remained at the P.C. According to this method the following procedure may be used to orient the transit at any point on the curve.

(*a*) Calculate deflections as for use at the P.C.; (*b*) when set up at any point on the curve, backsight at the last previous transit station with the vernier reading the deflection for the point sighted (as calculated under (*a*) above); to locate other points, use the deflection angles previously calculated for these points. When the point used as a backsight is the P.C., the backsight vernier reading is zero.

374. Elevation of Outer Rail.—On a railroad curve the velocity of movement of a train develops a horizontal centrifugal force. In order that the plane of the rails may be normal to the resultant of the horizontal and vertical forces acting on a car, the outer rail is elevated above the inner rail by an amount equal to approximately $0.00067V^2D$ expressed in inches, in which expression V is the train

[1] In turning deflection angles at points on a curve the backsights are taken with the telescope reversed, and the foresights with the telescope direct.

speed in miles per hour and D is the degree of curve in degrees. The amount of this elevation should not exceed 7 or 8 inches on account of the use of the track by slow trains. For a speed of 40 miles per hour, it equals, in inches, a fraction more than the degree of curvature in degrees. The elevation of the inner rail is maintained at grade.

Since the elevation of the outer rail must be attained gradually near the end of each curve, it is desirable that the centrifugal force be built up gradually, so that there is an approximate balance between the two at all points. This is accomplished by the use of a curve of varying radius.

375. Spiral Curves.—On railroad lines where trains are to be operated at high speed, it is common practice to insert between circular curve and tangent a curve of varying radius, called a *spiral,* in order that the degree of curvature and centrifugal force may be developed gradually. At the end of the spiral adjacent to the tangent its radius is very long, decreasing gradually until at the point where the spiral joins the circular curve the radii of the two are equal. Spiral curves are also called *easement curves* or *transition curves.*

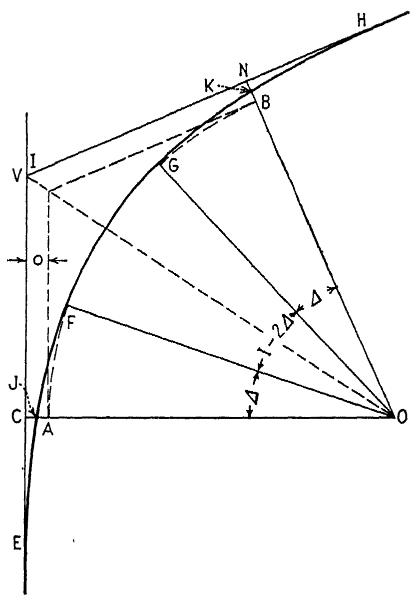


Fig. 375.

In order to provide room for the spiral, the circular curve is offset from the main tangent to the position $AFGB$ of Fig. 375. If the two spirals EF and GH are of equal length the offsets AC and BN are equal, and the distance $VC = VN = (R + o) \tan \frac{1}{2}I$, in which o is the length of the offset.

Many mathematical solutions of the spiral are available, and the reader is referred to these for exact values (see Ref. 12, p. 583). The following approximate and empirical solution is not greatly in error.

1. The central angle I and the degree of circular curve D are known.

2. The length of spiral L' is selected (Ref. 12, p. 583); for curves likely to limit train speed L' should be not less than 240 feet; for minor curves, L' may be 100 ft. or even less.

3. The length of the offset $o = AC = BN$ is computed. This may be assumed to be 6.50 ft. for $D = 10°$ and $L' = 300$ ft., varying directly as the degree of curve and as the square of the length of spiral. Thus for $D = 5°$ and $L' = 200$ ft., $o = \frac{5}{10} \times \frac{(2)^2}{(3)^2} \times 6.50 =$ 1.44 ft.

4. $VC = VN = (R + o) \tan \frac{1}{2}I$.

5. $EC = NH =$ one half of the spiral length minus a correction. For curves of dimensions common in railroad practice this correction has approximately the following values: for spiral angle $\Delta = 5°$, 0.06 ft.; for $\Delta = 10°$, 0.25 ft.; for $\Delta = 15°$, 0.50 ft. For exact formula, see Ref. 12, p. 583.

6. Spirals bisect the offsets AC and BN so that $CJ = \frac{1}{2} AC$ and $NK = \frac{1}{2}BN$.

7. Between E and J, perpendicular offsets from the tangent to the spiral vary in proportion to the cubes of the distances from E; between J and F, radial offsets from the circular curve to the spiral vary as the cubes of the distances from F; similarly for the other spiral GH.

8. Angle AOF = angle $BOG = \Delta = \frac{DL'}{200}$.

9. In the field, the points N, B, C, and A are located, and the direction of each offset tangent is established by means of another and equal offset from the main tangent. The simple curve $AFGB$ is located.

The necessary offsets are made to points on the spirals. For construction surveys it is usually sufficient to offset the circular curve, leaving the staking of the spirals to be done after the line is graded.

10. The alinement with spirals is along the line $EJFGKH$.

376. Location Survey; Field Work.—After the paper location has been made, the work of field location consists in laying off the line in the field so that it bears approximately the same relation to the preliminary traverse on the ground as does the paper location to the preliminary traverse on the map. This relation may be accomplished by scaling from the map the offsets from the stations on the preliminary traverse to the tangents of the paper location. In the field the tangents are established by chaining the scaled offsets from the various stations on the preliminary traverse. In this way a close agreement is always maintained between the paper and field

location of tangents. Adjoining tangents are then run to an intersection, the intersection angles are measured, and the curve notes are figured as explained in the preceding articles, the degree of curve for a given curve being the same as that assumed in the paper location.

Beginning at some initial point for which the chainage is taken as zero, the field location is extended in the manner described above, stakes being set at full stations and frequently at multiples of 50 ft. or even 25 ft., and hubs being set at all P.I.'s, P.C.'s, P.T.'s, and at all intermediate points that are occupied by the transit. Transit notes are kept up the page in the manner illustrated by the example of Art. 373*e*, and all important features such as roads, railroads, streams, property lines, etc., are shown in their proper relation to the located line by means of sketches on the right-hand page, the center line of the page being considered as the located line.

Profile levels are then run over the located line in the same manner as for the preliminary line, and from the data thus obtained a location profile is prepared showing the ground and grade lines and an alinement diagram, all as illustrated by Fig. 149*a*, p. 207.

In the light of a study of this profile and the preliminary map, grades are fixed. If minor modifications in alinement appear desirable, parts of the location are revised in the field. The line as finally located is plotted in both plan and profile; it is then termed the *final location*. On the final location map are shown all features of importance in the immediate vicinity of the line, including the position and character of objects to which hubs are referenced and the position of location bench marks.

The economics of grade location is quite beyond the scope of this text, but it is perhaps appropriate to state that for standard lines the grade seldom exceeds 1 or 2 per cent, and generally is less than 0.5 per cent; but for small branch lines in mountainous country the grade may reach 4 or 5 per cent.

377. Haul.—A primary consideration in fixing the location and grade of a roadway is the amount of *haul* that will later be necessary to transport excavated material from the cuts or borrow pits to the adjacent fills or to waste. The construction contract usually names a price per cubic yard to be paid for excavation of each class of material (earth, loose rock, solid rock, etc.) and for transporting this material for any distance up to a limit of *free haul*. Transportation of material beyond this distance is termed *overhaul* and is paid for at a rate fixed by the contract. The unit of measurement for overhaul is the *station yard*, one station yard being one cubic yard of material transported 100 ft.

The limits of free haul are determined by fixing (on the profile) one point in cut and one point in the adjacent fill, at the specified free-haul distance apart, such that the included quantities of excavation and embankment balance.

The overhaul distance is computed as the distance between the center of gravity of the remaining mass of excavation and the center of gravity of the resulting fill, less the limit of free haul.

In order to determine in advance the proper distribution of excavated material and the amount of waste and borrow, and as a basis for estimate of cost, a *mass diagram* is commonly employed. The abscissas in the mass diagram are the distances along the survey line, and the ordinates are the algebraic sums of earthwork quantities to each ordinate, considering cut volumes positive and fill volumes negative. Given the mass diagram, it is possible to determine by trial the following:

1. The earthwork distribution plan that will result in the minimum cost for overhaul.
2. The economical expenditure for overhaul.
3. The economical expenditure for borrow.

The use of the mass diagram is discussed in detail in texts on railroad location and earthwork, among which are Books 1, 4 and 6, p. 583.

378. Railroad Construction Surveys.—The construction surveys consist essentially in (1) staking out earthwork and structures preparatory to and during the process of grading and construction, and (2) making the measurements necessary to determine the volume of work actually performed up to a given date, as a basis for payment to the contractor.

Generally just prior to the beginning of construction, the located line is rerun, missing stakes are replaced, and hubs are referenced. Also right-of-way boundaries are established and right-of-way maps are prepared.

Prior to grading, the line is cross-sectioned and slope stakes are set, as described in Art. 139, p. 193. Also, if necessary, borrow pits are staked out and cross-sectioned as described in Art. 137. Cross-sections are plotted and volumes of earthwork are calculated as described in Chap. X. Lines and grades are also given for culverts, bridges, and other structures.

The general practice is to prepare a monthly estimate of work completed. In order to make such an estimate it is necessary to make a quantity survey near the close of each month.

Before trimming the slopes, stakes called *finishing stakes* are often set to grade at the outer edge of the roadbed on either side of the center line.

During the progress of laying track, alinement is accurately established by setting tacked stakes along the center line at full stations on tangents and usually at fractional stations on curves. Also stakes, usually to one side of the track, are set to the elevation of the grade of top of rail.

Staking out bridges, culverts, and foundations of all sorts is a detailed job, for which no full explanation is possible here. Stakes are set on all important lines, so that the limits of the work are clearly marked. Batter boards are convenient guides in the construction of culverts, bridge abutments, etc. All such stakes and batter boards must be set beyond the limits of the work so that excavation and construction will not disturb them.

HIGHWAY SURVEYS

379. General.—Most highway surveys are made along established roads as a basis for improvement of such roads. Only small changes in alinement are possible, as the work is usually limited to existing right-of-way lands. Purchase of small parcels of adjoining land may sometimes be possible, permitting some improvement of alinement, as for example, at sharp curves. But the general route is fixed beforehand and no reconnaissance is necessary. Frequently no preliminary survey is required, and the location survey may be run at once, subject to small changes and adjustments after further study.

Occasionally the problem of the location of a new highway is presented, and in such cases reconnaissance and preliminary surveys will be run as for a railroad location, using practically the same field methods.

For an example of the use of a contour map in the location of the route for a highway, see Art. 436*d*.

380. Location Survey.—A transit line is run either on the road center line or offset to one edge of the proposed roadway. This line is stationed in the usual manner, stakes being set every 100 ft., and sometimes every 50 ft. or less. If a topographic map was not made from a preliminary survey, this work is done in connection with the location, the necessary field measurements being made and the whole being plotted on the map of the line. In the field the transit line is fitted to the ground and the necessary curves are located.

381. Curves.—In highway practice, the degree of curve is defined as the central angle subtended by an arc 100 ft. in length, hence $R = \left(\frac{360°}{D°}\right)\left(\frac{100}{2\pi}\right)$. This gives radii for stated degrees of curve which are somewhat less than those obtained from the definition of degree of curve used in railroad work, previously defined. The

radius of a 1° curve on this arc basis is 5,729.58 ft. and the radius varies inversely as the degree of curve, which means, for example, that the radius of a 10° curve is 572.96 ft., to the nearest hundredth of a foot. On the chord basis, the radius for a 1° curve is 5729.65 ft. and the radius for a 10° curve is 573.68 ft. Field measurements with the tape must, of course, be made along the chords and not along the arc, and unless a correction is made for the difference between chord length and arc length, an error of greater or less magnitude will result. Common practice is to make no correction but to use chords so short as to reduce this error to a negligible amount. One authority recommends that 100-ft. chords be used for curves up to 5°, 50-ft. chords from 5° to 15°, 25-ft. chords from 15° to 30°, and 10-ft. chords for curves sharper than 30°. It may be assumed that such practice will give a sufficiently close approximation. Curves sharper than 20° should be used only in special cases.

Curves are located by the deflection-angle method in most cases, but a short curve may be staked in sufficient detail by measuring from the vertex to each end of the curve along the tangents and from the vertex to the mid-point of the arc along the line from vertex toward center of curve.

382. Grades.—If possible, an unpaved road should be planned with sufficient grade to give longitudinal drainage, and to accomplish this the grade should be not less than 1 per cent. For a paved road this is not necessary, as the crown on the road surface provides sufficient drainage. Main roads are designed, where possible, with grades flat enough to be climbed by automobiles without shifting gears, and this means that the grades should not exceed 5 or 6 per cent. Should such a rate of grade prove too expensive, steeper grades are used to reduce construction cost, the exact rate of grade selected depending upon the topography and the density of traffic to be handled. On very steep grades, say 15 to 20 per cent, safety of descent is probably the controlling factor.

In improving an existing road it is important to balance the earthwork quantities so that the excavated material will make all the fills, with no excess or deficiency, by reason of the fact that frequently there is no opportunity for waste or borrow of material along the line.

OTHER ROUTE SURVEYS

383. Survey for Irrigation Canal.—The plan of work for the location of a main irrigation canal is similar to that for the location of a railroad, but certain important differences are worthy of mention. The grades used are relatively very flat, and small errors in difference

in elevation are comparatively important. For this reason the engineer's level is used on reconnaissance, hubs being set every few hundred feet at the required grade elevation, the distances being measured by pacing or by stadia. By this method the errors in position of the line due to errors in elevation are much reduced. The reconnaissance is run from a controlling point at one end of the line, either the selected point of diversion from the river at the upper end of the canal line, or at the required position of the lower end of the line, selected high enough to place the canal above the area to be irrigated.

383a. Grade.—The grade to be used will be selected in such a way as to give the desired velocity of flow with the chosen cross-section. Formulas for this purpose will be found in Chap. XXVI. It is sufficient to say here that for the main canals a very small grade or slope is necessary, sometimes 1 ft. or less of fall per mile of distance. A velocity of 2 to 3 ft. per second is sufficient to prevent weeds and deposits of silt. Average loamy soil will not be eroded at those velocities, while heavy soil with much gravel and rock will be safe against troublesome scouring at higher velocities, say 5 or 6 ft. per second, depending upon the character of the material. Canals in rock or lined with concrete will safely carry water at velocities up to 15 or 20 ft. per second.

383b. Preliminary Survey.—The preliminary survey is run as for a railroad line, except as follows: The level party usually works ahead, setting stakes at grade, as a guide for the proper placing of the line. The transit (or plane table) party then runs a tape or stadia traverse along the line so staked, and takes sufficient topography to make possible a proper location. Any excess fall must be taken care of by drops or chutes, which are structures specially designed for that purpose.

383c. Location and Construction Surveys.—In principle, these are the same as the corresponding surveys for railroad location. Differences in detail result from the fact that instead of the broad, nearly flat roadbed required by the railroad, the finished cross-section will be of ditch or canal shape. In shallow cuts this takes the form of an excavated channel, having on each side an embankment constructed of the excavated material. In side-hill work the material dug out will be used to form a bank on the downhill side of the channel. Instead of a fill across low ground, such as might be used in railroad construction, use is made of a flume or an inverted siphon.

For the canal section either of two assumptions may be made: (1) that the cut and fill are to balance as nearly as may be, or (2) that the water section is to be entirely in cut. Whichever plan is used, stakes are set at the center to mark the located line.

384. Survey for Power Transmission Line.—While the surveying methods for the location of a transmission line are much the same as those for railroad location, with reconnaissance, preliminary, and location surveys, it is obvious that the controlling factors differ markedly. One of the most important considerations is economy of tower and insulator design. Where there is no change of direction at a certain tower, the only loads to be considered in the design of the tower are the vertical load due to the weight of the cables, the possible ice load, the wind load, and possible occasional loading caused by the breaking of a cable. At the end of a line of towers, or where a change of direction gives rise to similar conditions, there is a large horizontal force applied requiring special construction. Therefore the line is made as straight as possible, changes in direction being avoided wherever practicable to do so.

While construction is cheapest in level country, fairly heavy grades may be adopted to avoid changes in direction in the alinement or to avoid unnecessarily heavy cost of right-of-way. Further, to reduce the cost of right-of-way it is desirable to follow section lines or other property lines. If the line can be located near a highway or railway, construction cost is reduced, as is also the cost of patrolling and maintenance.

No curves are used in the alinement, a change of direction being made by an angle in the line at a tower.

384a. Field Work.—The line is run and stationed in the usual manner, with tower locations tentatively selected and marked by stakes. A careful study of map and profile gives the final locations of the towers. These locations are then marked on the ground, and the necessary stakes are set as a guide to the placing of the poles or of the tower foundations.

385. Problems.

1. Given: $I = 34°30'$, $D = 3°00'$, and P.C. = station $74 + 30.0$. Required: R, L, T, and E; also deflection angles, arranged in notebook form, for staking out this curve, using 100-ft. stations.

2. Given: $I = 92°30'$, $T = 425.00$ ft., and P.C. = station $25 + 10.0$. Required: R, D, C, E, M, L; also deflection angles, arranged in notebook form, for staking out this curve, using 50-ft. stations.

3. If the curve of problem 2 represents the center line of a highway curve, suppose it is desired to set alinement stakes along two curves, one of which is to be 10 ft. outside, and the other 12 ft. inside of the center line; required: L_1, L_2, D_1, D_2, E_1, E_2, and the deflection angles arranged in notebook form, for staking out these curves.

4. Given: $I = 60°40'$, $E = 125.5$ ft. Required: R, D, C, T, M, and L.

5. Two tangents AV and BV have an intersection angle of $45°00'$. A point C is located by the coordinates $VH = 270.2$ ft. and $HC =$

157.4 ft.; VH being measured along the tangent VA, and HC being measured perpendicular thereto. It is desired to connect the two tangents with a curve passing through the point C. Required: R, D, T, L, and E.

6. Having calculated the values in problem 5, change the value of D to that value which is a multiple of 10′ and which is nearest to the calculated value. Change all other elements of the curve to agree with the new value of D and compute the deflection angles.

7. Given the data of problem 1. Make the necessary computations for the insertion of a spiral of length 250 ft. at each end of the curve.

References

ARTICLES

1. Bishop, L. E., and discussion by others, "Economics of Canal Location in Uniform Countries," *Trans. Am. Soc. Civ. Eng.*, Vol. 74, p. 179, 1911.
2. Boardman, H. P., "Stadia Surveys for Railway Location," *Eng. News*, Vol. 45, No. 8, Feb. 21, 1901.
3. Busfield, J. L., "Surveys for Canadian Northern Railway Tunnel," Montreal, *Eng. Contr.*, Vol. 42, No. 17, p. 383, Oct. 21, 1914.
4. Laird, R., "Staking out Curves on Rough Ground," *Eng. Contr.*, Vol. 30, p. 357, Nov. 25, 1908.
5. Lavis, F., and discussion by others, "Methods of Location of the Choctaw, Oklahoma and Gulf Railroad," *Trans. Am. Soc. Civ. Eng.*, Vol. 54, p. 104, 1905.
6. McFetridge, W. S., "Some Extensive Railroad Surveys and Their Cost per Mile," *Proc. Am. Soc. Civ. Eng.*, Vol. 35, p. 308, 1909.
7. Millard, W. J., "Mule-back Reconnaissances," *Proc. Am. Soc. Civ. Eng.*, Vol. 37, p. 794, May, 1911. *Ibid.*, Oct. 1911, p. 1153.
8. Ross, H. V., "Pathfinding for Canada's New Transcontinental Railway," *Eng. News*, Vol. 55, p. 111, Feb. 1, 1906.
9. Trumbull, C. W., "Surveys for New York State Road Improvements," *Eng. Record*, Vol. 52, p. 650, Dec. 9, 1905.

BOOKS

1. Allen, C. F., "Railroad Curves and Earthwork," McGraw-Hill Book Company, New York, 1920.
2. Blanchard, A. H., (Editor) "American Highway Engineer's Handbook," John Wiley & Sons, New York, 1,579 pp., 1919.
3. Etcheverry, B. A., "Irrigation Practice and Engineering," Vol. II, McGraw-Hill Book Company, New York, 1915.
4. Fish, J. C. L., "Earthwork Haul and Overhaul," John Wiley & Sons, New York, 1913.
5. Grieves, Capt. L. C., "Military Sketching and Map Reading," United States Infantry Association, Washington, 147 pp., 1921.
6. Searles, W. H., and Ives, H. C., "Field Engineering," John Wiley & Sons, New York, 1925.

7. Sherril, Capt. C. O., "Rapid Reconnaissance Sketching," 100 pp., 1912.
8. Spalding, Major G. R., "Training Manual in Topography, Map Reading, and Reconnaissance," Government Printing Office, Washington, 151 pp., 1917.
9. Webb, W. L., "Railroad Construction," John Wiley & Sons, New York, 9th Edition, 1933.
10. Wellington, A. M., "Economic Theory of Railway Location," John Wiley & Sons, New York, 950 pp., 1887.
11. Williams, C. C., "Design of Railway Location," John Wiley & Sons, New York, 505 pp., 1917.
12. "Manual of the American Railway Engineering Association," American Railway Engineering Association, Chicago, 1929.

CHAPTER XXII

MINE SURVEYING

386. Divisions of Subject.—The subject of mine surveying will be discussed in this chapter under two main heads:

1. Underground surveying, as practiced in mining and tunnel operations; and

2. Mineral-land surveying, involving location and patent surveys.

387. Definition of Terms.—In discussing mining problems it will be necessary to use a few special geological and mining terms. Of these the most important are here defined:

Vein.—A relatively thin deposit of mineral between definite boundaries.

Strike.—The line of intersection of the vein with a horizontal plane; also the direction of that line expressed as a bearing.

Dip.—The vertical angle between the plane of the vein and a horizontal plane, measured perpendicular to the strike. A vertical vein has a dip of 90°. If the vein is nearly horizontal, the dip is a small angle.

Outcrop.—The portion of the vein exposed at the surface of the ground.

Heading.—A passage driven into the rock or ore ahead of the main excavations.

Patent.—The document, issued by governmental authority, granting and conveying public land.

UNDERGROUND SURVEYING

388. General.—Underground surveying differs from surface work in the following ways: The station is usually in the roof instead of in the floor of the workings; the object to be sighted and the cross-hairs of the telescope must be illuminated; distances are usually measured on the slope instead of along horizontal lines; and the transit tripod has adjustable legs to adapt its use to low workings or to very irregular or steeply inclined surfaces.

389. Stations.—When the station is in the roof, the transit may be centered in either of two ways: (1) by first plumbing from the station mark to a point on the floor and then setting up over this latter point, as in surface work; or (2) by centering the transit beneath a plumb bob suspended from the roof station. When the first method is employed, quite commonly the temporary floor point is a piece of lead into which a nail has been driven. There is always a chance that such a mark will be accidentally displaced during

the process of setting up the instrument. In setting up the transit beneath a suspended plumb bob it is necessary to have the plate and also the telescope level before the centering is done.

A station set overhead is more easily found, is less liable to disturbance, and is therefore more durable than one underfoot. It is set in the mine timbering or by driving a plug of hard wood into a hole ¾ to 1 in. in diameter, drilled several inches into the rock. The exact point is established by setting a marker called a *spad* in the timber or plug, just as a tack is driven into the transit hub in surface surveys. In the case of a roof station the object used must, of course, be something from which it is convenient to hang a plumb bob. Non-corrosive spads made especially for this purpose are sold by dealers.

390. Illumination.—Since the field of view is dark, on long sights the cross-hairs require artificial illumination. This may be accomplished by slipping a rolled piece of paper into the sunshade, then holding a miner's lamp, electric flashlight, or other source of light in front of and a little to one side of the objective end of the telescope. By moving the source of light toward or away from the end of the telescope, the cross-hair illumination is increased or decreased until both the cross-hairs and the object sighted are visible. Some transits are equipped with special sunshades which reflect light into the telescope in much the same manner as the rolled paper. Some transits are built with a hollow horizontal axis through which light is transmitted to a reflector within the telescope.

The signal or target is usually a plumb bob hung from the roof station. To illuminate the plumb bob, either a light is held to one side of it, or a piece of thin paper or tracing cloth is held behind it and illuminated by means of a lamp held beyond the paper. Also with short sights such an illuminated screen may be all that is necessary to make the cross-hairs visible. If the point sighted is not too far from the transit, good results are obtained with a piece of cardboard illuminated by a flashlight.

391. Distances.—In underground traversing, except in nearly level workings, the distances are measured on the slope, the horizontal and vertical distances being calculated from the slope distance and the vertical angle. For this purpose a steel tape is used that will reach from one station to the other. As the stations must ordinarily be placed rather close together on account of the character of the workings, a tape length of 100 or 200 ft. will usually be sufficient. Generally the tape is graduated to hundredths of a foot throughout.

The following procedure is convenient (Fig. 391). The transit is set at one station and the vertical distance from the station to the

horizontal axis of the transit is measured. Since this distance is short, possibly only a few inches, it is measured vertically from a roof station to the top of the telescope or from a floor station to the plumb-bob hook, and a constant previously determined for the instrument is added to carry the measurement to the horizontal axis. The height of instrument, or H.I., is positive if the instrument is above the station and is negative if the instrument is below the station. A plumb bob is hung at the next station, with a point on the plumb line marked by some form of clamping target at a known distance below the roof station. This distance is the *height of point*, or H.P. The vertical angle to the point so marked is measured, and the distance is taped

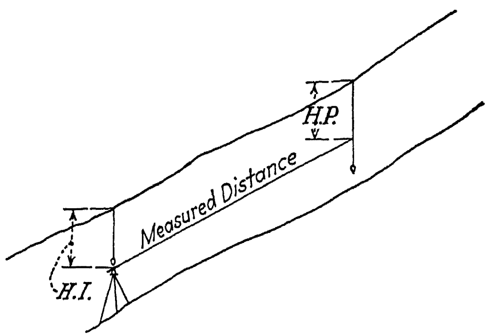


Fig. 391.

to the same point from the end of the horizontal axis, on which the point to be used is definitely marked. While the distance is being taped, the telescope must be pointing toward the plumb line at the next station.

In underground surveying it is desirable to use the method of angular measurement called "angles to the right" or "azimuth from back line" (Art. 229, p. 314), and to double the angle. Generally the compass can not be used for checking, and under such conditions this method is less liable to mistakes than is the deflection-angle method.

The next step is to calculate, from the measured inclined distance and vertical angle, the difference in elevation between center of instrument and point sighted; knowing that difference in elevation and the vertical distances of center of instrument and point sighted below their respective stations, the difference in elevation between the two stations may be found by algebraically adding the H.I. and the product of the slope distance and the sine of the vertical angle, and algebraically subtracting the height of point.

Example:

H. I.	−3.45 ft.
H. P.	−4.67 ft.
Inclined distance	94.78 ft.
Vertical angle	−17°42′

$$
\begin{aligned}
94.78 \times \sin 17°42' = &-28.82\\
&-\ \ 3.45\\
\hline
&-32.27\\
&+\ \ 4.67\\
\hline
&-27.60 \text{ ft.} = \text{difference in elevation between the two stations.}
\end{aligned}
$$

The problem is identical in principle with the one that occurs in surface surveying when a vertical angle is read to a point on a rod, either at the height of instrument or at some other height.

A special case occurs when, by leveling the telescope, the mark on the plumb line at the point sighted is set at the same elevation as the transit telescope. The vertical distance from such mark to the station plug above is then measured and used as in the general case illustrated above. The vertical angle is, of course, zero.

It is obvious that in any case the horizontal distance between the two stations is the measured slope distance multiplied by the cosine of the vertical angle or angle of inclination of the line taped. Unless the vertical angle is large, this reduction may be simplified by the use of a table of versed sines, as explained in Art. 86, p. 89.

392. Mining Transit.—The transit commonly used underground in mine or tunnel is the ordinary engineer's transit on an extension-leg tripod. It should have a full vertical circle and a sensitive telescope bubble, and preferably should be equipped with a striding level for the horizontal axis. An instrument with a horizontal circle about 5 in. in diameter is preferable to a larger one on account of greater ease in handling.

On account of the dirt and water frequently present underground, it is desirable that the vertical circle be fully enclosed and that the instrument be so constructed as to exclude water from the telescope, circles, compass box, and bearings, so far as possible.

When the slope of the underground workings requires the taking of sights along lines of large vertical angle, the transit as ordinarily constructed cannot be used on account of the fact that the horizontal plate will interfere with pointing the telescope. For such conditions an auxiliary telescope is attached either at one end of the horizontal axis or above the main telescope and at a distance therefrom somewhat more than one half of the diameter of the horizontal plate. In

either type the line of sight of the auxiliary telescope is parallel to that of the main telescope. Figure 392 shows a transit with a side telescope attached.

Other devices sometimes employed to take the place of the side or top telescope include overhanging bearings for the main telescope,

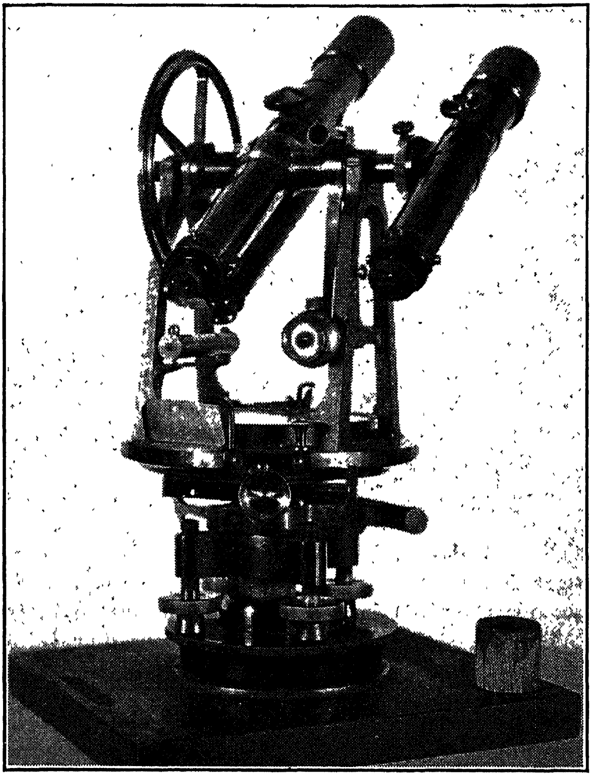


FIG. 392.—Mining transit with side telescope.

and an arrangement by which prisms are used to offset the line of sight to an objective mounted at the end of the horizontal axis.

For very steep sightings a prismatic eyepiece is a great convenience.

The center point should be definitely marked on the top of the telescope.

393. Use of Auxiliary Telescope.—The side telescope is offset from the vertical axis of the transit and this offset or eccentricity affects the observed values of horizontal angles read with the side telescope. Similarly the top telescope, being offset from the horizontal axis of the instrument, is eccentric in the vertical plane, and

this eccentricity affects the observed values of vertical angles read with the top telescope. The process of calculating the true angle from the observed angle is called *reduction to center,* and the value so found is called the *reduced value.* This term does not imply that the calculated value is numerically smaller than the one observed; it may be greater. The difference between the observed and reduced values will here be called simply the *difference.* Briefly stated, the difference, positive or negative as the case may be, is applied to the observed value to give the reduced value, which is the one that would have been obtained had sights been taken with the main telescope.

With the top telescope, horizontal angles will not require reduction to center, because the line of sight lies in the same vertical plane as does that of the main telescope, and the transit is so constructed as to give the horizontal angle between vertical planes through the center of the instrument and the points sighted. For similar reasons, with the side telescope no reduction to center is necessary for vertical angles.

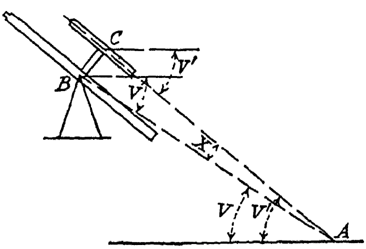


FIG. 393*a*.

Figure 393*a* illustrates the measurement of a vertical angle with the top telescope. The line of sight of the top telescope is offset an amount BC from the line of sight of the main telescope. If it were possible to use the main telescope the observed vertical angle would be V, the value desired. The use of the top telescope gives a reading V'.

$V' - V = X$, and $\sin X = \dfrac{BC}{AB}$ in which BC is the distance between the lines of sight of the two telescopes and hence is a constant for the instrument, and AB is the measured distance between the horizontal axis of the transit and the point sighted. Since BC is a constant, the angular difference X varies only with the distance AB. In practice, a table is prepared showing the values of the difference X for various distances AB. The values given in this table are then used as differences to be applied to the observed vertical angles. If the observed vertical angle is positive, the difference is added; if negative, the difference is subtracted. The field notes should indicate for what observations the top telescope was used, and whether the vertical angle read was positive or negative. The reduction to center should be calculated later, in the office, the field record showing only the values read in the field. The reduced

value of the vertical angle is, of course, used as if it had been read directly with the main telescope. Following is an example.

Example: The vertical angle observed with a top telescope is $V' = -15°23'$, and the distance between the lines of sight of the main and top telescope is $BC = 0.26$ ft. The inclined distance to the point sighted is $AB = 127.20$ ft. It is desired to find the true vertical angle.

$$\sin X = \frac{BC}{AB} = \frac{0.26}{127.20}$$
$$X = 0°07'$$

Then

$$V = -15°23' + 0°07' = -15°16'$$

With the side telescope a similar reduction to center is necessary for horizontal angles, as illustrated by Fig. 393*b*. If observations

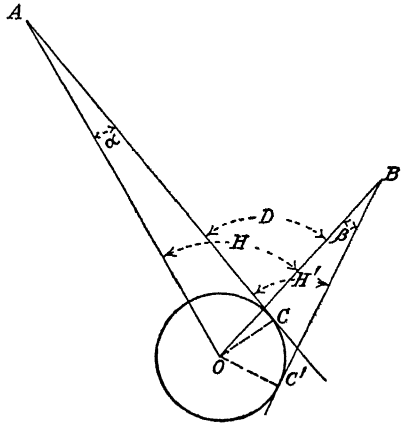


Fig. 393*b*.

could be made with the main telescope, the angle $H = AOB$ would be read. When sights are taken with the eccentric side telescope, the line of sight of the auxiliary telescope, which is offset a distance OC from the center O, is always tangent to the circle with center O and radius OC, as the instrument is revolved. The line of sight of the auxiliary telescope has a direction CA when sighting upon A and a direction $C'B$ when sighting upon B. The angle H' is the difference between these two directions, and is the angle through which the instrument is turned between the two sightings. This is the angle read on the horizontal circle of the transit.

To reduce the observed angle H' to the angle H at the center O, the values of the angles α and β must be used.

$$\alpha = \sin^{-1}\frac{OC}{AO} = \tan^{-1}\frac{OC}{AC}$$

Since AO and AC are practically equal, except in the case of very short lines, either distance or either function may be employed. Also

$$\beta = \sin^{-1} \frac{OC'}{OB} = \tan^{-1} \frac{OC'}{BC'}$$

Then

$$D = H' + \beta = H + \alpha$$

and

$$H = H' + (\beta - \alpha) \tag{1}$$

It will be seen that the difference to be applied to the observed angle H' to reduce it to the angle H is the difference between the angles α and β. In practice the values of α and β are taken from a table similar to that used in connection with the top telescope. Care must be taken to apply this difference with the proper algebraic sign.

Example: The horizontal angle observed with the side telescope of a mining transit is $H' = 73°19'$. The distance between the lines of sight of main and side telescopes is $OC = OC' = 0.31$ ft. and the distances to points sighted from the transit are $AC = 107.31$ ft. and $BC' = 69.31$ ft. It is desired to reduce the horizontal angle to center.

$$\begin{aligned} \alpha &= \tan^{-1} \frac{0.31}{107.31} = 0°10' \\ \beta &= \tan^{-1} \frac{0.31}{69.31} = 0°15' \\ H &= 73°19' + (0°15' - 0°10') \\ &= 73°24' \end{aligned}$$

The best procedure is to measure the angle a second time with a reversal of the instrument between the observations. The side telescope will be on the opposite side of the main telescope, and the angles α and β will enter into the difference with algebraic signs the reverse of those attached to α and β for the first observation. As a result, the mean of the two observations will be free from error of eccentricity of the telescope, and no reduction to center is necessary.

394. Adjustments of Auxiliary Telescope.—The usual adjustments of the transit having been made, it becomes necessary so to adjust the auxiliary telescope that its line of sight lies in the same plane with and parallel to that of the main telescope. The method of mounting the telescope varies with the make of instrument, and this influences somewhat the details of adjustment. If the auxiliary telescope is rigidly mounted upon the main telescope or upon the horizontal axis, the adjustment is made by moving the cross-hairs. If the auxiliary telescope is adjustable as a whole rela-

tive to the main telescope, advantage is taken of this feature in making the adjustment.

If the work of adjustment is done on the surface, the simplest plan is to sight the main telescope on some clearly defined point several miles away, to clamp horizontal and vertical motions of the transit, and then to adjust the auxiliary telescope until its line of sight strikes the same distant point. In case a short sight must be used, either underground or on the surface, two points are marked on a vertical surface, the distance between them being made equal to the distance between the lines of sight of the two telescopes. For a top telescope the two points are on a vertical line, and for a side telescope the two points are on a horizontal line. The line of sight of the main telescope is then directed toward one point, the horizontal and vertical motions are clamped, and the line of sight of the auxiliary telescope is adjusted until it strikes the other point.

395. Setting Up and Leveling the Transit.—From the discussion of Art. 211, p. 284, it is evident that the errors in horizontal angle due to errors in adjustment of horizontal axis and plate levels, and due to inaccurate leveling of the instrument, increase with the magnitude of the vertical angle; hence when the sights are steeply inclined special care must be taken that the transit is accurately adjusted and carefully leveled. As an aid to accurate observation, a sensitive striding level mounted on the horizontal axis is frequently employed. If the transit is not so equipped, it may be leveled by the use of the telescope bubble.

396. Connecting Surface and Underground Surveys.—The methods used to accomplish a connection between surface and underground surveys depend mainly upon the character of the opening from the surface to the underground workings. In the case of a tunnel which is horizontal or nearly so, or in the case of an incline at an angle of not more than 60° or 65° with the horizontal, no special methods are necessary, except that instead of the usual tripod for supporting the transit, some special form of support may be necessary.

Where headroom is very limited or where the transit can best be supported upon a shelf built from the side or the top of the workings, it is mounted upon a *trivet* instead of a tripod. A trivet consists of a modified tripod head with three short supporting pins.

For steeper inclines the transit with auxiliary telescope is used.

In the case of a vertical shaft, a vertical plane is defined by two plumb lines suspended in the shaft, in a plane of known azimuth determined by connection with the surface survey. Wire known as "electrician's banding wire" is recommended for use for the plumb

lines, with bobs weighing ten to forty pounds suspended in oil to reduce oscillation.

Underground, a transit is set up close to the wires and in line with them, that is, in the plane of known azimuth. An angle is then turned to some other line and two points are permanently set on this line, which is then used as a reference line of known azimuth. By this method the underground survey is referred to the same meridian as the surface survey. Great care must be taken in lining in the transit with the two plumb lines, because the distance between the plumb lines is necessarily short and a small error in orientation at the shaft will result in a considerable error in the calculated positions of points some distance removed from the shaft. This linear error in the position of any point is in a direction at right angles to a line from the shaft to the point, the displacement being equal to the azimuth error (in radians) multiplied by the distance from the shaft to the point in question.

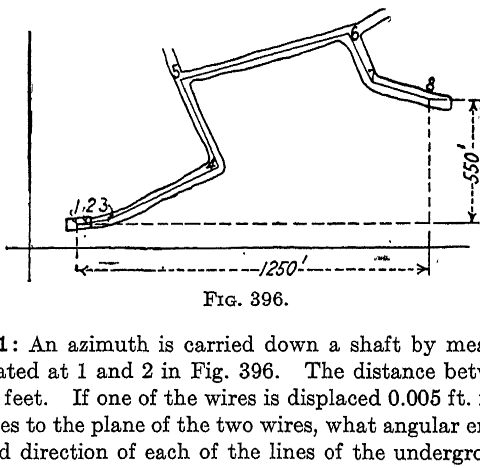


FIG. 396.

Example 1: An azimuth is carried down a shaft by means of plumb wires illustrated at 1 and 2 in Fig. 396. The distance between the two wires is 5.00 feet. If one of the wires is displaced 0.005 ft. in a direction at right angles to the plane of the two wires, what angular error results in the measured direction of each of the lines of the underground traverse to which the known azimuth marked by the wires is connected?

If A represents the angular error, then

$$\tan A = \frac{0.005}{5.00} = 0.001$$

and

$$A = 0°03'30'' \text{ (approximate)}$$

Example 2: Calculation of the latitudes and departures of the courses of the underground traverse of example 1 shows that station 8 in the figure is 550 ft. north and 1,250 ft. east of the shaft. What is the linear error in the calculated position of that station due to the above-mentioned inaccuracy in plumbing down the shaft?

The distance from shaft to station 8 is

$$\sqrt{550^2 + 1{,}250^2} = 1{,}365 \text{ ft. (approximate)}$$

The linear error in the calculated position of the station is in direct proportion to the distance from the shaft to the station. As the error is 0.005 ft. in 5.00 ft., then by proportion:

$$\frac{\text{error in position of station 8}}{\text{distance to station 8}} = \frac{\text{error in plumb lines}}{\text{distance between plumb lines}}$$

$$\frac{\text{error in station 8}}{1{,}365} = \frac{0.005}{5.00}$$

$$\text{Error in station 8} = 1.37 \text{ ft.}$$

The linear error in position of the station is also equal to the distance of the station from the shaft, multiplied by the sine or tangent of the angular error, hence

$$1{,}365 \sin 0°03'30'' = 1.37 \text{ ft.}$$

The preceding example illustrates how a relatively small error in the alinement of the plumb lines may produce a station error of a magnitude that is of real importance if a connection is to be made with other workings.

The coordinates of one of the plumb lines and the azimuth of the horizontal line joining them must be determined from the surface survey. When the transit is set up underground in the plane of the two plumb lines, the coordinates of the transit station may be found. Then the coordinates of other points in the underground traverse are determined in the ordinary way.

In the case of a shallow and wide shaft not too much filled with timbering, it is sometimes possible to set two stations at the bottom of the shaft from a transit set up at the surface. The two stations will be in a plane of known azimuth, from which the underground survey can be oriented. As in other cases of very steep sights, the transit should be in excellent adjustment and should be leveled with great care.

If it is possible, as is generally the case, the survey should be carried into at least two openings and should be closed within the mine. This gives a check on the work. At each entrance the survey should preferably start from a line of known azimuth and position, thus reducing the probable errors of the calculated coordinates of all underground stations.

Another satisfactory method is sometimes used when two vertical shafts form the entrances to the mine. A single plumb line is suspended in each shaft. On the surface a traverse tied to or including a line of known azimuth is run between the two plumb lines, and the length and azimuth of the straight line connecting the

two plumb lines are computed, as described in Art. 267*a*, p. 377. Underground a traverse is run from one plumb line to the other through the mine workings.

A reference meridian is arbitrarily chosen for the underground traverse, from which the length and bearing of the closing course are calculated, as in the surface traverse.

The lengths of the two closing courses, surface and underground, should be equal. The bearings, however, will not agree by an amount equal to the angle between the surface reference meridian and the underground assumed meridian. The azimuths of the underground courses are now corrected to agree with the surface reference meridian, and the proper coordinates of the transit stations are computed. A disadvantage of this method is that the check afforded consists in comparing the calculated lengths of the closing sides only. If in the underground traverse the error of closure should have its direction nearly perpendicular to the closing course, its length would not be greatly affected and the error would not be detected.

397. Computations.—From the length and vertical angle of each course the corresponding horizontal and vertical distances are calculated. From the horizontal distance and azimuth of each course the latitude and departure of that course are computed. The latitude, departure, and vertical distance for each course are the coordinate differences in a three-dimensional coordinate system. Such a system is very useful in all underground work. The three coordinates of each station are computed and are recorded for future use. Later extensions to the surveys, branch lines, etc., can then be fitted easily into the general scheme, and surveys and maps of different parts of the mine can be shown in proper relation to one another.

For calculating latitudes and departures from lengths and bearings, special tables, called *traverse tables*, are frequently used. These tables give directly the latitude and departure corresponding to a given bearing angle and a given length of line. To be useful in the calculation of a transit survey, the table must give values for angles varying by single minutes. If values of latitude and departure for each minute of angle are given for distances 1, 2, 3, 4, 5, 6, 7, 8, 9 along the traverse courses, then the value for any distance is found by simple addition, moving the decimal point as may be necessary.

398. Field Notes and Office Records.—On account of the dirt and water usually encountered underground, it is difficult to keep the notebook pages clean, and the ordinary field notebook soon becomes soiled and difficult to read. For this reason some form of loose-leaf field notebook is desirable in underground surveying. By placing

a few loose leaves in a metal or heavy cardboard binder and using them underground for one day only, a more legible record is secured.

The pages of the notebook should be numbered serially to guard against loss of pages, and should be bound into an office binder for filing. From the notes the latitudes, departures, and differences of elevation are calculated; these are then recorded in a special book. The three coordinates of each point are also tabulated.

The following form of record is convenient for the field notebook.

Sta.	B. S.	F. S.	H. I. + or −	Hor. Angle	H. P. + or −	Vert. Angle	Taped Dist.	Description of F. S. point

For the office book the same headings and the following additional headings are suggested.

Calc. Brg.	Vert. Dist.	Hor. Dist.	Diff. El.	Lat.	Dep.	Total		
						Lat.	Dep.	El.

399. To Give Line for a Connection.—In mining operations it is frequently necessary to drive an opening or connection between more

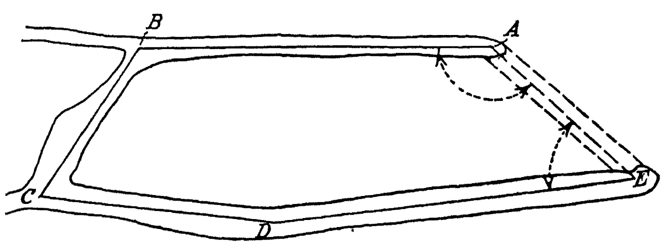


FIG. 399.

or less widely separated parts of existing workings, as between A and E (Fig. 399). This may be necessary for purposes of ventilation, drainage, haulage of excavated material, or to provide for the miners a second route out of the workings in case of accident. The connection is ordinarily in a straight line between two given points. The problem is to determine the length, direction, and slope of this line in order that the work of driving the connection may be properly directed.

Starting at one of the two given points as A, a transit traverse as $ABCDE$ is run through the existing workings to the other point E.

The length, azimuth, and slope of the connecting line AE are then computed by the usual method for traverse with one side of unknown length and direction (see Art. 267*a*, p. 377). From either or both of the given points A and E, a line of the calculated azimuth and slope is laid off with a transit, and thus line and grade for the connection AE are established. As the work of tunneling progresses, additional line and grade points are set at frequent intervals. Also in order that the progress of the work may be determined, measurements of distance are taken from given points to heading. Usually the connection is driven from both ends.

400. To Mark a Property Boundary Underground.—This also is a problem in supplying the missing parts of a traverse. A survey is

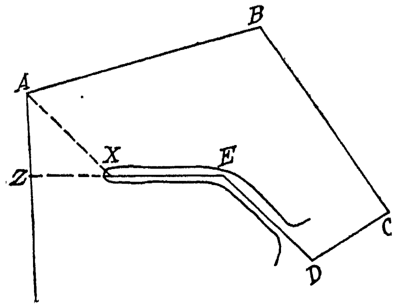


FIG. 400.

run from some point as A (Fig. 400), on the boundary line as marked on the surface, to some point as X, underground near the boundary. At A, the direction of the boundary line is observed. The problem is to find the distance XZ, in a desired direction, from the underground point X to the vertical plane which defines the boundary. From the data of the survey the coordinates of A and X are available and hence the azimuth and horizontal length of the line connecting A and X may be computed, as explained in Art. 267*a*, p. 377. This line forms one side of a triangle, the other two sides of which are ZA on the boundary line and XZ the line of known direction from X to the boundary. The length AX, also the direction of all three sides of the triangle, being known, the distance XZ from X to the boundary is computed, as explained in Art. 267*e*.

401. To Measure Difference of Elevation Down a Vertical Shaft.—This is best done by means of the steel tape. The elevation of a point at the mouth of the shaft having been determined by ordinary differential leveling, the distance is measured vertically to some convenient point further down, and so on to other points. If the distance between working levels is less than one tape length, the

points used are conveniently placed at the levels, and elevations from these points can be carried into the various parts of the mine. The points set in the shaft for this purpose should be marked by small nails driven into the shaft timbers, but more permanent points should be set in the various levels to serve as bench marks.

402. Tunnel Surveys.—Tunnel surveys are run for the purpose of directing the operations of tunneling between two or more given points, either below the ground or on the surface. The objects to be attained are the same regardless of the purpose of the tunnel, these objects being to determine by field measurements and computations the length, direction, and slope of a line connecting given points, and to lay off this line by appropriate field measurement; but the methods employed naturally vary somewhat with the purpose of the tunnel and the magnitude of the work.

For a short tunnel between two points on the surface, as, for example, a railroad tunnel through a ridge, a traverse is usually run between the terminal points, and the length, direction, and grade of the connecting line is computed. Usually the difference in elevation between the terminals is found by direct leveling. When practicable, the surface traverse between the terminals takes the form of a straight line. Outside the tunnel, on the center line at either end, permanent monuments are established. Also points are established in convenient surface locations on the center line, which points, with those just mentioned, serve to fix the direction of the tunnel on either side of the ridge. As construction proceeds the line at either end is given by setting up at the permanent monument outside the portal, taking a sight at the fixed point on line, and then setting points along the tunnel, usually in the roof. Grade is usually given by direct levels taken to points either in the roof or floor, and distances are measured from the permanent monuments to stations along the tunnel.

Essentially the same process is followed in mining except that if the tunnel is between two shafts, it is necessary to transfer elevation and direction down each shaft; also, if the tunnel is on a slope, this slope is ordinarily established with sufficient precision by laying off the vertical angle.

Railroad and aqueduct tunnels in mountainous country are often several miles in length and are neither of uniform slope nor direction. Tunnels of this character are usually driven not only from the ends, but from several intermediate points where shafts are sunk or adits are driven to intersect the center line of the tunnel. The surface surveys for the control of the tunnel work usually consist of an accurate triangulation system tied to monuments at the portals of the

main tunnel and at the entrances of shafts and adits, and also an accurate system of differential levels connecting the same points. With these data as a basis the length, direction, and slope of each of the several sections of the tunnel are calculated; and construction is controlled by establishing these lines and grades as the work progresses.

MINERAL-LAND SURVEYING

403. Subsurface Ownership Ordinarily Defined by Vertical Bounding Planes.—The result of the survey of the boundaries of a piece of ordinary land is a geometrical figure in a horizontal plane. The bounding lines are usually straight, but may be curved. The map of the survey shows this geometrical figure on a plane surface representing the horizontal plane into which all the points of the figure are projected.

The boundaries of land are marked on the surface of the earth by monuments, fences, or other objects, and ownership is often considered as applying to the surface only. But when we think of the construction of a building, and realize that no part of the structure may project over the property lines, or when we remember that similar limitations apply below the surface, or that wires entirely above the ground (supported by poles upon other property) may not run across property without permission of the owner, we realize that ownership of land implies ownership within vertical planes through the boundaries. This is the rule which usually controls, unless specifically modified by laws as explained in the succeeding article.

The owner of land may deed or grant to another person or to a corporation the ownership or rights above or below some specified elevation or level, as for the purpose of driving a tunnel perhaps many feet below the surface. Any such privilege, whether it is above or below ground, if it is distinct from ownership of the soil, is known as an *easement.*

There may be a seam of coal underlying a certain piece of land. The owner of the land may sell to a mining company, operating below adjoining land, the right to mine the coal under his property. Such a sale may be for a lump sum, or the amount to be paid may be based upon the quantity of coal taken out. In the latter case it is necessary for the surveyor to establish underground the boundaries of the property in question, as previously described, in order to make possible a measurement of the quantity of mineral removed.

Where the rule of vertical planes applies, the surveys of mineral lands do not differ from other land surveys, except in so far as the

shape or size of the parcel or the character of the reference points to be used for the survey may be fixed by law.

404. Lode Claims; General.—To encourage the development of mining, the United States Government has passed laws modifying in certain cases the usual rule of vertical planes, and specifying the manner in which the person discovering a mineral vein or lode on government land may acquire title to the vein and thereby profit by his discovery. It is provided that a mining claim of specified maximum dimensions may be located on the surface, and that after certain requirements designed to prove the serious intent of the claimant have been satisfied, the United States Government will give the claimant a patent carrying a clear title to the land claimed, this title carrying ownership of the vein beneath.

The federal laws dealing with lode claims are based upon the conception of a relatively thin vein or lode, limited between surfaces that are essentially plane. According to the law, the claim is to be located along the outcrop of the vein (its intersection with the ground surface) and is limited to a maximum length of 1,500 ft. and to a maximum width of 600 ft. Any state may by law reduce, but not increase, these dimensions. The outcrop must cross the end lines but not the side lines.

The ownership of a properly located lode claim carries with it ownership of the vein anywhere between vertical planes through the end lines. This holds even though the inclination of the vein from the vertical carries the vein underground beyond the side lines of the claim. The effect of this is to modify the usual rule of bounding vertical planes; the owner of the claim on the outcrop owns the vein even if it passes beyond a vertical plane through either of the side lines of his claim.

The end lines of the claim must be parallel straight lines, and the length of the claim must be measured along the center line of the claim. Except as limited above, the claim may be of any shape.

404a. Special Cases.—Figure 404*a* represents an ideal rectangular claim 1,500 ft. by 600 ft. *D* is the point of discovery, and the irregular line represents the outcrop of the vein, crossing the end lines but everywhere between the side lines. The center line and each side line are 1,500 ft. long. The end lines are parallel straight lines, each 600 ft. long.

Figure 404*b* shows a four-sided trapezoidal claim, such as is sometimes necessary on account of the shape of adjoining properties. Again the center line is 1,500 ft. long, but to secure the maximum width of 600 ft. between side lines, the length of each end line must be

$$\frac{600}{\sin 60^\circ} = 692.82 \text{ ft.}$$

or one half of this distance each way from the end of the center line. The 60° angle was selected merely for illustration. The angle might,

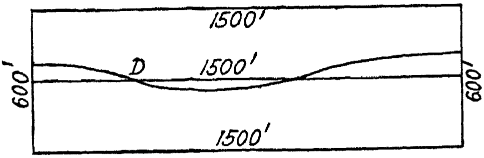


Fig. 404*a*.

of course, have any value less than 90°, but the smaller the angle, the smaller will be the perpendicular distance between the end lines of the claim.

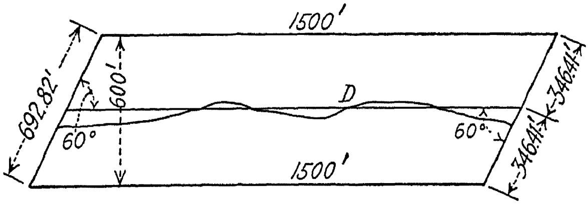


Fig. 404*b*.

Occasionally the topography is such that one or more angles in the center line will be necessary if the outcrop is surely to be kept within the side lines throughout the length of the claim. Figures

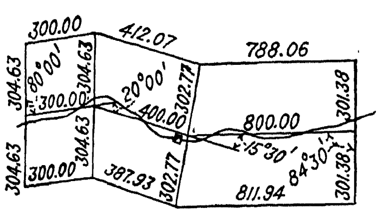


Fig. 404*c*.

404*c* and 404*d* illustrate such cases. The lengths of the end lines are found by the method just described, and the distances from the points where the center line breaks to the corners opposite them are found as illustrated by the following example.

Example: The side lines of a claim, as shown in Fig. 404*e*, lie parallel to the center line and 300 ft. from it. The center line deflects 20° at *A*. It is desired to locate points *B* and *C*. Corners *P* and *C* will lie on the

bisector of the large angle at A, and four equal right triangles will be formed as shown. Solving one of these triangles gives the distances $AB = AC$ and $BD = BE = CF = CG$. As illustrated by Fig. 404e,

$$AB = AC = \frac{300}{\cos 10°00'} = 300 + 300 \text{ exsec } 10°00' = 304.63 \text{ ft.}$$

$$BD = BE = CF = CG = 300 \tan 10°00' = 52.90 \text{ ft.}$$

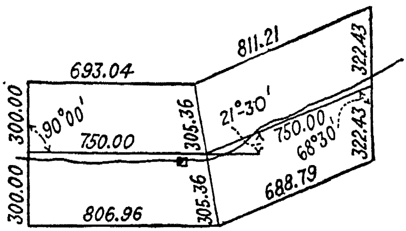


FIG. 404d.

The transit is now set up at A, the proper angle is turned off, and points B and C are set. By measuring the lengths of the side lines from corner to corner and comparing the measurements with the calculated lengths, the location of the corners is checked.

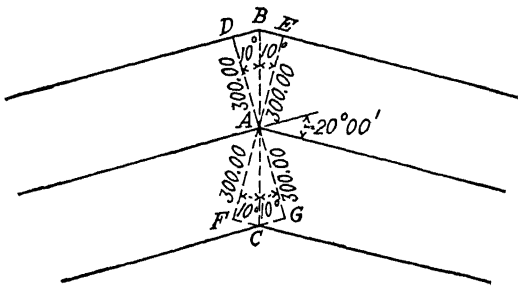


FIG. 404e.

Claims of the type illustrated by Figs. 404c and 404d are made necessary in cases where an irregular ground surface gives a curved outcrop. This condition will be accentuated if the dip of the vein is relatively small.

405. Field Work.—In locating a mining claim to secure the maximum dimensions and area allowed by law, the following procedure is suggested:

When the maximum allowable length of the center line of a claim is fixed, this line is so located as to follow the outcrop closely with as few breaks as practicable. The deflection angles are read at breaks in the center line, after establishing hubs at these points and at the two ends of the line. The sum of the various segments of the center line should be equal to the maximum allowable length of the

claim (as 1,500 ft.). The transit is set up at one end of the center line, the proper angle is turned from the center line, and the corners are set at the two ends of the end line. A similar procedure is followed at the other end of the claim and at breaks on the center line. By solution of the right triangles similar to those illustrated by the preceding examples, the direction and length of each line of the traverse bounding the claim are calculated. Then as a check on the location, a traverse is run through the points forming the boundary. Such care in the location survey is not a legal requirement, but it is desirable from the point of view of the locator.

The location survey may be made by the claimant or by someone employed by him. The final survey for patent must be made by a United States mineral surveyor, commissioned by the United States to do that work.

406. Problems.

1. A vein has a strike of N10°15′W and a dip of 43°40′. What will be the bearing of a drift in the vein having a grade of 2 per cent?

2. A vein has a strike of N27°30′E. A drift in the vein on a 3 per cent grade has a bearing of N30°20′E. What is the dip of the vein?

3. A transit has an auxiliary side telescope, the line of sight of which is offset 0.35 ft. from that of the main telescope. In measuring a horizontal angle, the sights being taken with the side telescope to the right of the main telescope, the following measurements were taken: distance $OA = 47.32$ ft.; distance $OB = 268.3$ ft.; angle $AOB = 135°42'$ (point B is to the right of point A). What is the corrected angle AOB?

4. If the line of sight of a side telescope is inclined to that of the main telescope by an angle of 01′, what error in azimuth will be introduced in measuring a horizontal angle between two points, if the sight to one is horizontal and to the other is inclined 68°? 85°? Disregard reduction to center.

5. From a given station A, at the portal of a tunnel, both a tunnel traverse and a surface traverse are run, with results as follows: for the tunnel traverse, A to B, azimuth = 310°22′, distance = 320.2 ft., vertical angle = +1°20′; B to breast, azimuth = 355°30′, distance = 286.1 ft., vertical angle = +2°01′; for the surface traverse, A to C, azimuth = 24°41′, distance = 416.8 ft., vertical angle = +2°54′; and C to D, azimuth = 343°16′, distance = 458.3 ft., vertical angle = +18°16′. A vertical shaft is to be sunk at station D, and the breast of the tunnel is to be connected with the shaft by a drift having a 2 per cent grade.

(*a*) How deep must the shaft be?

(*b*) What will be the azimuth of the drift?

(*c*) What will be the slope distance?

6. Given the traverse at top of page 605. Calculate the azimuth, length (slope distance), and the vertical angle of a line to connect station 28 to station 32.

Station	Object	Azimuth	Slope distance	Vertical angle	Object	Height of inst.
28	22	255°32′	138.07	−0°44′	+4.92	−1.09
22	28	75°32′	....	....	....	−4.92
	21	253°04′	167.48	−0°53′	+9.18	
21	22	73°04′	....	....	....	−6.22
	31	344°58′	115.78	−80°32′	+2.93	
31	21	164°58′	....	....	....	−6.78
	32	73°32′	304.02	−0°10′	0.00	

7. Three bore-holes have been sunk to a vein of ore. The depth of these holes at the points A, B, and C, and the surface measurements connecting them, are as follows: elevation of surface at A = 4,750, depth of hole = 3,500 ft.; at B, elevation = 4,920, depth = 2,860 ft.; at C, elevation = 4,790, depth = 2,080 ft.; azimuth of AC = 60°22′, distance = 1,320 ft.; azimuth of AB = 80°30′; azimuth of CB = 140°20′. distance = 920 ft. Required the strike and dip of this vein (see Ref. 13).

REFERENCES

1. COOKE, L. H., "Measurement of Horizontal Angles with Steep Sights," *Trans. Inst. Min. and Met.*, Vol. 31, p. 389, 1922.
2. DAVIS, JAMES L., MAY, EDW. A., and TIDD, ARTHUR W., "Methods of Alignment in Three Aqueduct Tunnels," *Eng. News*, Vol. 67, p. 1183, 1912.
3. FISHER, F. L., "Organization of Mine-surveying Work," *Eng. and Min. Jour.*, Vol. 111, No. 2, p. 58, Jan. 8, 1921.
4. HAERTTER, J. H., "Surveying at Lytle Colliery," *Mines and Mining*, Vol. 29, p. 108, October 1908.
5. HARRINGTON, D., "Colliery Surveys," *Mines and Mining*, Vol. 30, pp. 94, 159, 235, 305, 337, 439, 1909.
6. KUHN, L., "Transfer of Surface Alignment Underground by the Use of Plumb Bobs," *Eng. and Min. Jour.*, Vol. 55, p. 179, Feb. 25, 1893.
7. MARKS, E. O., "A Mining Model," *Trans. Inst. Min. and Met.*, Vol. 23, p. 394, 1914.
8. MCFARLANE, G. C., "Air Currents and Shaft Plumb Lines," *Engr. and Min. Jour.*, Vol. 91, p. 318, Feb. 11, 1911.
9. MEARS, F., "Surveys, Construction Methods, and a Comparison of Routes," (Cascade Tunnel), *Trans. Am. Soc. Civ. Eng.*, Vol. 96, p. 926, 1932.
10. MORRISON, E. C., "Mine Surveying," *School of Mines Quarterly*, Vol. 29, p. 34, November 1907.

11. Priest, C. G. and Whyte, W., "Mine Survey Records and Calculations," *Trans. Inst. of Min. and Met.*, Vol. 23, p. 400, April 1914.
12. Riter, G. W., "Mine Survey Notes," *Trans. Am. Inst. Min. Eng.*, Vol. 41, p. 790, 1912.
13. Simons, T., "Calculation of Strike and Dip," *Eng. and Min. Jour.*, Vol. 97, No. 15, p. 753, Apr. 11, 1914.
14. Weeks, W. S., "A Graphical Method for the Correction of Steel Tapes," *Univ. Calif. Publ. Eng.*, Vol. 1, No. 9, pp. 261–268, 1916.
15. Whitehead, W. E., "Steep Sights in Underground Surveys," *Trans. Inst. Min. and Met.*, Vol. 31, p. 263, 1922.

CHAPTER XXIII

USE AND ADJUSTMENT OF THE PLANE TABLE

407. General.—The essential features of the plane-table instrument (Fig. 409*b*) are (1) a plane drawing board supported by a tripod, and (2) a line of sight having its vertical plane fixed with relation to a straightedge ruler which rests upon the drawing board. These features make it possible to plot the position of any object sighted from the station occupied by the instrument, as follows: With the straightedge through the plotted point representing the station, the line of sight is directed at the object and a line is drawn on the map; this line represents the direction from the station to the object. If along this line the measured distance between station and object is plotted to scale, the relative position of the object is definitely located on the map. Other features of the instrument such as the telescope, the stadia hairs, and the vertical arc are used to increase its usefulness; these features are described later in this chapter.

The term *plane table* as it is ordinarily used is somewhat ambiguous, for it refers to all of the features mentioned above, whereas in fact the board with its supporting tripod is more accurately termed the plane table, and the line of sight together with the straightedge ruler is called the *alidade*. In this book, as in common practice, however, the term *plane table* will refer to both the table and its accompanying alidade.

By means of the plane table, points on the ground to which observations are made are immediately plotted in their correct relative positions on the drawing, all angles being plotted graphically. The plane-table method is more especially adapted to securing the details of the map, and on extensive surveys the primary points of horizontal and vertical control are established by other more precise methods. With its accessories, the plane table (often employed in conjunction with other surveying instruments) is adapted to mapping, regardless of scale, precision required, or magnitude of the survey.

408. Relation between Transit and Plane Table.—It is helpful to a clear understanding of the principles which underlie the use of the plane table, to consider the similarity between angular measurements made with the transit, together with the office procedure

of plotting the notes, and the corresponding operations with the plane table. To this end the methods of traversing with the transit should be reviewed (Arts. 222 to 229) also the methods of plotting traverses should be recalled (Arts. 256 to 260).

The plane-table board may be said to take the place of the graduated horizontal circle of the transit, and orientation consists in turning the table until some line on the paper becomes parallel with a corresponding line on the ground, and clamping the table in this position. After orientation, the direction of any line is observed by turning the alidade until the line of sight coincides with the line (just as with the transit, the upper motion is rotated) and graphically the direction of the line is given by the position of the straightedge on the paper (just as with the transit, the numerical value of the azimuth is given by the vernier reading). If the table is oriented at any point A (Fig. 413), whose corresponding location on the paper is a, the direction of the line AB on the ground is determined by pivoting the straightedge about the point a on the paper until B is sighted. The corresponding line ab on the paper is then established by drawing a line along the straightedge and by laying off to scale the measured distance AB. Thus it is seen that with the plane table there is employed what is virtually a combination of transit and drafting-room methods, but no record of numerical values is secured. The plane table is therefore suitable for mapping only.

409. Instruments.—Three distinct types of tables are in common use, namely; the *Coast Survey*, the *Johnson*, and the *traverse* tables.

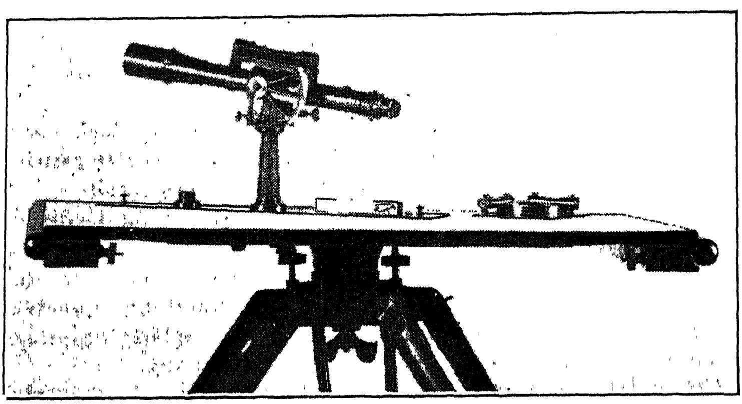


Fig. 409*a*.—Plane table, Coast Survey type.

409a. Coast Survey Table.—This is the most stable of the three types and is suitable for triangulation work (Art. 415), using sights

possibly several miles in length, and for the relatively high degree of accuracy required by the U. S. Coast and Geodetic Survey on its shoreline charts, or for large-scale city work. The board is 24 by 30 in. (Fig. 409*a*) and is securely attached to a metal casting below, so arranged that the table can be accurately leveled by three leveling screws. By means of a clamp and tangent-screw the board may be fixed in any position in azimuth. The paper on which the map is made is held in position by metal spring clamps, thus permitting the use of a sheet larger in size than the board. The tripod is of heavy and rigid construction.

409b. Johnson Table.—This table, shown in Fig. 409*b*, was devised by Willard D. Johnson of the U. S. Geological Survey. It is well adapted to the requirements of most plane-table work.

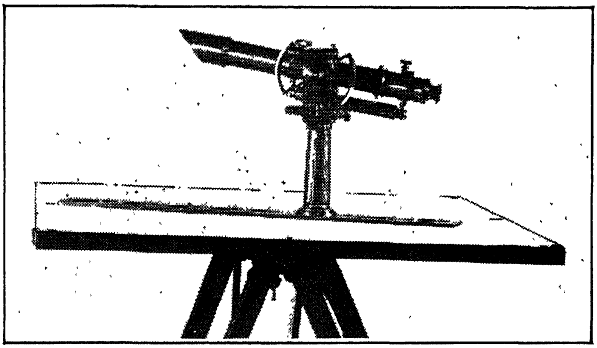


Fig. 409*b*.—Plane table, Johnson type.

The table consists of a board 24 by 30 in., gained into the under side of which is a circular brass plate into which a threaded opening has been cut. By this means the table can be screwed to the upper casting of the tripod head. This head consists of two castings which form a ball-and-socket joint (Fig. 409*c*), held by two clamp screws. By loosening clamp *A* the ball is free to rotate in its socket, and thus the table may be brought to a level position. By tightening the clamp the table is fixed in this plane. By loosening clamp *B* the table can be rotated about the vertical axis. The table is sufficiently stable and precise in its controls to be suitable for perhaps nine tenths of topographic mapping work.

409c. Traverse Table.—The table, usually 15 by 15 in. in size, is attached to a simple tripod movement (Fig. 409*d*) such that it can be rotated about the spindle and can be clamped in position. There is no leveling device, but by estimation the table is approximately leveled by adjusting the tripod legs. A box-trough compass,

with needle generally 4 in. long, is gained into the board near one edge. The traverse table is suitable for: (*a*) military reconnaissance sketches; (*b*) traverses for small-scale maps, and for mapping roads to be used as control data on larger plane-table sheets; and (*c*) mapping ridges, valleys, and relatively inaccessible areas to fill in a topographic map being drawn on a larger sheet.

410. Alidades.—An alidade, in its original meaning, is a combined sight and straightedge ruler. In addition to this meaning, the term

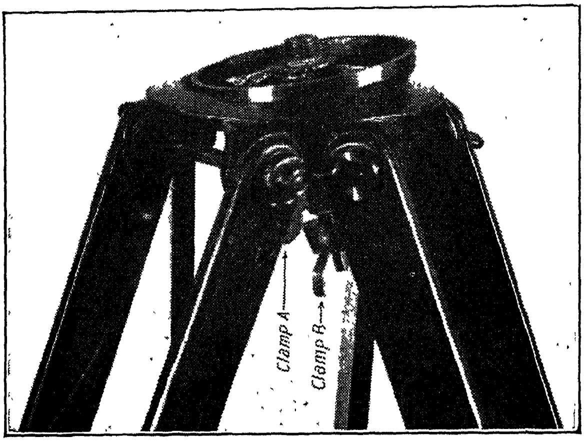


Fig. 409*c*.—Johnson head.

is now applied to the upper motion of the transit which consists of that portion of the instrument attached to the inner spindle, including the verniers, the standards, and the telescope.

410a. Peep-sight Alidade.—One type of alidade used in plane-table work consists of a peep sight, mounted on a ruler. The peep sight is formed by two sight vanes, either fixed or folding, similar to those employed on the surveyor's compass. The ruler generally consists of a brass plate, 6 to 10 in. long, one edge of which is beveled and graduated to a suitable scale. It is often employed by topographers as an auxiliary to the telescopic alidade and is almost indispensable for sighting details while sketching.

410b. Telescopic Alidade.—The telescopic alidade is designed to afford greater accuracy in the control of the table, and especially to make possible the stadia method of measuring distances. The base of the alidade usually consists of a brass ruler or straightedge approximately 18 by 3 in., beveled on one edge. Upon this ruler sometimes is mounted at one end a circular level, and at the other

end a box compass with a 4-in. needle (Fig. 409*a*). In the center of the ruler is mounted a column which supports the telescope, vertical arc, either a striding or an attached level, and in addition many instruments are provided with a Beaman stadia arc and a vernier-control bubble.

There are two distinct types of telescopic alidades in use. In one type the telescope tube is rigidly attached to or is an integral part of the horizontal axis, as in the engineer's transit. This may be called the *fixed-tube* type. The telescope tube of the other type is fitted into a cylindrical sleeve which is an integral part of the horizontal axis. In this type, which may be called the *tube-in-sleeve* type, the telescope may be turned about its axis in the sleeve much as may

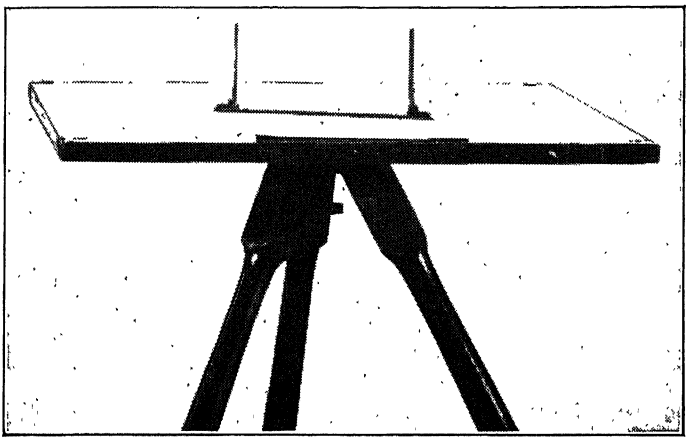


FIG. 409*d*.—Traverse table.

the telescope of the wye level be turned in its wyes. On the telescope are turned two shoulders perhaps 5 in. apart, upon which rests a striding level.

With either type the vernier of the vertical arc may be fixed or movable, or there may be an auxiliary level tube attached to the vernier arm as with the transit. Because the plane table is relatively unstable as compared with the transit, a control level mounted on the vernier arm greatly facilitates the measurement of vertical angles, rendering unnecessary an initial reading and index correction. If many vertical angles are to be read this level is essential. On the ruler of some instruments, instead of the circular level two plate levels are mounted at right angles to each other. A separate combined level and compass is sometimes used, two plate levels and a small circular compass being mounted on a brass plate 5 in. square

(Fig. 409*a*). This device, called a *declinatoire*, is used only in orienting and in leveling the table.

411. Setting Up and Orienting the Table.—When the plane table is being set up, the tripod legs should be spread well apart and firmly planted in the ground, and the table should be approximately waist-high, so that the topographer may bend over the board without resting against it. The board is leveled by whatever device is provided, but since few tables are sufficiently rigid to remain level as the alidade is shifted about, no special attempt is made to see that the board is perfectly level each time an observation is made.

For plotted angles to be theoretically correct, the plotted position of the station at which the plane table is set should be exactly over the corresponding point on the ground. Practically, the degree of care exercised in bringing the plotted point over the ground point depends upon the scale of the map. On small-scale work the plane table is set over the station without any attempt being made to place the plotted point vertically above the ground point; on very large-scale mapping the table is roughly set up, approximately oriented, and then, by plumbing, the table is shifted bodily until the point on the paper is practically over the station point. In any case, the aim is to set up with sufficient care so that the plotted position of lines drawn from the station will be correctly shown within the scale of the map.

Orientation may be accomplished by any of the following methods:

1. *By the magnetic needle.* For rough mapping at small scale, orientation by the magnetic compass is often sufficiently exact. This method is susceptible to the same errors as those encountered when using the surveyor's compass, but possesses the advantage over other methods in that an error in the plotted direction of one line introduces no systematic errors in the lines plotted from succeeding stations. The table is oriented by rotating it until the fixed bearing (usually magnetic north) is observed. The table is then clamped, and all mapping at the station is carried on without disturbing the board. If the compass box is attached to the alidade or to a movable plate, the straightedge is alined with a meridian which was drawn on the paper at the station which was first occupied, and the table is turned until the needle reads north.

2. *By backsighting along an established line the position of which has previously been plotted.* The method is equivalent to that employed on the azimuth traverse with the transit. Its advantage over the use of the magnetic needle lies in the increased precision obtainable; a disadvantage is that an error in direction of one line is transferred to succeeding lines. It is the method generally employed on intermediate- and large-scale mapping. If the plane

table is set up at B (Fig. 413) on the line AB which has previously been plotted on the paper as ab, the straightedge of the alidade is placed along the line ba and the board is oriented by rotating it until the line of sight falls at A. The table is clamped in this position and is not again disturbed.

3. By the *three-point* and *two-point problems*, which will presently be explained (Arts. 417 and 418).

412. Method of Radiation.—When the table has been oriented, the direction to any object in the landscape may be drawn on the

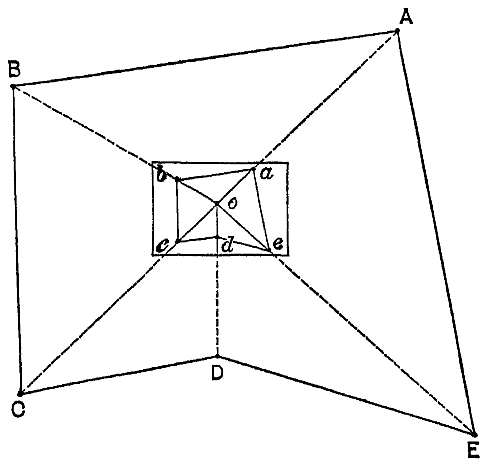


Fig. 412.—Radiation with plane table.

map by pivoting the alidade around the plotted position of the plane table station, pointing the alidade toward the distant object, and drawing a line along the edge of the ruler. Thus in Fig. 412, the plane table is shown in position over station O in the center of the field. The plotted position of the plane-table station is indicated at o on the plane-table sheet. The alidade is pivoted about this point, and as sights are taken to the points A, B, C, etc., rays are drawn along the edge of the ruler. The distances are measured and are then plotted to scale along the corresponding rays. Thus the positions of the objects sighted are located on the map. This procedure is termed *radiation*.

413. Method of Traversing.—In the traverse method the same principles are involved as in traversing with the transit. As each station is occupied the table is oriented, sometimes with the compass needle, but more generally by taking a backsight to the preceding station, the straightedge of the alidade being placed along the posi-

tion of the preceding traverse line and the board being turned until the line of sight cuts the preceding station. With the board clamped in this position the straightedge is pivoted about the plotted position of the station occupied, and a sight is taken to the following station and its position is plotted as in the radiation method just described.

Thus in Fig. 413, a series of traverse stations *A*, *B*, *C*, *D*, etc., is represented. Let the plane table be set up first at station *A*, and let *a* representing *A*, be plotted in such position that the other stations will fall within the limits of the sheet. With the straightedge passing through *a*, a sight is taken to station *B* and *b*, its position on

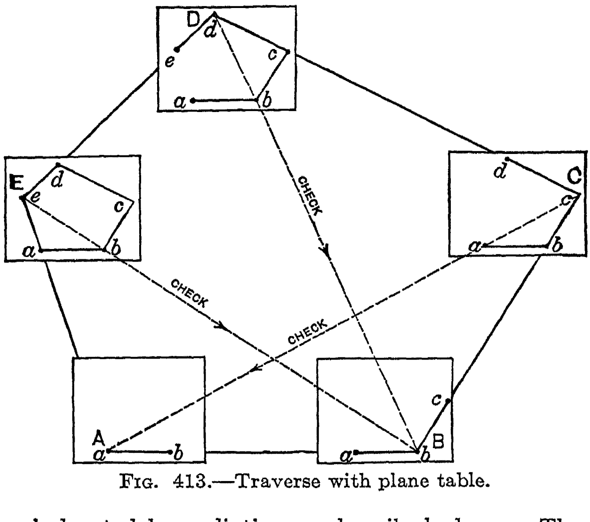


FIG. 413.—Traverse with plane table.

the map, is located by radiation as described above. The instrument is next set up at station *B*, the alidade is placed along the ray *ba*, the table is oriented by rotating the board until the line of sight points at station *A*, and the table is clamped in this position. A foresight is taken to station *C*, and its position is plotted at *c*. By a similar procedure the positions of the stations *D* and *E* are plotted at *d* and *e*. If the traverse stations form a closed figure, then the plotted traverse should also form a closed figure. Any error of closure indicated by the initial and the final plotted positions of the point *a* is an indication of the precision of the work and is a final check on the work.

At any station a portion of the traverse may be checked if two or more of the preceding stations are visible and are not in the same straight line with the station occupied. Thus, if the plane table at

station C is oriented by sighting at B and, in addition, station A can be seen, a ray drawn from c toward station A should pass through a, provided the traverse between the two stations is correctly drawn. In order that the check be reliable, the angle between the traverse line and the check line should not be small.

414. Method of Intersection.—This method is similar to that employed with the transit in locating an object by angles taken from each end of a line of known length, and makes use of the principle that if the angles and one side of a triangle are known, the remaining sides may be determined. It is a useful method for locating objects when distances to them are not otherwise conveniently obtainable.

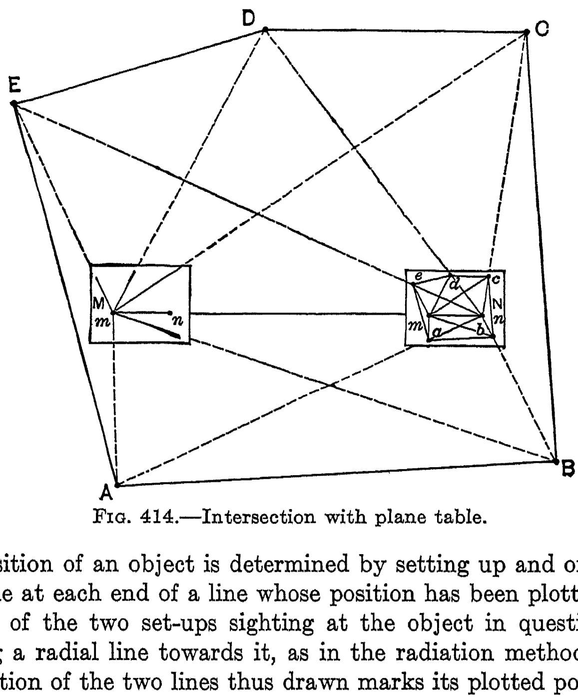


FIG. 414.—Intersection with plane table.

The position of an object is determined by setting up and orienting the table at each end of a line whose position has been plotted, and at each of the two set-ups sighting at the object in question and drawing a radial line towards it, as in the radiation method. The intersection of the two lines thus drawn marks its plotted position.

Thus the plotted positions of the objects A, B, and C (Fig. 414) may be determined as follows: The table is set up and oriented over station M, the point m on the sheet marking the plotted position of the station. The alidade is sighted on station N by pivoting about the point m, a ray is drawn, and the distance from M to N is scaled along the line to locate the point n on the sheet. The alidade is centered about the point m and rays of indefinite length are drawn toward the objects A, B, and C. The table is then moved

to station N, is set up, and is oriented by backsighting to station M. The alidade is pivoted about n and rays are drawn toward the objects A, B, and C. The intersections of these rays with the corresponding rays drawn from m mark the plotted positions of the objects as at a, b, and c. Distances to the objects are not measured, but may be scaled from the map.

415. Method of Graphical Triangulation.—The method of graphical triangulation, about to be described, achieves the same results as does triangulation with the transit, as explained in Chap. XXVIII. But the procedure is quite different in that the plotted positions of the distant signals are determined graphically on the plane-table

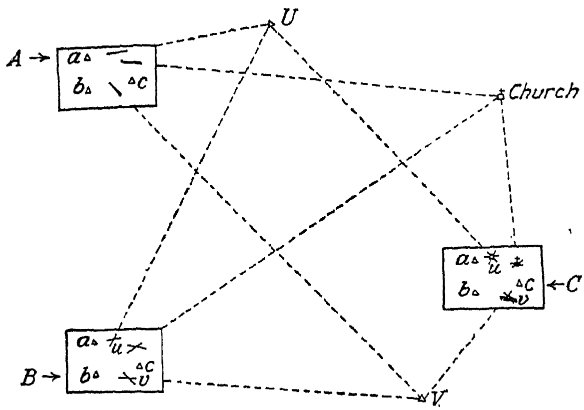


FIG. 415.—Graphical triangulation.

sheet instead of by the use of transit angles, office computations, and plotting methods, as in the case of the transit method. The peculiar advantage of the use of the principles of intersection and of resection (the latter to be explained in the following article) in plane-table mapping makes it desirable to locate the positions of many definite landmarks which are widely visible and suitably situated, such as flagstaffs, church spires, lone trees, etc. Accordingly, at the same time that the topographer locates the signals of distant stations, he also locates landmarks whose positions are not to be occupied by the instrument, but which will be useful in subsequent work.

In employing this method at least two and preferably three stations in the field, as A, B, and C (Fig. 415) whose positions are known, must be capable of being occupied by the instrument, and must be marked by signals. Their positions are plotted on the plane-table sheet at a, b, and c prior to going to the field. The field pro-

cedure is as follows: The table is set up, say at A, oriented by sighting at B and C, and rays are drawn towards stations U and V and, say, the church spire. The table is then set up and oriented at stations B and C in succession, and the same objects are again sighted. The correct plotted positions of the desired stations may be determined by the intersection of two rays, as by those drawn from stations A and B, but proven locations of points are secured when the three rays drawn toward a given object are found to pass through a point, as shown by the third rays drawn from station C.

This method is most advantageous where the terrain offers unobstructed sights, considerable relief, and many well-defined objects. It is more especially employed in intermediate- or small-scale mapping.

416. Principle of Resection.—It has already been explained in Art. 414 that by the method of intersection, the plotted position of an object is located on the plane-table sheet by rays drawn to it from two or more stations. Now let it be supposed that the plane table itself is the object to be located; that is, the position of the ground point over which the table has been set up is to be plotted on the map. This may be accomplished *after the board has been oriented* by drawing rays from two or more visible objects through the corresponding plotted positions of the objects on the map; and the plotted position of the plane table will be defined by the point where these two or more rays cross one another. This procedure is termed *resection*, and the rays are said to *resect* at the plotted position of the plane-table station on the map. The fact should be emphasized that the principle of resection is used only after the table has been oriented.

As explained in Art. 411, four methods of orientation are commonly used: (1) by the magnetic needle, (2) by backsighting along a line the direction of which has been plotted but the length of which may be unknown, (3) by the three-point problem, and (4) by the two-point problem. In the last two methods, orientation and resection are accomplished in the same operation, as will be shown in later paragraphs.

These methods of locating the position of a station which has not previously been plotted are often of the greatest convenience, for they enable the topographer to select advantageous stations for his work which otherwise could not be used.

416a. Resection: Orientation by Magnetic Needle.—When this method of orientation is used, the method of resection is as follows: If P be the station over which the plane table is set up but the plotted position of which is unknown, and if A and B be two visible stations the plotted positions of which are a and b, then p, the plotted position

of P, is determined by resecting through a in the direction of A and through b in the direction of B. The plotted position of the plane table will be defined by the point where these two (or more) rays cross one another. The method, of course, would be utilized only for small-scale or rough mapping for which the relatively large errors due to orienting with the compass needle would not impair the usefulness of the map.

416b. Resection: Orientation by Backsight.—If the table is oriented by backsighting along a line (Art. 411), the method of resection is as follows: Suppose that the topographer wishes to occupy station C (Fig. 416), a convenient summit, toward which a single ray has been drawn from station B, perhaps during a previous day's work. Also, suppose that stations A and B are visible from station

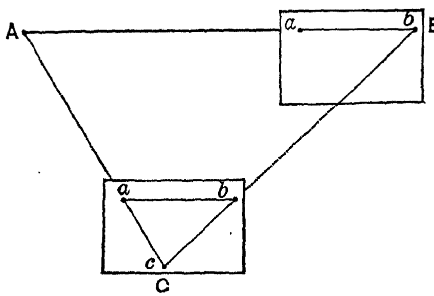


FIG. 416.—Resection.

C, but that station C has not been located on the map. Under these conditions, the topographer occupies station C, and he orients the table by placing the alidade rule along the ray drawn from station B and turning the board until station B is sighted. Then he pivots the alidade about the point a on the map, sights at station A, and draws a ray to cross, *i.e.*, to resect, the ray drawn from B. The point c where these two rays cross marks the plotted position of the occupied station. As in the method of intersection, so in the method of resection the position of the plotted point will be indefinite if the angle between the intersecting or resecting rays is small.

417. Three-point Problem.—Frequently the topographer wishes to avail himself of an advantageous position in the field, which has not been located on the map and towards which no ray from located stations has been drawn, and at the same time orientation by use of the compass needle is not sufficiently accurate. When three located stations are visible the three-point problem, of which there are several solutions, offers a convenient method of orienting and

resecting in the same operation. In the United States, Lehmann's solution is the one most commonly used by experienced topographers. This solution involves a trial orientation of the plane table; the resecting rays from three known points will not intersect at a common point unless the trial orientation happens to be correct. Rules are given by Lehmann as a guide to the adjustment of the table to the correct azimuth. The mechanical or tracing-cloth solution (Art. 417*c*), while somewhat simpler to understand, is not as satisfactory nor as expeditious under the usual field conditions.

417a. Lehmann's or Coast Survey Method.—The following paragraphs taken from "A Plane-table Manual" by D. B. Wainwright (Ref. 8, p. 631), describe clearly the application of this method.

When the table is imperfectly oriented, the lines drawn from the three projected points, when sighting on the corresponding actual points, will not intersect at one point unless all four are on the circumference of a circle (see Fig. 417*c*, indeterminate position). Except in this case, the lines will form a small triangle called the *triangle of error* (Figs. 417*a*, 417*c*, 417*e*, and 417*f*), or two of the lines will be parallel, intersected by a third (see Fig. 417*d*, station on range between two fixed points, and Fig. 417*b*, station on prolongation of range line). The solution of the three-point problem determines the location of the station occupied and orients the table simultaneously.

The relative positions of the three fixed points with reference to the new station have an important bearing on the strength of its determination.

In the following statements in regard to the different groupings met with in practice, for the sake of brevity the term "fixed points" will be understood to mean points already determined and plotted on the sheet; the "great triangle" referred to is one formed by the three fixed points, and the "great circle" is the circle passing through them.

1. When the new station is outside the great circle, the determination of a position will be weak when the middle point, as seen from the new station, is the farthest of the three and the angles are small (new station below the circle in Fig. 417*c*).

2. The determination increases in strength for given angles as the middle point approaches the new station (Fig. 417*a*).

3. When one angle is small or 0° (points in range), the determination will be strong, provided the two points making the small angle or range are not too near each other when compared to the distances to the new station and to the third point; provided also the angle to the third point is not too small (Fig. 417*b*).

4. When the new station lies on or near the great circle, its position is indeterminate (Fig. 417*c*).

5. When the new station is within the great circle, the strength of its determination increases as it approaches the center of gravity of the great triangle (Figs. 417*c*, 417*d*, and 417*e*).

There are a number of graphic solutions, but many of them are better suited to the drafting room with its appliances than to conditions which exist in the field. The method following solves by

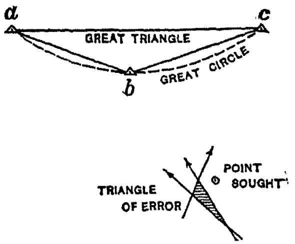


FIG. 417*a*.

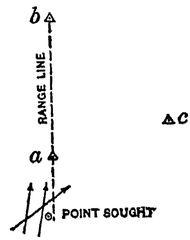


FIG. 417*b*.

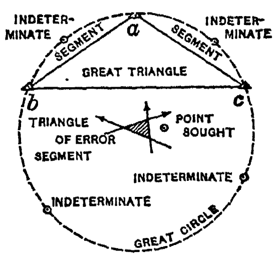


FIG. 417*c*.

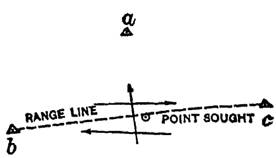


FIG. 417*d*.

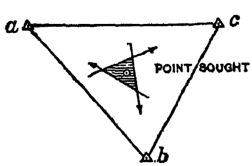


FIG. 417*e*.

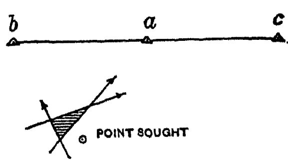


FIG. 417*f*.

estimation instead of by actual construction. It utilizes certain geometrical relations of the lines and points to serve as guides in making an estimate. It is most rapid in the hands of an experienced topographer, but for those having only occasional use for a graphic solution the tracing-cloth method is recommended.

417b. Rules for Lehmann's Method.—The directions for correcting the orientation after the first trial are stated in the form of rules.

The term "point sought" will be understood to mean the true position on the sheet of the projected point of the station occupied. The sur-

veyor is assumed to be facing the signals, and the directions right and left are given accordingly.

Rule 1.—*The point sought is always distant from each of the three lines drawn from the three fixed points in proportion to the distances of the corresponding actual points from the station occupied, and it will always be found on the corresponding side of each of the lines drawn from the fixed points.*[1]

The simplest case for the application of this rule occurs when the station to be determined is within the triangle formed by the three fixed points; the point sought must then be within the triangle of error to satisfy the conditions (Fig. 417*e*).

Although Rule 1 is sufficient in itself for the solution of the problem, there are two subordinate rules which materially assist the topographer in reaching a decision as to the proper location of the point sought with reference to the lines from the fixed points.

Rule 2.—*When the point sought is without the great circle, it is always on the same side of the line from the most distant point as the intersection of the other two lines* (Fig. 417*a*).

Rule 3.—*When the point sought falls within either of the three segments of the great circle formed by the sides of the great triangle, the line drawn from the middle point lies between the point sought and the intersection of the other two lines* (Figs. 417*c* and 417*f*).

Application of Rules.—In practice the topographer first decides the relation of the new station with reference to the fixed points, whether it is within the great triangle or in one of the segments of or outside the great circle. He then determines the position of the point sought with reference to one line (if within one of the segments of or without the great circle, by Rule 2 or 3); it then follows from Rule 1 that it must be on the corresponding side of the other two lines. Finally, he estimates the relative distances of the three actual points from him and marks the position of the point sought a proportionate distance from each of the three lines. Using this point, he reorients the table.

417c. Tracing-cloth Method.—A simple procedure, known as the tracing-cloth method, is as follows: A piece of tracing cloth or tracing paper is fastened on the plane table over the map. Any convenient point on the tracing cloth is chosen to represent the unknown point over which the plane table is set, and from it rays are drawn to the three distant and known points. Then the cloth is loosened and is shifted over the map until the three rays pass through the plotted positions of the corresponding known points. The intersection of the rays marks the point sought. It is pricked through onto the plane-table map, and the table is oriented in the usual manner.

[1] That is, if it is on the right side of one line it will be on the right of the other two.

418. Two-point Problem.—The purpose of the two-point problem is to orient the table and to locate the point sought when two stations only are visible, and when it is impossible or undesirable to occupy either of them, as when the signals are inaccessible or at a considerable distance. To accomplish this, it is necessary that two set-ups be made, as follows:

The positions of the two known stations A and B are plotted on the sheet at a and b (Fig. 418a). The table is set up at some point C, by estimation is oriented as nearly as possible, and the point c' is located by resection through a and b. A sight is taken toward the point P whose plotted position is desired, and a ray is drawn toward it. The table is taken to station P, and is oriented

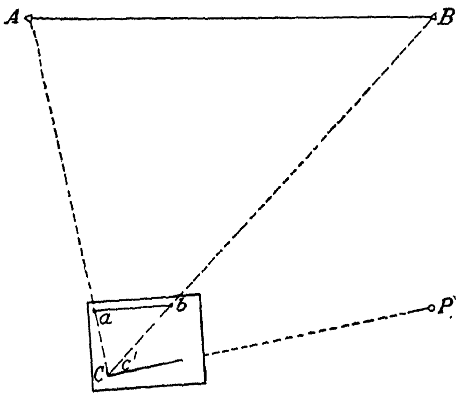


Fig. 418a.

by sighting along the ray $p'c'$ toward C (Fig. 418b). By resection through a the point p' is marked. Through this point the alidade is sighted toward station B, and a ray is drawn intersecting the line $c'b$ at the point b'. The direction of the line ab' is now parallel to the direction of the line joining the two stations A and B, therefore the angle bab' is a measure of the error in the attempted orientation of the table at station C. A convenient method of turning the table through this small angle into its true position, is as follows: The alidade is placed along the line ab' and a signal is set on the line or some definite point, as Z, is selected on the line of sight. Next, the alidade is placed along the line ab and the table is turned until the signal or selected point is sighted. The table is now oriented, and by resection through a and b the true plotted position of the table is located at p, as shown in Fig. 418c.

418a. Verification.—The procedure described above offers no verification for the location of p, hence topographers are reluctant to use the two-point problem. Verification is possible, however, by a simple construction (in which the scale only is employed) now to be described. Before the table is oriented at station P, locate the center o of the circle through the three points a, b, and c' and draw the radius oc' and a short arc (Fig. 418d). (The other arcs shown in the figure need not be drawn.) Place the alidade along this radius and set a signal or select a point on the line of sight, as when sighting along the line ab'. After the board is correctly oriented, center the alidade at o, sight the signal or object

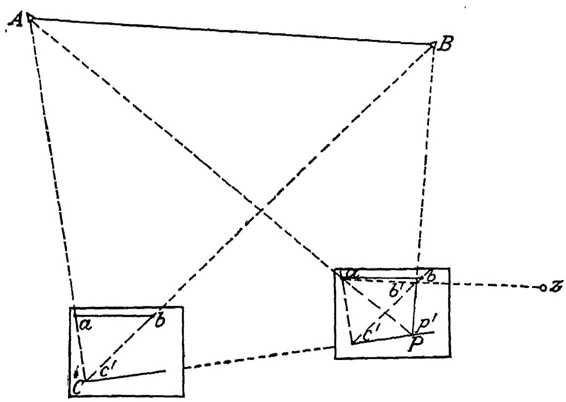


FIG. 418b.

previously sighted, and draw the ray oc''. Now scale the chord distance $c''c$ equal to $c'c''$. The location of the point c thus determined is the correct plotted position[1] of the point C and hence a third located point is obtained, by means of which the plotted position of the occupied station may be verified.

418b. A Special Case.—A special case of the two-point problem should be mentioned. On occasion it is possible to set the table in range with the two points A and B, as at C (Fig. 418e). In this case the board at C can readily be oriented and the correct location of the point d is effected by resection as follows: At C, any convenient point c on the prolongation of the line ab is chosen and a ray cd is drawn toward the station D. The table is set up at D and is correctly oriented by

[1] Proof: The angle $c'oc''$ is equal, by construction, to the error in orientation $c'ac = b'ab$. But since the angle $c'oc''$ is subtended at the center of the circle and the angle $c'ac$ is subtended at a point on the circumference, it is obvious that the correct location of the point c will be found at a point along the circumference such that the arc distance $c'c$ = angle $c'oc$, is twice the arc distance $c'c''$ = angle $c'oc''$. Therefore, the point c is found on a radius at a distance from c'' equal to the distance $c'c''$.

use of the ray previously drawn. The point d is then located by resection on the points A and B.

419. Measurement of Difference in Elevation.—The methods of fixing on the plane-table sheet the *horizontal projection* of ground

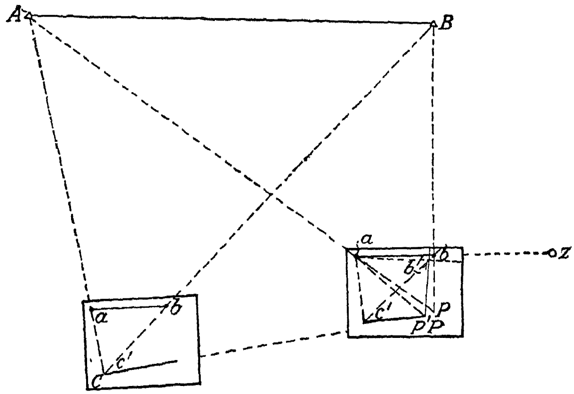


FIG. 418*c*.

points have been described. Since the plane table is used principally in the work of topographic mapping, many *elevations* of ground points are determined by methods exactly similar to those employed in the case of the transit alidade when used in leveling work, namely, by direct leveling, stadia leveling, and trigonometric leveling.

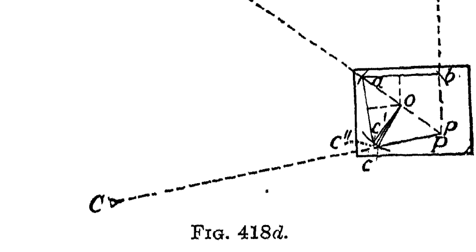


FIG. 418*d*.

Two conditions peculiar to the plane-table instrument should be mentioned. First, the table is not so precise nor so stable in its controls as is the horizontal plate of the transit. Accordingly, unless

a control level is attached to the vernier arm, in measuring the vertical angles in stadia or trigonometric leveling the index correction is likely to be much larger with the plane table, and it is necessary to determine the index error of each sight by leveling the telescope. If this attachment is provided, and its bubble is centered after a point is sighted, the vertical angle indicated by the vernier reading is then correct and leveling the telescope is unnecessary. The attachment is thus particularly useful on the vertical arc of the plane-table alidade.

Second, the elevation of a distant point whose location has been determined by the method of intersection may be secured readily by the method of trigonometric leveling (Art. 99, p. 115), in which the horizontal distance may be scaled directly from the map. The con-

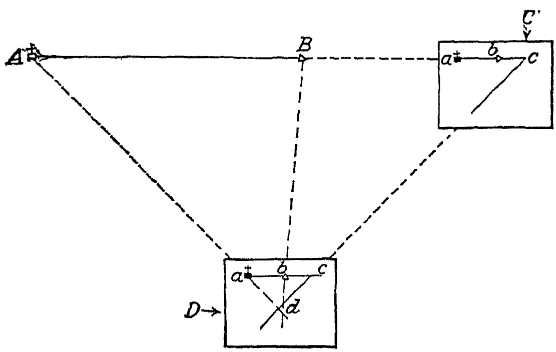


FIG. 418e.

ditions as to distance and as to the accuracy required will determine whether or not the corrections for refraction and for the earth's curvature should be applied.

420. Adjustments of the Plane-table Alidade.—The adjustments of the telescopic alidade introduce no principles of adjustment that have not been explained as they apply to the transit and the wye level. Observations made with the plane-table instrument are not required to be as precise as those made with the engineer's transit; accordingly, the adjustments in general need not be as refined, and in one or two cases they are omitted entirely. The telescope of the alidade is not reversed in altitude as is that of the transit, and hence no appreciable error is introduced through any lack of perpendicularity between the line of sight and the horizontal axis, nor through any lack of parallelism between the line of sight and the edge of the ruler. Also, it may be assumed without appreciable resultant errors that the edges of the ruler are straight and parallel, and that the horizontal axis is parallel to the plane of the ruler.

(1) *To Make the Axes of the Control Levels Parallel to the Plate or the Ruler.*

Test and Correction.—Same as for the plate levels of the transit (Art. 209, Adjustment 2), except that reversal of the plate or ruler is effected by a carefully marked guide line on the plane-table sheet. The plane-table board is not reversed.

(2) *To Make the Vertical Cross-hair Lie in a Plane Perpendicular to the Horizontal Axis.*

Test and Correction.—Same as for the corresponding adjustment of the transit (Art. 209, Adjustment 1).

(3) *To Make the Axis of the Telescope Level Parallel to the Line of Sight.*[1]

Test and Correction.—Same as for the telescope level of the transit (Art. 209, Adjustment 6).

For the alidade having a telescope which may be rotated about its axis in a sleeve, *i.e.*, *tube-in-sleeve type*, the following two adjustments take the place of that described in Adjustment 3. It will be noticed that these adjustments are practically the same as the corresponding adjustments of the wye level (Art. 117).

(3*a*) *To Make the Line of Sight Coincide with the Axis of the Telescope Sleeve.*[2]

Test.—Sight the intersection of the cross-hairs on some well-defined point. Rotate the telescope carefully in the sleeve through 180°. Usually the limits of rotation are fixed by a shoulder and a lug. If the intersection of the cross-hairs remains on the same point the line of sight is in adjustment.

Correction.—If the cross-hairs have apparently moved away from the point, bring each hair halfway back to its original position by means of the capstan-screws holding the cross-hair ring, as in the corresponding adjustment of the wye level (Art. 117, Adjustment 4). By means of the tangent-screw, set the intersection of the cross-hairs again on the point and repeat the test.

(3*b*) *To Make the Axis of the Striding Level Parallel to the Axis of the Telescope Sleeve (and Hence Parallel to the Line of Sight.)*[2]

Test.—Place the striding level on the telescope, and bring the bubble to the center of the tube. Remove the level carefully, turn it end for end, and replace it on the telescope barrel. If the bubble returns to its central position, the level is in adjustment.

[1] Applicable to an alidade with telescope rigidly fixed to the horizontal axis, *i.e.*, the fixed-tube type.

[2] Applicable to an alidade with telescope designed to rotate about its axis, *i.e.*, the tube-in-sleeve type.

Correction.—If the bubble is off center, move it through one half of the displacement by means of the adjusting screw at one end of the tube. Bring the bubble to the center by means of the tangent-screw and repeat the test.

(4) *To Make the Vernier Read Zero When the Line of Sight Is Horizontal.*[1] Same as for corresponding adjustment of the transit (Art. 209, Adjustment 7).

(4*a*) *To Adjust the Auxiliary Level on the Vernier Arm so that Its Axis Is Parallel to the Axis of the Telescope When the Vernier Reads Zero.*[2]—Same as Adjustment 7*a*, Art. 209.

421. Sources of Error.—The sources of error which affect the plane-table method are, in the main, those which affect transit and plotting work, and the discussion relating to those subjects need not be repeated here. Three sources of error, however, should be considered, namely, (1) setting over a point, (2) drawing rays, and (3) instability of the table.

1. *Setting Over a Point.*—Because plotted results only are required, it is not necessary to center the plane table over a ground point with any greater precision than is required by the scale of the map. For scales smaller than 1 in. = 50 ft., the table is usually set up by estimating with the eye when the plotted position of the station is over the ground point. For scales equal to or larger than 1 in. = 50 ft., the plotted position of the station on the map will be placed vertically over the ground point by first setting the table up roughly, and then shifting it bodily until a plumb line held from the under side of the board and just beneath the plotted point comes over the ground point.

2. *Drawing Rays.*—The accuracy of plane-table mapping depends largely upon the precision with which the rays are drawn. Consequently when sighting the alidade, either in orientation or in plotting the position of points, its direction should be controlled or indicated by rays of considerable length, for the same reason that accuracy in plotting angles is increased if the size of the protractor or of the base used in laying off tangents is increased. The map would soon become illegible, however, if the full lengths of all rays were drawn; hence only enough of each ray is drawn to insure that the plotted position falls upon it, with one or two additional dashes drawn near the ends of the alidade ruler to mark its direction. Fine, thin lines are desirable in plane-table mapping both for accuracy and legibility; hence, sharply pointed, hard (6H to 9H) pencils are used. A needle is employed for plotting station points.

[1] Applicable to an alidade having a fixed vernier.

[2] Applicable to an alidade having a level attached to the vernier arm.

3. *Instability of the Table.*—If it is manipulated with care, the plane table can be oriented with considerable precision; however, a principal source of error in its use arises from the fact that its initial position is subject to continual disturbance by the topographer while he is working. Errors from this source may be kept within reasonable limits (1) by planting the tripod firmly in the ground, (2) by setting the table approximately waist high so that the topographer can bend over it without leaning against it, (3) by using the alidade and scale with greater care to prevent the exertion of undue pressure upon or against the table, and (4) by testing the orientation of the board occasionally and correcting its position if necessary. This test is always applied before plotting the position of a new instrument station.

422. Field Checks.—An important advantage of the plane-table method is that it provides many convenient opportunities for verifying the plotted positions of points on the map. This may be accomplished, after the table has been set up and oriented, by drawing a ray toward any visible object whose position has previously been plotted. If this ray passes through the plotted point, the verification is effected, just as the third ray intersecting at a point provides a check in the method of intersection (see Art. 414). The plotted position of the plane table itself may also be verified by resecting from distant visible objects whose plotted positions are known to be correct. When sketching, these checks should always be applied at each station to guard against faulty orientation and mistakes in observations.

423. Plane-table Sheet.—The plane-table sheet is exposed to outdoor conditions. This fact requires the paper to be given special preparation to prevent undue expansion or shrinkage. It is the practice of the U. S. Geological Survey to paste two sheets of Paragon paper together with a piece of muslin cloth between and with the grain of the paper in one sheet laid transverse to the grain of the other. This produces an excellent sheet, but it is not sufficiently flexible to be used in the field if a sheet larger than the plane-table board is desired. Accordingly, the U. S. Coast and Geodetic Survey uses a single sheet of Whatman's cold-pressed, hand-made antiquarian paper 52 by 30 in. mounted on muslin. Obviously, only the best drawing papers should be used. These can be seasoned, *i.e.*, rendered more resistant to changes in humidity of the air, by exposing them alternately to very moist and very dry atmospheres for a considerable period of time.

If a sheet is to receive the plotting from several days' work a cover-sheet of some smooth, tough paper is used to protect it during the field work. The cover is torn away to expose the sheet as the work progresses.

424. Comparative Merits of Transit and Plane Table.—The comparative merits of the two instruments as used in topographic mapping have been the subject of much debate (see references at end of chapter) but the arguments in most cases have been based upon experience with one instrument only. If a topographer has used both instruments or if different parties have worked under similar conditions, one party using a plane table and the other using a transit, the evidence seems to favor the plane-table method.

It is obvious that comparisons are not valid unless the instrumentmen are equally competent, and this factor has a very different significance in the case of the two instruments. The time required for a man to gain an equal proficiency in the use of the plane table and of the transit will be in the ratio, at least, of months to weeks.

As compared with other methods of mapping, the plane-table method has these advantages: (1) relatively few points need be located; (2) contours and irregular objects as woods, streams, ponds, etc., may be more accurately represented; (3) since numerical values of angles are not observed, the consequent errors and mistakes due to reading, recording, and plotting are avoided; (4) since all plotting is done in the field, omissions in the field data are avoided; (5) the useful principles of "intersection" and "resection" are made convenient; (6) checks on the positions of points on the finished map are more readily obtained; and (7) a large part of the office work is eliminated.

The disadvantages are: (1) the plane table is cumbersome; (2) numerous unhandy accessories must be carried; (3) more time is required in the field; and (4) the usefulness of the method is limited to relatively open country (*i.e.*, where visibility is fair) and to weather conditions free from rain, cold, and high wind.

Some of these disadvantages are, at times, overcome by unusual adaptations of the plane table to adverse conditions. Thus, in regions subject to much rainfall, mapping with the plane table is effected by the use of celluloid sheets, instead of drawing paper. Also, although the instrument is somewhat more cumbersome than the transit or level, it can be used in regions of rugged topography; or with a long-legged tripod the instrument may be raised above underbrush, cornstalks, or other obstructions to visibility, as shown in Fig. 424.

In view of the preponderance of the advantages listed above, it is obvious that this method merits a wide use. It is not more generally employed by practising engineers because (1) most offices are not equipped with plane-table instruments, and (2) the success of the method depends to a large degree upon the special training and

ability of the topographer; and except for those employed on government and state surveys, but few men have the proper qualifications.

It is true, on the one hand, that points (as for example, corners of buildings, poles, isolated trees, etc.) can be located more rapidly and more accurately with the transit than with the plane table; on the other hand, it is clear that irregular lines and areas (as for example, drives, streams, woods, and contours) are more quickly and accurately mapped with the plane table than with the transit. The matter may be summarized in the statements (*a*) that if a large number of *points* are to be plotted on a map the transit method is

Fig. 424.

superior, and (*b*) that if *irregular lines and areas* are to be represented on a map the plane-table method is superior in accuracy, speed, and economy.

425. Problems.

1. In the two-point problem, derive the proof that the line *ab'*. (Fig. 418*c*) before the table is oriented, is parallel to the direction between the stations *A* and *B*.

2. Demonstrate the proposition in Rule 1, for the solution of the three-point problem (Art. 417*b*) that "the point sought is always distant from each of the three lines drawn from the three fixed points in proportion to the distances of the corresponding actual points from the station occupied."

3. Determine the accuracy of the plane-table method by running a closed traverse. Several points should be sighted from three or more traverse stations. The accuracy of the work will be measured by the

error of closure and by the triangles of error formed by the intersections upon distant points.

4. Compare the precision of different instruments and tables as follows: From the ends of a carefully measured base line locate by the method of intersections a number of definite points. By means of tracing cloth compare the results and determine the probable error in locating points by each combination of table and alidade.

5. Verify the solutions of the two-point and three-point problems in the drawing room, as follows: Use a large sheet of paper 24 by 30 in. to represent the field and a small sheet 3 by 5 in. to represent the table. A straightedge will serve as a line of sight to connect stations.

6. Make a plane-table map of a portion of the campus using all methods given for locating points and orienting the board. Be careful to apply the various checks indicated.

REFERENCES

1. BARRY, P. J., "Combined Methods Useful in Topographic Mapping," *Eng. News-Record*, Vol. 86, May 13, 1921.
2. BROWN, W. N., "Plane Table Has Another Champion," *Eng. News-Record*, Vol. 83, p. 674, Oct. 2, 1919.
3. EMORY, LLOYD T., "Plane-table Survey in Asia Minor," *Eng. Record*, Vol. 68, p. 437, Oct. 18, 1913.
4. "Topographic Survey of Flint," (Mich.), *Eng. News-Record*, Vol. 84, p. 700, Apr. 8, 1920.
5. HIGGINS, A. L., "Graphical Solution of the Three-point Problem," *Engineering*, Vol. 97, p. 93, Jan. 16, 1914.
6. REABURN, D. L., "Methods of Making Topographical Surveys and Their Costs," *Eng. News*, Vol. 66, p. 186, Aug. 10, 1911.
7. "The Topographical Survey of Staten Island," *Eng. Rec.*, Vol. 56, p. 79, July 20, 1907.
8. WAINWRIGHT, D. B., "A Plane-table Manual," Sp. Publ. 85, U. S. Coast and Geodetic Survey, Government Printing Office, Washington, D. C.
9. WESTLAND, C. R., "Notes on Plane-table Work," *Can. Eng.*, Vol. 29, July 8, 1915.
10. WILLIAMS, L. H., "Plane Table Advocated for Drainage Work," *Eng. News-Record*, Vol. 80, p. 416, Feb. 28, 1918.
11. Engineers on the comparative merits of the plane-table and the transit methods of topographic surveying, *Eng. News-Record*, Vol. 83, p. 95, July 10, 1919; *ibid.*, Vol. 83, p. 333, Aug. 14, 1919; *ibid.*, Vol. 85, p. 1165, Dec. 16, 1920; *ibid.*, Vol. 86, p. 41, Jan. 6, 1921; *ibid.*, Vol. 86, Jan. 13, 1921; *ibid.*, Vol. 86, p. 313, Feb. 17, 1921.

CHAPTER XXIV

TOPOGRAPHIC MAPPING

426. General.—A topographic map shows by the use of suitable symbols (1) the configuration of the earth's surface, called the *relief*, which includes such features as summits, depressions, hills, valleys, plains, etc.; (2) other natural features, such as trees or timber, streams, ponds, lakes, etc.; and (3) the physical changes wrought upon the earth's surface by the works of man, such as houses, villages, highways, railways, canals, and cultivation. Many maps and plats of various sorts may show such objects as are listed under (2) and (3) above, but such a map may not properly be called a topographic map unless, in addition, it represents the shape of the ground surface. Therefore, the distinguishing characteristic of a topographic map, as compared with others, is the representation of the terrestrial relief.

The uses made of topographic maps are innumerable, but in general it may be said that they are a necessary aid in the design of any engineering project which requires a consideration of land forms, of elevations, or of gradients. Examples of such projects are water-power plants, irrigation and drainage works, aqueducts, highway and railway construction, city improvements, and landscape developments.

In addition to the requirements of engineering construction works, there exists the need for topographic maps to supply the general information necessary to the studies of geologists, economists, and others interested in the broader aspects of the development of natural resources. This class of surveys is largely in the hands of governmental organizations, the principal example being the topographic map of the United States being constructed by the Topographic Branch of the U. S. Geological Survey. This is published in quadrangle maps, which generally include territory 15′ in latitude by 15′ in longitude. Other organizations which have as a part of their work the execution of topographic surveys are the U. S. Coast and Geodetic Survey, the Corps of Engineers of the U. S. Army, and the U. S. Bureau of Reclamation.

427. Representation of Relief.—Relief may be represented by relief models, hachures, shading, or contour lines. Of these, the contour

line has by far the widest use, and therefore in the following paragraphs the purpose will be to indicate the principles which underlie the interpretation, construction, and use made of contour lines on topographic maps.

Shading and hachures were the principal symbols used on the topographic maps in European countries down to the beginning of the present century. In the United States the contour line has been used from the time of the earliest work of the Topographic Branch of the U. S. Geological Survey, and its superiority in accuracy of representing relief has caused its universal adoption where fidelity of representation is the most important consideration.

428. Relief Model.—This symbol is a representation of ground forms done in three dimensions to suitable scales. Plastic materials, such as wax or clay, which will retain a shape given them while in a plastic state, are used.

As the name implies, a relief model is a miniature of the terrain it represents. It is the most legible of all methods of representing relief, but its usefulness is greatly limited because of its size and bulk. It is a great aid, however, in many of the special studies of the geologist, the geographer, and the mining engineer.

429. Shading.—Shading in black and white or in brown is a method of showing relief in plan as it would appear from a point vertically above and with parallel rays of light flooding the landscape from a given angle, causing shadows to lie upon the less-illuminated areas. The method is pictorial in effect, and is useful in showing the general features where the relief is high and the slopes are steep.

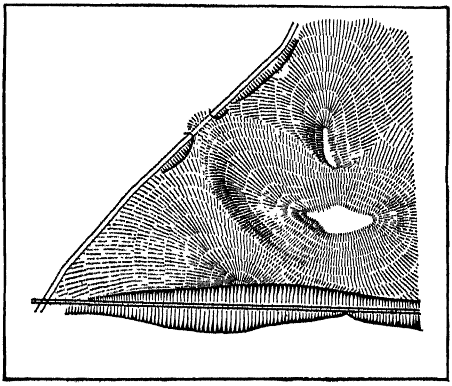


Fig. 430.—Hachures.

Because of this effect, this symbol is often used in combination with hachures or contour lines, to render the map more legible.

430. Hachures.—Hachures show relief more accurately but less legibly than does shading. The symbol consists of short, nearly parallel lines whose spacing, weight, and direction produce an effect similar to shading but capable of more definite handling. The lines are drawn parallel to the steepest slopes, and in the best practice a standard scale of lengths and weights of lines is used to represent the various degrees of inclination of slopes. The method is illustrated in Fig. 430, which is a representation of a portion of the relief shown by contour lines in Fig. 432.

431. Contour Lines.—A contour line on the ground may be defined as an imaginary line passing through all points of equal elevation. It may be thought of as the trace formed by the intersection of a level surface with the ground surface. The shore line of a still body of water is the best visible example of a contour line on the ground.

If on the drawing are plotted the positions of several ground points of equal elevation, say 720 ft. above sea level, the line on the map joining these points is called a *contour line.* A contour line may therefore be defined as a line, either on the map or on the ground, which joins all points of equal elevation. It will be a useful distinction, however, in the discussion which follows, if the ground line is called a *contour,* and the corresponding map line is called a *contour line.*

The use of the contour line has the great advantage that it permits the representation of relief with much greater facility and with far greater accuracy, than do other symbols. It has the disadvantage that the map is not so legible to the layman.

432. Characteristics of Contour Lines.—The principal characteristics of contour lines may be illustrated by reference to a contour map such as the map shown in Fig. 432. For the purpose of this discussion the slope of the river surface may be disregarded. If the stage of the river at the time of the field survey was at an elevation of 510 ft., the shore line on the map marks the position of the 510-ft. contour line. If we imagine the river to rise through a 5-ft. stage, the shore line is represented by the 515-ft. contour line, and similarly the successive contour lines at 520 ft., 525 ft., etc. represent shore lines which the river would have if it should rise further by 5-ft. stages.

The characteristics of contour lines are as follows:

1. The horizontal distance between contours is inversely proportional to the slope. Thus on steep slopes (as at the railroad embankments and river banks) the contour lines are spaced closely.

2. On uniform slopes, the contour lines are spaced uniformly.

3. Along plane surfaces (such as those of the earthwork in Fig. 432) the contour lines are straight and parallel to each other.

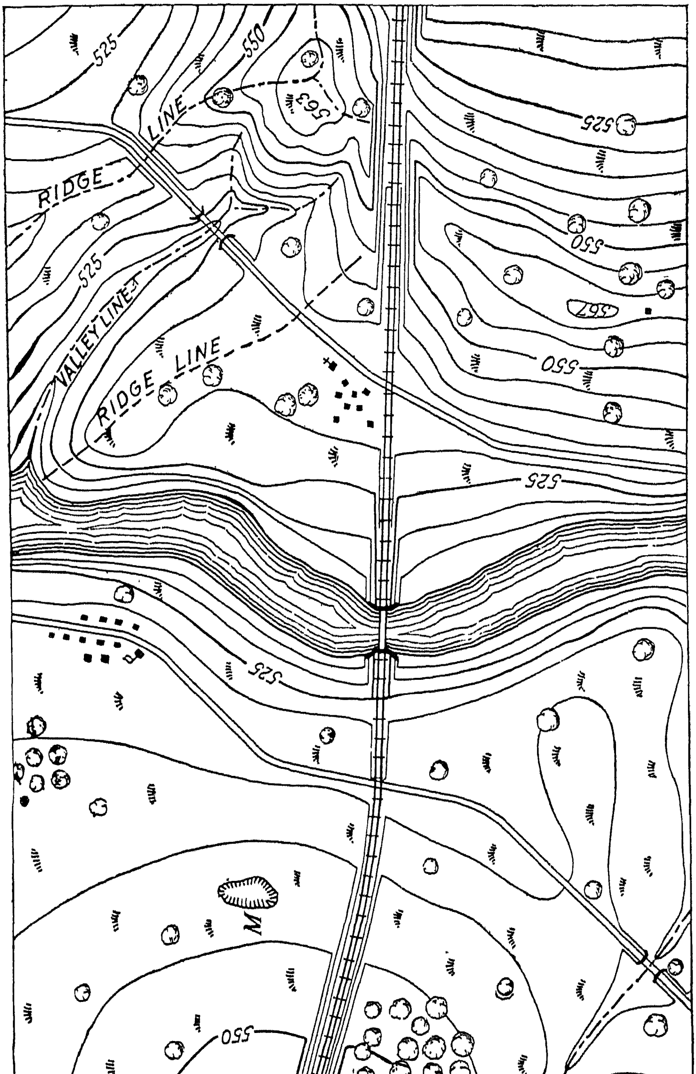


FIG. 432.—Contours.

4. Since contour lines represent level lines they are perpendicular to the lines of steepest slope and therefore are perpendicular to ridge and valley lines where they cross such lines.

5. Evidently since all land areas may be regarded as summits or islands above sea level, all contour lines must close upon themselves either within or without the borders of the map. Therefore, a closed contour line on a map always indicates either a summit or a depression. The elevations of adjacent contour lines will usually indicate which condition is represented, but if a summit or a depression is represented by a single contour line, as at M in Fig. 432, it would be impossible to tell which feature is represented. To prevent confusion, therefore, a depression is always shown either by water lines, if water is present, or if not, by a hachured contour line, or *depression contour*, as shown at M.

6. Since contour lines represent contours of different elevation on the ground, they cannot merge or cross each other on the map, save the rare exceptions, that in the first case they represent a vertical surface, as the abutments of the railroad bridge in Fig. 432, and in the second case, they indicate an overhanging cliff or a cave.

7. A single contour line can not lie between two contour lines of higher or lower elevation.

433. Contour Interval.—The vertical distance between contours is called the *contour interval.* The choice of interval depends upon the purpose and scale of the map, and upon the character of terrain represented. For small-scale maps of rough country, the interval may be taken as 50 ft., 100 ft., or more; for large-scale maps of flat country, contour intervals as small as ½ ft. are sometimes employed. For maps of intermediate scale, such as are used for many engineering studies, the interval is usually 2 or 5 ft.

434. Contour-map Construction.—The construction of a topographic map normally consists of three operations, as follows: (*a*) the plotting of the horizontal control, or skeleton upon which the details of the map are hung, (*b*) the plotting of details, including the map location of points of known ground elevation, called *ground points*, by means of which the relief is to be indicated, and (*c*) the construction of contour lines at a given contour interval, the ground points being employed as guides in the proper location of the contour lines.

The common methods of plotting both horizontal control and details have been described in an earlier chapter and will not be considered further. Methods of plotting contours will now be discussed.

Regardless of the number of ground points whose plotted positions are known, it is evident that any contour line must be drawn, to some degree, by estimation. This condition requires that the drafts-

man use his skill and judgment to the end that the contour lines may best represent the actual configuration of the ground surface.

Contour lines are shown for elevations which are multiples of the contour interval. Usually each fifth contour line is made heavier than the rest, and sometimes these lines are drawn first, in order to facilitate the location of intermediate contour lines. However, the location of intermediate lines should be considered just as important as that of the fifth lines, and an excessive degree of conformity between the contour lines is a sure sign of an inaccurate contour map.

Since contour lines ordinarily change direction most sharply where they cross ridge and valley lines, and since the gradients of ridge and valley lines are generally uniform, these lines are important aids to the correct drawing of the contour lines. Because of this fact, special care is taken in the field to locate the valley and ridge lines. Examples of such locations are shown plotted at points *a* (Fig. 434*a*). The stream lines are drawn through those points which represent valleys, and the contours are spaced along them before any attempt is made to interpolate or draw the contour lines. This procedure aids the draftsman in his interpretation of the data. For example, in the square bounded by the points *D*-5, *D*-6, *E*-5, and *E*-6, the contours are made to show the head of the valley, the existence of which is indicated by no other fact than that of the valley line previously drawn in the square below. In the figure shown, the ridge line is somewhat indefinite and but little aid would result from the attempt to sketch it on the map before drawing the contour lines.

434a. Interpolation.—Consider the two contour points *A*-2, *B*-2 (Fig. 434*a*), whose elevations are 848.0 and 852.0 ft. respectively. The contour interval for this map has been taken as 2 ft. and the 848 and 852-ft. contour lines pass through the corresponding points. Assuming that the slope is uniform, the 850-ft. contour line passes through a point midway between the two points considered. If a 1-ft. interval were used, then the additional 849 and 851-ft. contour lines would be drawn through the quarter points of the line from *A*-2 to *B*-2. This process of spacing the contour lines proportionally between plotted points is called interpolation.

The procedure is usually not so simple as in the case cited above. It is noticed that the elevations at the corners of the other squares in the figure are mostly of such values that the contour lines do not pass through them. For this condition the interpolations may be made by estimation on the map, by arithmetical calculations, or by graphical means.

1. *Interpolation by Estimation.*—Since each contour map is the result of more or less interpretation by the draftsman, it is thought in many cases to be not inconsistent with the other methods of its construction if the interpolation is made by careful estimation accompanied occasionally by approximate mental calculations. This method is most commonly used on intermediate- and small-scale maps.

2. *Interpolation by Calculation.*—Where considerable accuracy is desired in the map, the errors of estimation may be eliminated by

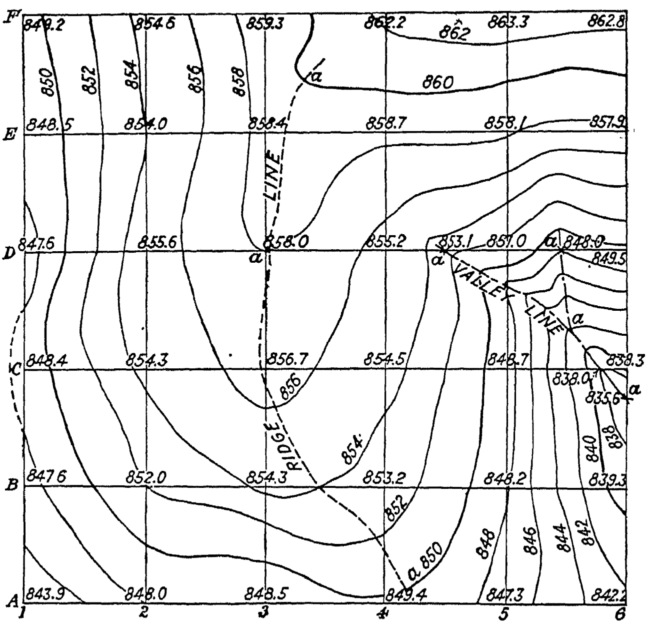


FIG. 434*a*.

simple arithmetical calculations. For example, the elevations of points *E*-6 and *F*-6 (Fig. 434*a*) are 857.9 and 862.8 ft. respectively. The contour interval is 2 ft., hence the difference in elevation between the point *E*-6 and the 858-ft. contour line is 0.1 ft. Then since the total difference in elevation is 4.9 ft., the proportional part of the distance from *E*-6 to *F*-6 to locate the 858-ft. contour is $\frac{0.1}{4.9}$ of the map distance between these points. Similarly, the proportional parts for the 860- and the 862-ft. contour lines are respectively $\frac{2.1}{4.9}$ and $\frac{4.1}{4.9}$ of the distance from *E*-6 to *F*-6. These calculations are facilitated

by the use of a slide rule. The computed map distances are plotted to scale.

3. *Interpolation by Graphical Means.*—The calculations indicated in the previous paragraph become laborious if many interpolations are to be made, and accordingly various means of graphical interpolation are in use. One of these is shown in Fig. 434*b*. A number of parallel lines are drawn at equal intervals on tracing cloth, each fifth or tenth line being made heavier or of a different color from the rest and being numbered as shown. Now if it be desired to interpolate the position of, say, the 52 and 54-ft. contours between *a* with elevation of 50.7 and *b* with elevation of 55.1, the line on the

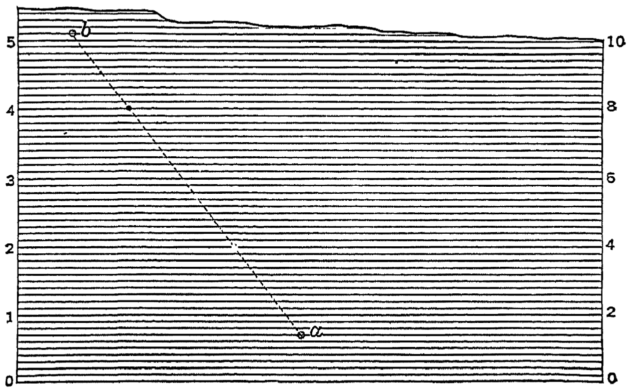


FIG. 434*b*.

tracing cloth corresponding to 0.7 ft. (scale at left end) is placed over *a* and the tracing is turned about *a* as a center until the line corresponding to 5.1 ft. (scale at left end) covers *b*. The interpolated points are at the intersections of lines 2.0 and 4.0 (representing elevations 52 and 54) and the line *ab*, and may be pricked through the tracing cloth. Had the known points been much closer together, the figures at the right end of the tracing would have been used, or in other words, the value of each space would have been doubled; or if the scale were small, the contour interval large, and the topography rugged, each space might represent 1 ft. Thus by assigning different values to the spaces, a single piece of tracing cloth prepared in this way can be made to suit a variety of conditions.

435. Systems of Ground Points.—The typical systems of ground points commonly used in drawing topographic maps are the *trace-point, coordinate-point, controlling-point, cross-section-point,* and *cross-section-controlling-point* systems.

1. *Trace-point System.*—If a number of points on a given contour have been located on the ground and their corresponding positions plotted on the map, the contour line may then be drawn through these plotted points.

2. *Coordinate-point System.*—By this method a system of squares or rectangles is plotted, and the elevations of the corners are recorded beside them. Also, the positions of valley or ridge lines are shown. Following the principles stated in Art. 434, the valley and ridge lines are sketched, the contour crossings are interpolated on these lines and on the sides of the squares, and the contour lines are drawn.

3. *Controlling-point System.*—The significance of valley and ridge lines has already been mentioned. It has also been noted that where a uniform slope exists between two contour points the intermediate contour lines on the map may be located by interpolation. Hence, if a system of points is provided which locates the summits,

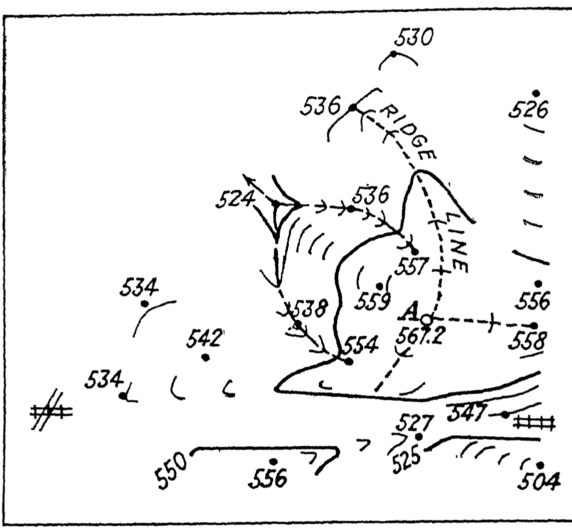


Fig. 435*a*.

depressions, valley and ridge lines, and all important changes in slope, a contour map of the region may be drawn. Such a system is illustrated in Fig. 435*a*. Here are shown the points used in drawing the summit and immediate vicinity illustrated in the map of Fig. 432.

In this sketch the ridge and valley lines have been drawn, the contours spaced along them and also spaced between other controlling points, and the fifth contour lines have been sketched. From

this procedure it is evident how much aid is supplied by this information and how simple is the additional interpretation required to complete the map.

4. *Cross-section-point System.*—This system is most frequently used to draw the maps required in connection with route surveys. The field surveys locate the positions of contour points along lines normal to the route traverse line. In Fig. 435*b* a transit traverse is represented as a straight line along which the 100-ft. stations are shown. The cross-section lines are dashed and the contour crossings are shown as dots. These dots obviously lie on the contour lines themselves, and hence the latter may be drawn on the map as in the case of the trace-point system. The number of points on a given contour line will generally be much less in the cross-section than in the trace-point system, and hence, the draftsman will be called upon for a much greater amount of interpretation in the former than in the latter case.

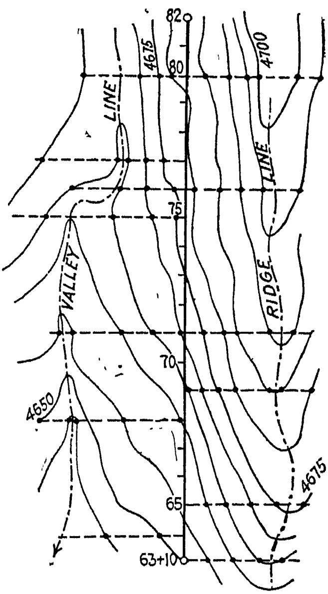


Fig. 435*b*.

5. *Cross-section-controlling-point System.*—This system is similar to the cross-section-point system, except that on the cross-sections the elevation is determined in the field only at points where the slope changes. From the field notes the cross-sections are plotted and contour points are interpolated.

436. Contour-map Studies.—Many uses are made of contour maps, and no attempt will be made here to indicate their scope further than to describe the procedure of drawing cross-sections and profiles from contour maps, and to describe three common studies made upon contour maps: (1) earthwork, (2) reservoir areas and volumes, and (3) route location.

436a. Cross-sections and Profiles from Contour Maps.—Fig. 436*a* shows a contour map, the purpose of which is to estimate the earthwork necessary to grade a portion of the area to a uniform slope. The full lines represent contours before earth is removed, and the dotted lines represent contours after earthwork has been completed. Below the map is a cross-section along the line *AB*.

The horizontal position of each point marking the intersection of contours with the line *AB* is first projected to *CD*, the base line of the cross-section. The elevations are then read from the contours and the appropriate distances are scaled up from the base line. A line drawn through the points thus plotted defines the cross-section. In the figure the full line of the cross-section shows the original surface and the dotted line shows the surface after grading has been completed.

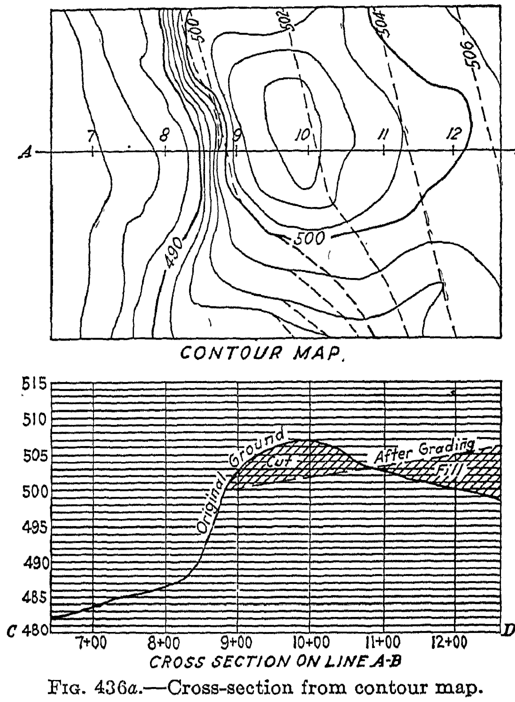


FIG. 436*a*.—Cross-section from contour map.

In practice, parallel lines along which the cross-sections are to be taken are drawn on the map, and the distance between them is scaled. One person scales horizontal distances to contour crossings and reads contour elevations, while a second person plots the data on regular cross-section paper in the manner described in Art. 153.

It should be noted that in practice the horizontal scale to which the profile line is plotted is frequently not the same as that used on the map, and also that the horizontal alinement is usually both curved and straight. Therefore, for these conditions the mechanical means of plotting the profile by projecting the points where the center

line crosses the contour lines down or up from the map to the profile paper can not be used. The usual procedure is to mark the 100-ft. station points on the map, and from them to scale the distances to contour crossings and then to plot these distances on profile paper as if the elevations and stations were being taken from profile notes.

Figure 436*d* shows the profile of the ground line and of the grade line for a roadway construction. The manner of drawing the profile is similar to that for the cross-section described above.

436b. Earthwork.—Earthwork quantities may be estimated from contour maps by three methods: (1) cross-sections, (2) horizontal planes, and (3) equal-depth (or equal-height) contours. A discussion of the probable errors involved is given in Art. 165.

1. *Volume by Cross-sections.*—When cross-sections have been plotted in the manner just described, volumes of earthwork between adjacent cross-sections may be determined by the use of average end areas (Art. 161).

2. *Volume by Horizontal Planes.*—For preliminary estimates for grading areas, especially where the graded surface is itself more or less irregular, the more common practice is to utilize the topographic map directly as a basis for calculations of volume. On the map are shown the contours for the natural ground and also contours for the proposed graded surface. This method consists in determining the volumes of earth to be moved between the horizontal planes marked by successive contours.

The light, full lines of Fig. 436*b* represent contours of the original ground and the dash lines represent contours of the proposed graded surface. The heavy, full lines are drawn through points of no cut or fill. Thus the line *abcdefa* bounds an area that is entirely in fill and the line *dghjked* bounds an area that is entirely in cut, as will be seen by studying the figure. The "no cut or fill" lines are seen to pass through the points of intersection between full contours and the corresponding dash contours (as at *a*, *b*, *d*, *e*, *h*, and *j*). The conditions surrounding the problem make it possible to estimate the position of the lines where the cut or fill runs out between contours (as the lines *bcd*, *efa*, and *jke*). The cross-hatched portions are the horizontal sections of earth cut or filled at the contour elevations. Thus F_1 represents the horizontal section of earth filled at elevation 96. It is evident that the volume of earthwork between two successive contours is a solid whose altitude is the contour interval and whose top and bottom bases are the horizontal projections of the cut or fill at the contour elevations (as between the 94 and 96-ft. contours, for fill *abcdefa* the height is 2 ft. and the bases are F_2 and F_1). Where the cut or fill runs out between contours (as

along line *bcd*) the height of the end volume will be less than the contour interval. This height may be estimated by assuming the slope of the ground to be uniform between contours (thus point *c* is estimated to be at elevation 97.2 and the volume above the 96-ft. contour is a solid whose base is F_1 and whose altitude is $97.2 - 96 = 1.2$ ft.). The end volumes may be considered as pyramids.

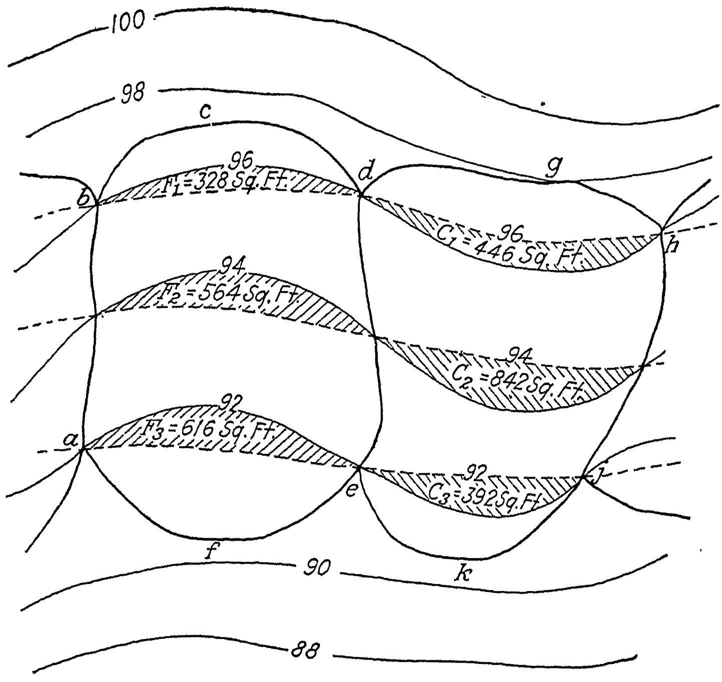


FIG. 436*b*.

Example 1: It is desired to determine the volume of earthwork in fill bounded by the line *abcdefa* (Fig. 436*b*). The intermediate volumes are to be calculated by the method of average end areas; the end volumes are to be considered as pyramids. The areas of fill at the contours are as shown in the figure. The point *c* is estimated to lie 1.2 ft. above the 96-ft. contour and the point *f* is estimated to be 1.6 ft. below the 92-ft. contour. For solution, see tabulation on page 645.

3. *Volume by Equal-depth Contours.*—This method consists in calculating volumes between upper and lower surfaces bounding certain increments of cut or fill. In either case, horizontal projections of the inclined areas are taken from the map, usually with the planimeter, and the volume between any two successive areas is

determined by multiplying the average of the two are as by the depth between them.

Elevation	Base area, square feet	Altitude, feet		Volume, cubic feet
$c = 97.2$	0			
		1.2	$\frac{1}{3} \times 1.2 \times 328$	131
96	328			
		2.0	$\frac{1}{2} \times 2.0 \times 892$	892
94	564			
		2.0	$\frac{1}{2} \times 2.0 \times 1{,}180$	1,180
92	616			
		1.6	$\frac{1}{3} \times 1.6 \times 616$	328
$f = 90.4$	0			

Total. . . . . 2,531 cu. ft.
or 94 cu. yd.

Figure 436c represents the topographic map of a tract a portion of which is to be graded by filling. The light, full lines represent contours of the original ground and the dash lines represent contours of the proposed fill. Above the dash 102-ft. contour the fill drops abruptly to the natural ground. Along the bank thus formed just above and paralleling the 102-ft. contour, actually there would be 100, 98, and 96-ft. contour lines, but to avoid confusion of lines these are not shown.

The heavy, full lines drawn through points of equal fill are sometimes called *lines of equal fill* (or cut). The outer line marks the limit of the fill and passes through points of zero fill as defined by the intersection of full contours with dash contours of corresponding elevations; the next line encloses the area over which the fill is a minimum of 2 ft. and passes through points of intersection between a full contour and a dash contour whose elevation is 2 ft. greater; and so on. Along the side of the bank above the dash 102-ft. contour the heavy lines are seen to be close together and nearly parallel.

The fill between the graded surface and the surface 2 ft. below is represented by the solid whose altitude is 2 ft. and whose upper and lower surfaces are shown in horizontal projection by the line of zero fill and the line of 2-ft. fill respectively; likewise the lines of 2 and 4-ft. fill define the volume of fill between the depths of 2 ft. and 4 ft. from the graded surface. The volume below the innermost line of equal fill may be considered as a pyramid whose base area is that bounded

by the line and whose altitude is estimated, being always less than the full contour interval. Though volumes are usually determined by multiplying the contour interval by the average of areas of successive surfaces of equal cut or fill, when there is a large reduction between successive areas the prismoidal formula is sometimes used.

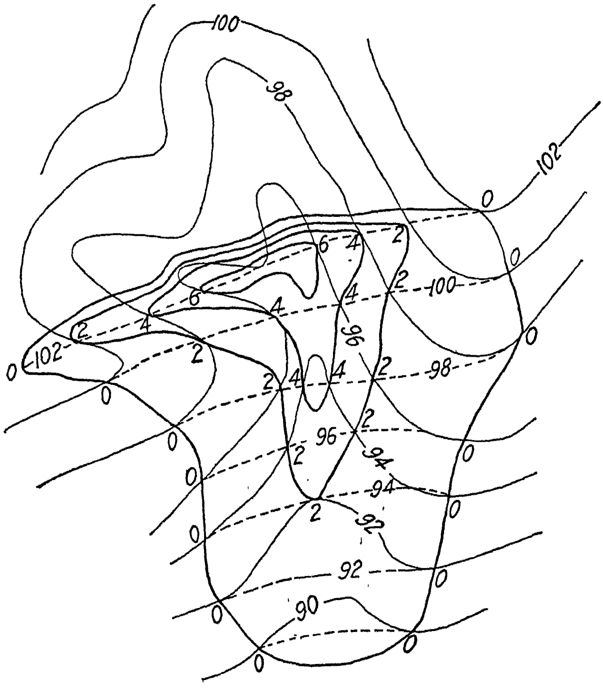


FIG. 436c.

Example 2: An estimate of volume of earthwork in fill is to be made from a contour map similar to that of Fig. 436c. Lines of equal fill are drawn, and the areas of the horizontal projections of surfaces of equal fill are determined by measurement with a planimeter. The altitude of the pyramid below the innermost surface of equal fill is estimated to be 1 ft. The computations are tabulated at the top of p. 647.

4. *Earthwork for Roadway.*—Figure 436*d* represents the contour lines for a proposed roadway drawn dotted over the existing contour lines of the map of the region. Above the contour map are shown a profile of the ground along the center line and the grade of the proposed roadway. The side slopes of the earthwork have the ratio of $1\frac{1}{2}$ to 1. The roadway is represented as having a width of 36 ft. in cut and 30 ft. in fill. From a study of these two drawings the following observations may be made.

Fill, feet	Area, square feet	Altitude, feet		Volume, cubic feet
0	101,000			
		2.0	$\frac{1}{2} \times 2 \times 134{,}000$	134,000
2	33,000			
		2.0	$\frac{1}{2} \times 2 \times 50{,}000$	50,000
4	17,000			
		2.0	$\frac{1}{2} \times 2 \times 22{,}000$	22,000
6	5,000			
		1.0	$\frac{1}{3} \times 1 \times 5{,}000$	2,000
7	0			

Total........... 208,000 cu. ft.
or 7,700 cu. yd.

1. The 840-ft. contour line of the proposed roadway crosses the roadway at a point on the map vertically beneath the point on the profile where the grade line crosses the 840-ft. elevation line; and similarly for the other gradient contours.

2. On the fill or in the cut at any station, the distance out from the edge of the roadway to a contour line is given by the difference in elevation between that which the contour line represents and the elevation of the grade at that station, multiplied by the side-slope ratio. Thus at station 76 + 40 the elevation of grade is 840.0 ft. and the elevation represented by the first contour line out from the edge of the fill is 838.0 ft., hence, the distance out is 2 ft. times 1½ or 3 ft. (the lateral scale is exaggerated in the illustration, for clearness).

3. If the grade line were level, the contour lines in cuts and on fills would remain parallel to the center line, because the distance out from the edge of the roadway to a given contour line would remain constant. But if the grade line is not level, the contour lines will make an angle with the edge of the roadway such that in the distance between two contour crossings of the roadway, a contour line which at one crossing was at the edge of the roadway will at the crossing of the next adjacent contour line be out a distance equal to the contour interval times the side-slope ratio. Thus, the 844-ft. contour line which crosses the roadway at station 73 + 30 is so inclined in direction that at station 74 + 80 where the elevation of grade is 842 ft., the 844-ft. contour line is out from the edge of the roadway a distance equal to 2 ft. times 1½ or 3 ft.

4. The toe of a slope is drawn on the contour map by connecting the points where the dotted lines intersect the corresponding full lines.

5. If a line is drawn across the roadway normal to the center line, a cross-section of the proposed roadway can be plotted as explained in Art. 153, p. 210. By plotting a sufficient number of such sections it is

evident that the quantity of earthwork can be computed by the method of average end areas as explained in Art. 161.

6. The volume of earthwork may also be estimated by the methods explained in (2) and (3) of this article.

436c. Reservoir Areas and Volumes.—Another common study made on contour maps is that of determining the capacity of a reservoir. The position of the flow line and the area of the drainage basin can also be readily found. This procedure can be indicated in miniature by reference to the fill across the valley in Fig. 436*d*. If water be imagined to stand at the elevation of 834 ft., the area of

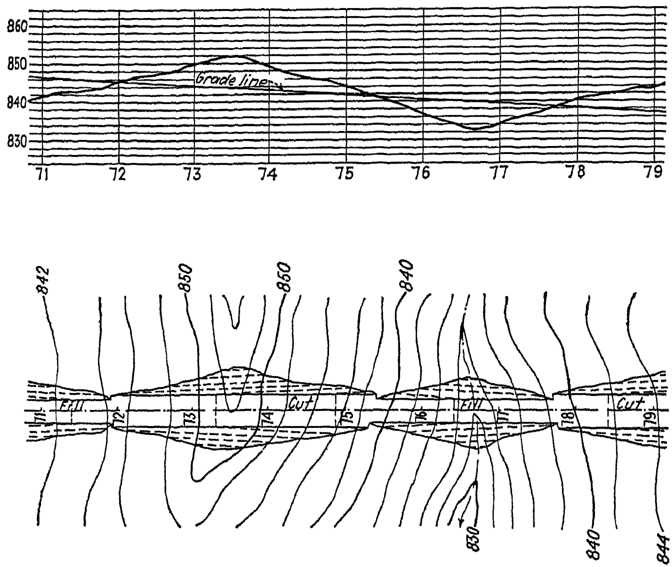


Fig. 436*d*.

the surface is represented by the area enclosed within the full and dotted 834-ft. contour line. If the water be imagined to rise through a 2-ft. stage to the elevation of 836 ft., the extent of its surface would be represented by the area enclosed within the full and dotted 836-ft. contour line. The volume of water which caused the 2-ft. rise is given by the product of the average of the two surface areas and the vertical distance of 2 ft. Similarly the volume of water required to cause a rise of the water surface from 836 ft. to 838 ft. may be found. By a similar procedure the volume of any reservoir may be estimated.

The flow line marking the outline of the submerged area of a proposed reservoir is, of course, given by the contour line representing the maximum stage of the impounded water. Also, the drainage area may be estimated by sketching on the map the watershed line and by measuring its extent with a planimeter.

436d. Route Location.—The use of a contour map in locating a proposed route for such projects as highways, railways, drainage ditches, canals, etc., is illustrated in Fig. 436*e*.

The attempt in a route location is to fix the center line of the proposed construction so that the subgrade will conform as nearly as

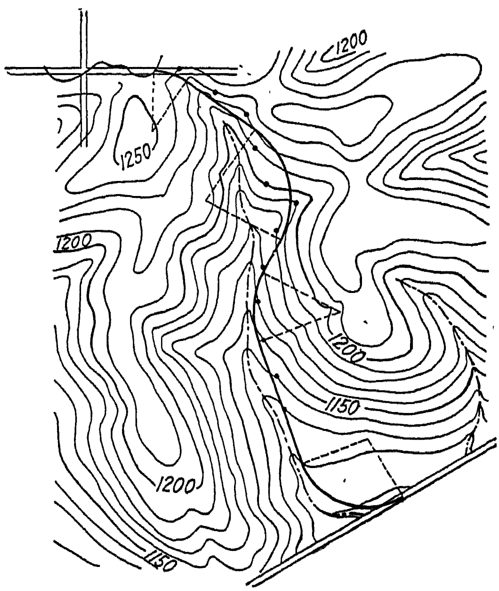


Fig. 436*e*.

may be practicable to the original ground surface. This desired condition could be attained without difficulty if there were no limitations as to the amount of curvature, radius of curves, or distance. But a proper design of any project imposes more or less severe restrictions as to these factors. It is not the purpose of this article to discuss the nature of these limitations, but to illustrate the use of a contour map in the study of a route location.

It may be supposed that a proposed highway is to be located in the valley shown in Fig. 436*e* and further, that the new location

begins at some point in the existing diagonal highway. The proposed route is projected on the map up the valley until the steep slopes require a careful study of the ground. Thus the location may be assumed to be satisfactory to the point shown by a dot where the projected line crosses the 1,130-ft. contour line. From this point the direction of a route which will have a suitable grade can be studied as follows: A pair of dividers is set at a map distance equal to the horizontal projection of the distance measured on the given gradient from one contour to the next. For example, if the contour interval is 10 ft., and the desired gradient is 4 per cent, then the horizontal projection of the slope distance between two adjacent contours is $10 \div 0.04 = 250$ ft. Now if the feet of the dividers are set to a map distance of 250 ft. and if one foot is placed on any given contour line and the other foot placed on the one next adjacent, the line joining the two feet of the dividers represents a line on the ground which has the desired gradient of 4 per cent.

In the illustration, the dots shown on the successive contour lines have been thus located by stepping from one line to the next with a pair of dividers set at a map distance of 250 ft. The first point was taken as that where the projected location crossed the 1,130-ft. contour line. This series of points marks on the ground surface the location of a 4-per-cent grade line up the valley. Hence, the route is made to follow this line as closely as other limitations as to radius of curves, etc., will permit.

The profile of this projected location may now be plotted and grade-line studies made upon it, from which desirable changes will be indicated on the map location. Thus an indefinite number of projected locations might be made on the map, but the process is not carried far because, finally, the location of the line must be fitted to the ground in the field.

437. Finishing the Map.—The subject of drafting in its general aspects has been discussed in Chap. IV, and the methods of plotting have been explained and map symbols have been given in Chap. XV. The following paragraphs deal with these subjects only as they apply to the symbols on topographic maps.

There is a noticeable tendency among topographic draftsmen toward restraint in the matter of ornate and elaborate titles and symbols. While the use of certain symbols and colors is amply justified by the character and importance of many maps, these aids should be employed skilfully, because of the impression they give as to the quality of the map as a whole.

437a. Colors.—The standards most generally used for the hues, tints, and shades of colors are those employed by the cartographers of

such organizations as the U. S. Geological Survey and the U. S. Coast and Geodetic Survey.

The colors which can be purchased in the market do not match the standards mentioned, and the draftsman must either be content with an approximation or he may alter the commercial colors by mixing them.

The fundamental principles of mixing colors are quite simple, but to secure the desired tint or shade of a line in practice may require much patient experimentation. The number of colors used by the map draftsman is relatively few.

The three primary colors are red, blue, and yellow. Any other color can be produced by mixtures, in varying amounts, of these primary colors. Thus a mixture of red and blue produces purple, blue and yellow yield green, and yellow and red yield orange. These different varieties of colors are called *hues*. If a color mixture having a given hue is thinned by adding water, the color is said to change in *tint*. Thus a hue may be given a light tint by adding water, or a darker tint by allowing the water to evaporate, or by adding more pigment. If black pigment is added to a mixture of a given hue, the color is said to change in *shade*. Thus various shades of a hue are secured by adding various amounts of black pigment. Any clean water may be used in mixing water colors, but distilled water should be used with all inks.

In addition to black, three colors most commonly used on topographic maps are as follows: burnt sienna (reddish brown) for all land forms, *i.e.*, contour lines, hachures, sand, etc.; prussian blue for all water features, *i.e.*, streams, lakes, marsh, etc.; and green for trees and growing crops.

437b. Flat Tints.—If a considerable portion of a map is to be covered by symbols, it is sometimes best to use a *flat tint*, or tinge of color spread uniformly. This is especially true in the cases of water and timber areas.

Flat tints may be applied to drawing papers but should not be used on tracing cloth because of the resultant distortion. Such tints should be very light and evenly applied, but the procedure of tinting is too complex to be fully described here. Colored tints are sometimes applied to tracings by the use of pencils. This may be done on tracing papers, but on cloth the coloring materials frequently spread through the fabric and ruin tracings. The effect of a tint may be produced on a tracing by the use of a soft lead pencil.

437c. Water Colors and Inks Compared.—As regards ease in handling, inks are preferable on line drawings, *i.e.*, those executed with ruling and lettering pens; water colors, however, are preferable

if a flat tint is to be spread with a brush. As regards the quality of color, water colors are more readily mixed to secure variations in hue, shade, and tint, and they are not so vivid as inks, which condition is usually considered an advantage on maps. As regards permanency, water colors do not fade as do many inks; but the inks are waterproof whereas, of course, water colors are not. With the care that should be given to the preservation of a permanent drawing, however, this latter consideration should not be given much weight.

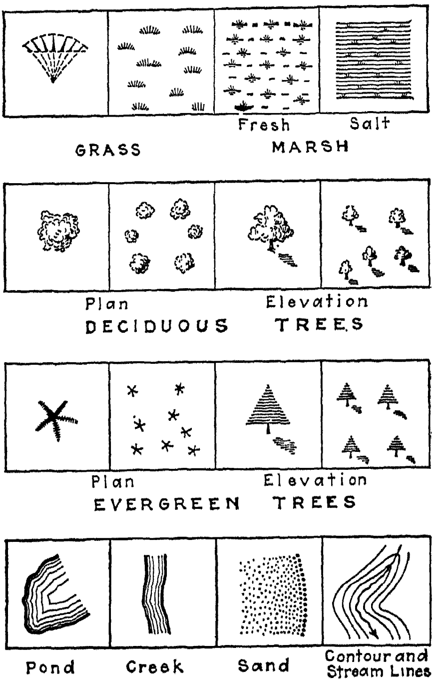


FIG. 437.

437d. Drawing the Symbols. *Grass.*—The methods described in this article are illustrated in Fig. 437. The symbol for grass consists of a series of lines drawn radially toward a point. The tops of the lines begin on an arc of a circle, and the bottoms of the lines terminate on a ground line parallel with the bottom border line of the map. The symbol may be composed of three, five, or seven lines or blades. The central blade is a straight vertical line; the others are symmetrical on either side and may be slightly curved, concave outward. The arc and ground lines may be lightly penciled by the beginner, these lines to be erased after the symbol is inked.

The separate symbols are distributed evenly over the area but they should be irregularly spaced so as not to give the appearance of rows. The size and spacing of the symbols will depend upon the scale of the map and the area to be covered.

Fresh Marsh.—The symbol for fresh marsh consists of the grass symbol beneath which the water surface is shown by either a single or a double line drawn slightly longer than the base of the grass tuft. The water-surface lines are drawn with a ruling pen, and should be accurately parallel with each other and to the base of the map. Other water lines may be sparingly filled in between the grass tufts. If the map is inked in colors, the symbol is drawn in blue.

Salt Marsh.—This symbol is shown by the use of closely and evenly spaced lines drawn with a ruling pen parallel to the base of the map. On these lines is drawn the grass symbol, spaced as in a field or in fresh marsh. The colored symbol is drawn in blue.

Trees.—Tree symbols may be drawn either in plan or in elevation. The latter practice is better adapted to reconnaissance sketches or elevation drawings of a terrain, but on most topographic drawings the symbol shown in plan is more suitable. It is common practice to differentiate between deciduous and evergreen trees.

The symbol in plan for deciduous trees is executed by first drawing an outline as a scalloped, broken line to represent the two or three main branches of a tree. The inside area is then sparingly filled in with small scalloped, broken lines. Assuming that the source of light is from the upper left-hand quadrant, the lower right-hand quadrant of the tree in plan would appear to lie in shadow; accordingly the lower right-hand quadrant of the symbol may be shaded. In elevation, the tree symbol is shown as a fairly even, symmetrical, scalloped outline, beneath which the trunk is represented by a heavy vertical line. Again, the lower right-hand area of the symbol is shaded, and the shadow of the tree on the ground may also be sketched on the map.

The size of tree symbols is varied on maps of different scales. On very large-scale maps, if many trees are to be drawn, the symbol obscures other features. Hence on such maps the outline only is drawn; or in some cases the trunk only may be indicated as a dot, and the diameter of the trunk and the kind of tree may be recorded beside it, as for example, 20-in. maple. On intermediate-scale maps the symbols can be made of size to show very nearly the horizontal projection of the tree represented, but on small-scale maps no attempt is made to draw the symbols to scale.

The colored symbol is drawn in green. If a large area is to be shown as covered with forest, or if many other details are to be shown on the map, a flat tint of green should be used.

To represent evergreen trees in plan, the symbol is drawn as bold lines radiating from a central point. The separate symbols should each be composed of five or six lines, and should be fairly symmetrical and uniform in shape. The representation in elevation is drawn as a series of closely spaced horizontal lines beginning with a dot at the top, and grad-

ually increasing in length toward the bottom. Beneath the last of these lines the trunk is shown as a heavy vertical line. The area in shadow may be sketched on the map. The colored symbol is drawn in green.

Water Lines.—To draw the water-line symbol for lakes and ponds, the draftsman begins by sketching at the shore line, two hair lines spaced as closely as they can be drawn without touching each other. Each succeeding line is drawn in close conformity with the one preceding and the spacing between the lines is increased uniformly outward from the shore. The conformity between adjacent lines is effected by drawing each line so that it makes a series of smooth, intersecting arcs of curves, all of which arcs are drawn concave toward the shore. The excellence of the total effect depends upon (1) the regularity and the rate with which the spacing increases from the shore outward; (2) the uniformity of the space between two given lines; and (3) the smoothness of the arcs. The lines may be drawn to fill in the water area completely, or if the space is large it may be left without lines near the center.

For rivers the method is the same as that for lakes, but for small streams, although the principle is the same as that for larger bodies of water, the execution is varied somewhat to produce the desired effect within smaller scope. The principal device is that of drawing each shore line as a heavy single line, which gives the effect of two closely spaced lines, and the lines in the center may be broken occasionally to heighten the effect of an open water surface. The width of stream is drawn to scale, as nearly as may be, down to the single-line symbol. If contours are shown in close proximity to the water lines, there is some difficulty in distinguishing between them unless the map is drawn in colors. This confusion may be reduced to some extent by making the stream or shore lines wavy lines in which the lengths and amplitudes of the waves are very small.

Sand.—A sand bar or flat is represented by dots evenly spaced over the area. The edge of the flat or the shore line is indicated by a line of closely spaced, relatively large dots. Just behind this row, a line of dots, smaller in size, is drawn, each dot being placed in a space between two dots on the first line. A third row of dots still smaller in size is placed behind the second row, and so on, until the desired spacing for the entire area is reached. Beyond that line the appearance of rows should be avoided. This may be effected by first dividing the area into irregular triangles having curved sides. Then the space within each triangle is filled in by carefully spacing the dots uniformly over the area.

Contour Lines.—Contour lines are drawn as fine, smooth, freehand lines of uniform width. Each fifth line is weighted slightly heavier than the others to facilitate the reading of elevations on the map. If the contours are spaced closely on the map only the fifth contour lines need be numbered; but on areas of low relief where the contour lines are spaced widely apart each line may be numbered with its elevation. The line is broken to leave a space for the number. On large-scale maps

where the contour lines extend over large areas, it is difficult to maintain an even weight of line unless a *contour pen* is used (see Art. 61*e*, p. 64).

438. Tests for Accuracy of Topographic Maps.—A topographic map can be tested for the accuracy of its dimensions, both in plan and in elevation, by methods described in the following paragraphs. It is assumed in this discussion that the errors in the field measurements may be disregarded, and that a graphical scale is provided on the map such as to render negligible the errors in the map due to atmospheric variations as regards the effect of moisture and temperature upon the paper.

1. *Tests for Dimensions in Plan.*—These tests consist in comparing distances scaled from the map and distances measured on the ground between the corresponding points.

The precision with which distances may be scaled from a map depends upon two factors, *i.e.*, the scale of the map and the size of the plotting errors. Thus, if for a given map the scale of which is 1 in. = 100 ft., it is known that the error in position of any one point with respect to any other is $\frac{1}{40}$ in., then the error in the map distance between any two points is 0.025 in. or 2.5 ft. on the ground.

Some surveys are made for the purpose of estimating areas, as for example, of a reservoir site. The errors in areas scaled from such maps can readily be determined from a consideration of the errors in the scaled distances, if it is remembered that the per cent of error in the area of a figure is equal to the sum of the per cents of error in the length and the breadth of the area (Art. 31). As an example, it may be assumed that a given area on a map is 8 by 24 in., the scale of which is 1 in. = 400 ft.; also that the average error in plotting and scaling the map distances is ± 0.03 in. Accordingly, if it is assumed that the errors in scaled distances are independent of the lengths of the lines, the errors in the scaled dimensions of this area are $0.03 \times 400 = 12$ ft. in each side. The per cent of error in the area is then

$$\frac{12}{8 \times 400} + \frac{12}{24 \times 400} = \frac{1}{267} + \frac{1}{800} = \frac{1}{200} = 0.5 \text{ per cent.}$$

2. *Tests for Elevations.*—Tests for elevations can be applied by comparing the elevations taken from the map of a number of points chosen at random, and the elevations of the corresponding points determined by field levels. Such points are usually taken at 100-ft. stations along traverse lines chosen in such a manner as to cross a considerable number of the typical features of the terrain represented on the map.

A further application of this test consists in plotting the two profiles to identical coordinate scales on the same sheet, the values for one being read from the map and the values for the other being provided by the field level notes. These profiles provide a graphical record of the agreement between the map profile and the corresponding ground profile.

As in the case mentioned above, the comparison between separate 100-ft. station points can be made; also, the presence of systematic errors will be evidenced if the map profile is for an undue proportion of its length above or below the ground profile. Careless work in spots will be made evident by wide divergences between the profile lines at such places. This test is, therefore, more searching than that of random points only.

439. Choice of Map Scale.—From a consideration of the tests described in the previous article it is possible to choose a map scale consistent with the purpose of the survey if the size of the plotting errors is approximately known. For example, if it is known that with reasonable care in plotting, the average error in distance between any two definite points on the map is $\frac{1}{40}$ inch, and if it is known that the purpose of the survey will be met if the average error in scaled map distances is 10 ft., these conditions are satisfied by a map scale of 1 in. = 400 ft.

Also, let it be assumed that it is desired to estimate the area within the flow line of a proposed reservoir with a permissible error not greater than 5 acres, that the area is roughly 4 mi. long and $\frac{1}{4}$ mi. wide, and that the errors in map distances will not exceed $\frac{1}{40}$ in. From these assumptions the area contains roughly 640 acres, is 1,300 ft. in width by 21,000 ft. in length, and the allowable percentage of error in the area is $\frac{5}{640}$ or $\frac{1}{128}$. It follows that the permissible ratio of error in the perimeter is $\frac{1}{256}$. Hence the permissible errors in scaling the two sides of a rectangle which approximates the extent of the area on the map are $\frac{1{,}300}{256} = 5$ ft., and $\frac{21{,}000}{256} = 82$ ft., respectively. Therefore, the smallest permissible error in the map is 5 ft., and for a map in which the plotting error is $\frac{1}{40}$ in., the scale of the map should be 1 in. = 200 ft.

440. Specifications for Topographic Maps.—The principles stated in Arts. 438 and 439 provide the criteria by which the accuracy of topographic maps may be specified. Thus, the accuracy in horizontal dimensions may be required to be such that the average errors in scaled dimensions between definite points chosen at random shall not exceed a given value, or that the per cent of errors in areas scaled from the map shall not exceed a stated value.

The accuracy of contour lines may be specified by assigning maximum values to: (1) the average error in elevations taken from the map; (2) the maximum error indicated by random test profile lines; and (3) the ratio of the length of the map profile which lies $\left\{\begin{matrix}\text{above}\\\text{below}\end{matrix}\right\}$

the ground profile to the length which lies $\left\{\begin{matrix}\text{below}\\\text{above}\end{matrix}\right\}$ the ground profile.

For example, the accuracy of a given topographic map might be specified as follows: The average error in distances between definite points as scaled from the map shall not exceed 8 ft.; the average error in elevations read from the map shall not exceed 1 ft.; the maximum error indicated by random test profiles shall not exceed 4 ft., and the ratio of the length of the map profile which lies above the ground profile to the length which lies below the ground profile must lie between the values of 25 and 75 per cent.

441. Office Problems.

Problem 1. Profile from Topographic Map

Object.—To plot the profile for a proposed highway or similar route from data of a contour map. It is assumed that governing points and maximum rate of grade are given, and that width of roadbed and side slopes are fixed.

Procedure.—(1) Sketch in pencil a route between governing points that appears favorable. (2) Set bow dividers to measure 100 ft. or some multiple thereof at the scale of the map. From the point of beginning of the route step off distances and read elevations as indicated by the contours. (3) Plot the corresponding profile. (4) Fix the grade line, making such readjustments of the proposed route as seem necessary to secure the most favorable location. (5) Calculate the volumes of cuts and fills by the second method described in Art. 164, p. 222. Check calculations by the first method in Art. 164. (6) On each of the cuts and fills of the profile show the volume in cubic yards.

Problem 2. Volume of Earthwork from Contours

Object.—To determine volumes of earthwork from a topographic map showing contours before and after grading. It is assumed that a map showing contours of the original ground is assigned and that other conditions attached to the problem such as area to be graded, slopes of the finished surface, etc., are given.

Procedure.—(1) On the assigned map sketch contours of the proposed ground surface. (2) By the method explained in (2) of Art. 436*b* draw lines of no cut and fill. With the planimeter measure the horizontal sections of earth cut or filled at each contour elevation. Calculate the volume between successive contours and the total volume for each cut and fill. (3) Solve the same problem by the method in (3) of Art. 436*b*. (4) Note the difference in volumes given by the two methods. (5) Finish the sketch by showing all construction lines for the first method in black and all those for the second method in red. On the drawing show the volumes obtained by each method.

CHAPTER XXV

TOPOGRAPHIC SURVEYING

442. General.—It has been stated that the distinguishing characteristic of a topographic map is the use of symbols to represent the relief of the ground surface, and that the contour line is the symbol most commonly used for this purpose. Correspondingly, the distinguishing feature of a topographic survey is the determination of the position, both in plan and in elevation, of the system of ground points which are necessary to the construction of the map.

Considering the range of uses for topographic maps (Art. 426) and considering the variation in character of the areas covered, it is plain that topographic surveys must vary widely. Some are simple in plan and execution, covering but a few acres; others are complex in plan, difficult in execution, and extend over hundreds of square miles.

In this chapter no attempt will be made to deal with the procedure of making the refined measurements and the necessary computations required in that special class of surveys which extend over wide areas, but rather to provide the topographer and the engineer with practical aid in planning the ordinary topographic survey, and to provide the student with a clear conception of the principles and methods pertaining to this work.

443. Classes of Surveys.—Topographic surveys fall into three classes, namely, those for large-scale, intermediate-scale, and small-scale maps. The range of map scales within each designation is not definite, yet the classification is a useful one. The ratios commonly accepted as belonging to each of these classes are as follows:

Large scale: $\frac{1}{120}$ to $\frac{1}{1{,}200}$; *i.e.*, 1 in. = 10 ft. to 1 in. = 100 ft.

Intermediate scale: $\frac{1}{1{,}200}$ to $\frac{1}{12{,}000}$; *i.e.*, 1 in. = 100 ft. to 1 in. = 1,000 ft.

Small scale: $\frac{1}{12{,}000}$ to $\frac{1}{240{,}000}$; *i.e.*, 1 in. = 1,000 ft. to 1 in. = 20,000 ft.

444. Control.—The necessary measurements for so small a survey as that of a building site may often be made from a single set-up of

the instrument; but if the area is more extended or if visibility is obstructed, the instrument must be placed at two or more stations, which must be located with respect to each other. If the area to be surveyed is small so that a relatively few stations will suffice, these may all be located by a simple traverse or a triangulation system connecting them. This system of measurements, consisting of either a traverse or a triangulation system, which locates the instrument stations is termed the *control*. It provides the skeleton of the survey which is later clothed with the located positions of such objects as roads, houses, trees, streams, ground points of known elevation, and contour lines which fill in the completed map.

The system of measurements which locates the instrument stations in plan, together with the monuments which fix the positions of these points in the field, is called the *horizontal control;* and the system of measurements (lines of levels) which locates the instrument stations in elevation, together with the bench marks established in the field, is called the *vertical control.*

In many cases the area is of such wide extent that the required number of instrument stations cannot conveniently be included in a single traverse or a simple triangulation system, or else the measurements, which may be sufficiently accurate for surveys of small areas, will not be sufficiently accurate for the more extended survey because of the accumulative effect of the errors involved. In these cases the desired accuracy for the entire survey may be secured if a few stations evenly distributed over the tract are connected by a system of measurements of relatively high precision, called the *primary control.* Within this system, subordinate systems of measurements to locate the instrument stations need not be made with such high precision because the distance in which the accumulative effects of the errors can take place will be not farther than from one primary control station to another. This system of measurements to locate instrument stations within and under the primary control is called the *secondary control.* The extensive surveys executed by the U. S. Coast and Geodetic Survey include four grades of control, called first order, second order, third order, and fourth order, corresponding respectively to primary, secondary, tertiary, and quaternary control according to the more usual designation.

It should be noted that the terms *primary* and *secondary* are purely relative, so that the degree of accuracy used on a secondary traverse for one survey might be sufficient for a primary traverse on another. This fact may be noted by inspection of Table 444, which gives approximate values of the limits of permissible errors for control measurements suitable to the different map scales. It will be seen, for example, that the

TABLE 444.—TOPOGRAPHIC SURVEY CONTROL DATA (APPROXIMATE VALUES)

Scale of map	Kind of control, for given scale	Triangulation: Length of sight, miles	Triangulation: Average error of closure in triangles	Triangulation: Distance between bases, miles	Triangulation: Probable error in base measure	Triangulation: Maximum discrepancy between bases	Traverse: Length of traverse, miles	Traverse: Maximum error of angles	Traverse: Maximum linear error of closure	Levels: Length of circuit, miles	Levels: Maximum error of closure
Small	Primary	10 to 200	1″	100 to 500	$\frac{1}{1,000,000}$	$\frac{1}{25,000}$	50 to 500	2″	$\frac{1}{25,000}$	50 to 500	4 mm. $\times \sqrt{\text{kilometers}}$ or 0.017 ft. $\times \sqrt{\text{miles}}$
	Secondary	5 to 20	3″	50 to 200	$\frac{1}{500,000}$	$\frac{1}{10,000}$	25 to 200	5″	$\frac{1}{10,000}$	......	......
	Tertiary	1 to 10	6″	10 to 100	....	....	10 to 100	30″	$\frac{1}{5,000}$	10 to 100	0.05 ft. $\times \sqrt{\text{miles}}$
	Quaternary	½ to 2	1′ or graphical	2 to 10	....	....	1 to 2	2′ or compass	$\frac{1}{1,000}$	1 to 10	0.1 to 0.5 ft. $\times \sqrt{\text{miles}}$
Intermediate	Primary	1 to 5	10″ to 20″	5 to 50	$\frac{1}{10,000}$ to $\frac{1}{40,000}$	$\frac{1}{1,000}$ to $\frac{1}{4,000}$	1 to 20	10″ to 1′	$\frac{1}{1,000}$ to $\frac{1}{5,000}$	1 to 25	0.05 to 0.3 ft. $\times \sqrt{\text{miles}}$
	Secondary	½ to 2	Graphical	1 to 5	....	....	1 to 5	30″ to 3′	$\frac{1}{500}$ to $\frac{1}{2,500}$	1 to 5	0.1 to 0.5 ft. $\times \sqrt{\text{miles}}$
Large	Primary	1 to 5	2″ to 10″	2 to 20	$\frac{1}{20,000}$ to $\frac{1}{80,000}$	$\frac{1}{2,000}$ to $\frac{1}{5,000}$	1 to 5	30″ to 1′	$\frac{1}{5,000}$ to $\frac{1}{20,000}$	1 to 10	0.05 ft. to 0.1 ft. $\times \sqrt{\text{miles}}$
	Secondary	¼ to 1	5″ to 20″	1 to 5	....	....	½ to 3	30″ to 2′	$\frac{1}{1,000}$ to $\frac{1}{5,000}$	1 to 3	0.05 ft. to 0.1 ft. $\times \sqrt{\text{miles}}$

accuracy desirable for the secondary control for intermediate-scale maps corresponds roughly to that for quaternary control for small-scale maps; also the accuracy indicated for the secondary control for large-scale maps corresponds roughly to that for primary control for intermediate-scale maps, etc.

The various classes of control apply either to triangulation or traverse methods; and often a combination of the two methods is used, as when the primary control consists of a triangulation system and the secondary control consists of traverses connecting the triangulation stations. Figure. 454*b* illustrates this combination. The case of a secondary traverse within a primary traverse is shown in Fig. 452*b*.

444a. Horizontal Control.—The relative positions of a number of points on the earth's surface can be determined if they are intervisible to the extent that a suitable system of triangles can be formed. The measurements include the observed values of the angles in the triangles formed and the length and true azimuth of one side in the system, called the *base line*. From these measurements the lengths and directions of all other sides can be computed. Such a system of triangulation is shown in Fig. 454*a*, and the methods are described in Chap. XXVIII.

Also, the horizontal-control stations may be located by a traverse or by a series of traverses as shown in Fig. 452*a*. Under certain conditions it is advantageous to combine the methods.

444b. Vertical Control.—Vertical control is supplied by the establishment of bench marks suitably distributed over the area. The methods of leveling may be one or another of the various kinds described in previous chapters.

445. Details.—For most topographic maps the ground points are of primary importance

and the survey is planned with particular regard to them. As indicated in Art. 435, various systems of points are used in drawing the contour lines, and corresponding to each of these systems there are one or more field methods designed to locate the points most expeditiously. In addition to the ground points, the field measurements locate all other points or objects necessary to the purpose of the survey.

446. The Map.—The field measurements indicated above enable the draftsman to plot the horizontal control stations, the ground and other detail points, and the contour lines, and by the use of other symbols to complete the map. Most maps can be drawn conveniently on a single sheet of paper, but those which represent relatively large areas are better drawn on several sheets. In this case they may be unified by the use of a system of coordinate lines, parts of which will appear on each sheet and by means of which the positions of all points on the several sheets may be referred to a common origin. For example, the small-scale quadrangle maps of the U. S. Geological Survey are drawn on sheets on which there have been projected the meridians of longitude and parallels of latitude, which system of geographical coordinates serves to unify the map as a whole.

If a map represents but a few square miles of territory, the system of coordinates is rectangular in form because the curvature of the parallels of latitude and the convergency of meridians are inappreciable on such a map.

447. Monuments.—The need for the field work of some surveys extends over comparatively short periods of time, as for example, the control for minor topographic surveys for which there is no use beyond the map itself. On the other hand, if the survey is to extend over a long period of time, or if the field stations are intended for more general use than merely the resultant map (for example, surveys along boundary lines, or surveys of rivers and harbors), the positions of the field stations must be permanently marked. The permanence of such a mark depends upon the visible evidence left at the spot; the marker is called a *monument*. The more permanent forms of monuments are cast-iron or concrete posts planted in the ground, or bronze tablets imbedded in rock or in the masonry of structures. Such a monument may mark a traverse station, a triangulation station, or a bench mark. In many cases a single monument serves all three purposes.

448. Choice of Field Methods.—The choice of field methods is governed by (*a*) the intended use of the map, (*b*) the area under consideration, (*c*) the map scale, and (*d*) the contour interval.

448a. Intended Use of Map.—It is evident that the purpose which the survey is to meet is a prime consideration in planning the work. For example, the earthwork estimates to be made from a topographic map by a landscape architect must be determined from a map which represents the ground surface much more accurately, both in horizontal and vertical dimensions, than one to be used in estimating the storage capacity of a reservoir. Also, the kind of features to be shown and the relative degrees of precision desired in the plotted positions on the map of the different kinds of objects are determined by a knowledge of the use to be made of the map. For example, a survey for a bridge site would be more detailed and more accurate in the immediate vicinity of the river crossing than in areas remote therefrom.

448b. Area.—It is more difficult to maintain a desired precision in the relative location of points over a large area than over a small area.

448c. Scale of Map.—The consideration of consistent accuracy between the field work and the scale of the map is subject to some misapprehension and merits close attention. It is sometimes considered that if the errors in the field measurements are not greater than the errors in plotting that the former are unimportant. But since these errors may not be compensating, it is believed that the ratio between errors in plotting and errors in the field should be, perhaps, three to one before it may properly be assumed that the field errors are negligible on the map.

The ease with which accuracy may be increased in plotting as compared with a corresponding increase in the accuracy of the field measurements, points to the desirability of reducing the total cost of a survey by giving proper attention to the excellence of the work of plotting points, of interpolation, and of interpretation in drawing the map.

The choice of a suitable map scale is discussed in Art. 439.

448d. Contour Interval.—The choice of a proper contour interval is based upon three principal considerations: (1) the desired accuracy of elevations read from the map, (2) the characteristic features of the landscape, and (3) the legibility of the map.

1. *Accuracy.*—Let it be assumed that two maps are equally accurate, so that the average error in elevations, read from the map, of points chosen at random is one half of a contour interval. Assume one map to have a contour interval of 5 ft. and the other 2 ft. It is evident that the average error in elevations of points chosen at random on one map is $2\frac{1}{2}$ ft., and on the other, 1 ft. Therefore, the contour interval may be thought of as the scale by which the

vertical distances or elevations are measured on a map; the more refined the scale, that is, the smaller the interval, the more refined will be the measurements of the elevations of chosen points.

2. *Features.*—Often field conditions exist where characteristic features require the use of a contour interval which would otherwise be inappropriate. Thus, if the shape of the terrain is such as to show much variation within a small area, or in other words, if the topography is of *fine texture*, then a smaller contour interval is required to show the greater complexity of configurations. On the other hand, if the landscape is composed of large, regular forms, or is of *coarse texture*, then a larger interval may be used.

3. *Legibility.*—A map otherwise excellent may be rendered useless and its appearance disfigured by a mass of contour lines which obscures other essential features. Topographers have attempted to state the proper relation between the scale of the map and the contour interval in the form of rules; but the variables of scale, of character of ground, and of purpose, are too complex to have their relationship so specified. It may be said that, in general, contour lines should not be spaced on the map more closely than 30 to the inch, although the legibility of the map depends largely upon the fineness and precision with which the lines are drawn. The lithographed maps of the U. S. Geological Survey and of the U. S. Coast and Geodetic Survey yield good results with much closer spacing. Table 448 represents good practice for usual conditions.

TABLE 448.—RELATION BETWEEN SCALE OF MAP, SLOPE OF GROUND, AND CONTOUR INTERVAL

Range of scale	Slope of ground	Interval, feet
1 in. = 10 ft. to 100 ft.	Flat	0.5 or 1
	Rolling	1 or 2
	Hilly	2 or 5
1 in. = 100 ft. to 1,000 ft.	Flat	1, 2 or 5
	Rolling	2 or 5
	Hilly	5 or 10
1 in. = 1,000 ft. to 10,000 ft.	Flat	2, 5 or 10
	Rolling	10 or 20
	Hilly	20 or 50
	Mountainous	50, 100, or 200

449. General Field Methods.—The principal instruments used are the engineer's transit, the plane table, the engineer's level, and the hand level or the clinometer.

Points of horizontal and vertical control having been established, the details which are to be mapped may be located by any of the several methods previously described in chapters dealing with chaining and with the use of the transit and the plane table. Ground points are either located directly upon the contours (they are then called contour points) or chosen without regard to the actual location of the contours. The elevations of these ground points are determined either by direct leveling or by trigonometric leveling, and their positions are determined with respect to points of horizontal control in the same manner as other details which are to be mapped.

In the location of topographic details, the stadia is used extensively for both large- and intermediate-scale maps. It is sufficiently precise except for maps of very large scale, say larger than 1 in. = 20 ft.

For intermediate- or large-scale mapping where contours are located directly with transit or plane table, the level is frequently used as an auxiliary instrument for directly determining the position of the contours on the ground.

For small-scale mapping, a relatively small number of ground points are located by triangulation methods with the plane table, elevations being determined by scaled distances and vertical angles being measured with the telescopic alidade.

For large- and intermediate-scale mapping, ground points are commonly located by the method of radiation, and their elevations are usually determined by trigonometric leveling, the transit-stadia or plane-table-stadia method being used.

Where the country is comparatively flat, elevations of ground points are determined by direct leveling.

Sometimes, where the tract to be mapped is wooded, where the topography is smooth, or where for other reasons it is expedient, the tract is divided into squares in checker-board fashion, and by direct leveling the elevations of the corners of the squares and of other critical points along the sides of the squares are determined.

On route surveys, such as the preliminary surveys for highways, railroads, and canals, contour points are located along lines transverse to the main traverse, distances from traverse to contour points being measured with the tape and elevations of contour points being determined usually with the hand level.

Where the system of ground points is located on the contours by direct leveling and the positions of these contour points are determined by radiation or intersection with plane table or transit, it is called a *trace-point* system.

Where the system of ground points is at the corners of squares and at critical points along their sides, all ground points being located by linear measurement and their elevations being determined by direct leveling, it is called a *coordinate-point* system.

Where the ground points form an irregular system along ridge and valley lines and other critical features of the terrain and are located by radiation or intersection with transit or plane table, elevations being determined by trigonometric leveling or sometimes by direct leveling, it is called a *controlling-point* system.

Where the ground points form a system on lines transverse to a main traverse, the points being located by linear measurement and direct leveling, usually with the hand level, it is called a *cross-section-point* system.

Where the ground points form a system on lines transverse to a main traverse, elevations being determined only at points where the slope changes, it is called a *cross-section-controlling-point* system.

INTERMEDIATE-SCALE SURVEYS

450. General.—In the following articles will be described the general field procedure that might normally be employed on a topographic survey, for a map of intermediate scale, of a tract of considerable extent where points of horizontal and vertical control are located in favorable positions for securing the map details which are obtained by the transit-stadia or plane-table methods. An application of the general principle will be made to a specific example. With modifications the same procedure might be employed on large-scale surveys. Obviously the character of the country, the extent and purpose of the survey, and the contour interval and the scale of the map have an important influence upon the choice of methods and the field procedure.

451. Horizontal Control; General.—The horizontal control may consist of a traverse system, a triangulation system, or a combination of the two. For an extensive survey there is first established a primary system, and this is extended by a secondary system. On surveys of less extent only the primary system is necessary.

452. Primary Traverse.—If primary control is to be established by a traverse for a survey for an intermediate-scale map, the transit-tape method is generally used.

The route is chosen with regard to the position of the stations to be located and to ease of travel. Because of the accumulative effects of the errors in transit-tape traversing, it is desirable to provide checks at intervals not greater than 15 miles. Hence, it is best if possible to arrange for closed circuits whose lengths will not exceed 12 to 15 miles. If closed circuits cannot conveniently be

secured, checks for distance should be applied as the work proceeds and checks for azimuth by astronomical observations should be applied at intervals not exceeding 15 miles. If the area is such that a single closed circuit will suffice, the traverse is run near the perimeter of the tract. If several circuits are required, they are preferably arranged to divide the tract into areas approximately equal in size. The routes are taken along roads where these are suitably located, and elsewhere the traverse follows ridge or valley lines in so far as these natural control lines make possible a proper distribution of stations. Where they exist, advantage can sometimes be taken of the government land lines, because in flat or rolling country the roads usually follow these lines and the system provides an even and regular subdivision of the area.

If it is assumed that the length of the traverse will not exceed 15 miles, the requirements of most surveys will be met by ratios of precision of $\frac{1}{1,000}$ to $\frac{1}{10,000}$ which apply to map scales of 1 in. = 1,000 ft. (or more) to 1 in. = 100 ft., respectively.

As regards the personnel, instruments, and methods, the principles stated in Chap. XIII apply to this work.

452a. Example of Primary Traverse.—As an example of the procedure indicated in the preceding article, reference may be made to the survey illustrated in Figs. 452*a* and 452*b*.

The conditions assumed to govern this survey are that the area is approximately 3 by 7 mi. in extent, that the scale of the map is 1 in. = 500 ft., and that the country is such that control may properly be established by the traverse method.

The field survey is planned by the aid of an existing small-scale map (not shown) on which is drawn, as nearly as may be, the outline of the tract. Because of its size, the intermediate-scale map is to be drawn on a series of sheets, each to represent an area 7,500 by 12,500 ft., or 15 by 25 in. in size. The origin of a system of coordinates is chosen so that the entire area will lie north and east of this point. From the small-scale map it is seen that a highway and railroad crossing is situated near the southwest corner of the tract. Accordingly, the center of this intersection is arbitrarily given the coordinates of latitude 5,000 ft. north, and longitude 5,000 ft. east (see Fig. 452*a*), thus placing the origin approximately 1 mi. south and 1 mi. west of this intersection.

A coordinate map projection is then laid out to fix the coordinates of each corner of each map sheet, as shown by the straight dash lines in Fig. 452*a*.

The routes for the traverses are chosen along the highway and railway lines in such a manner as to provide closed circuits not over 10 mi. in length, which traverses follow roughly the perimeters of the various sheets. The transit party then establishes the traverses along the chosen

routes, being governed by the condition that the permissible error of closure is $\frac{1}{3{,}000}$. The transit-tape-azimuth traverse is employed, for which the procedure is described in Art. 227, p. 312. Permanent monu-

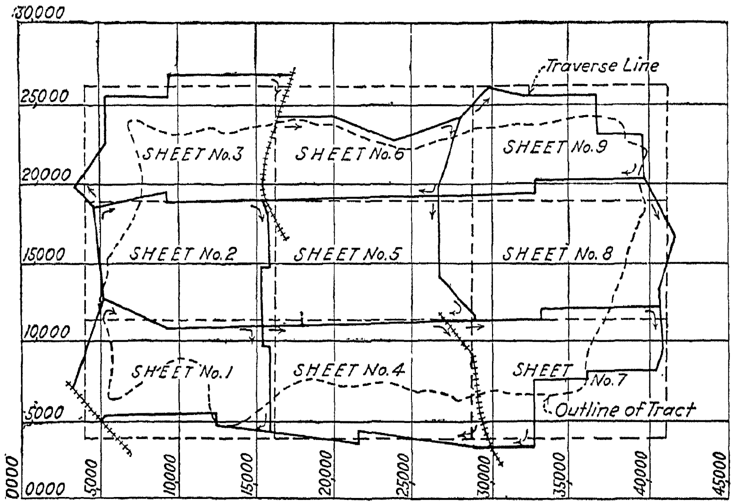


FIG. 452*a*.

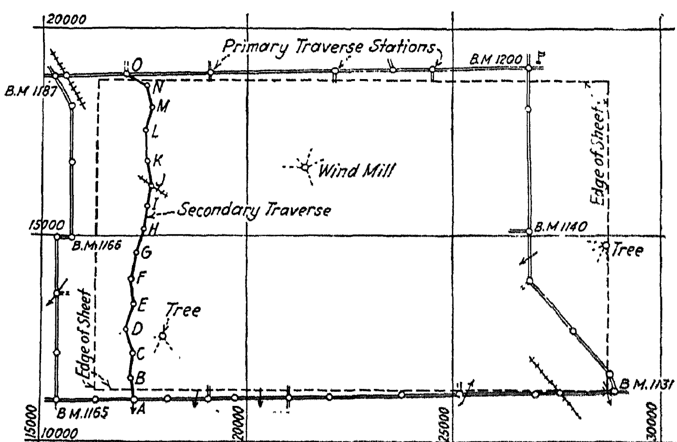


FIG. 452*b*.—Sheet 5 of Fig. 452*a* enlarged.

ments are placed at intervals of not over 1 mi., and are carefully referenced to nearby permanent objects. All streams, bridges, houses, and road crossings are located with reference to the traverse line. The trees and windmill shown in the enlarged reproduction of Sheet No. 5 (Fig.

452*b*) are examples of prominent objects to which azimuth angles have been read from transit stations on the traverse, to serve as checks on the work.

453. Secondary Traverse.—Wherever a secondary traverse is required to fix the positions of the instrument stations from which to locate the details, two general methods may be used: (1) the secondary traverse may be carried forward simultaneously and in connection with the survey for location of details; or (2) it may be run before and separate from the location of details. A considerable amount of time is saved by the use of the first method, provided no serious errors or mistakes are made, and provided the accumulation of errors between primary control points is not so great as to require unduly large adjustments of the secondary-traverse measurements to effect a closure. If the details are to be located by the plane-table method, it will usually be desirable to run the secondary traverse before the location of details is begun because, in this case, there is no opportunity to adjust the secondary traverse to the primary stations if the details are mapped as the secondary traverse proceeds.

The route of the secondary traverse is selected with particular regard to the location of instrument stations that will be best situated for observing details. The route is frequently chosen along a ridge or a valley line, and in all cases the length is made such as to avoid an unduly large accumulation of errors.

Since the secondary traverse is ordinarily not more than three or four miles in length, the needs of most surveys will be met if the ratios of error do not exceed $\frac{1}{3{,}000}$ and $\frac{1}{500}$ corresponding to the map scales of 1 in. = 100 ft. and 1 in. = 500 ft. (or more), respectively.

The personnel, instruments, and methods for traversing of this nature have been described in Chap. XIII.

It should be noted that the methods here described as being applicable to secondary traverses for surveys for intermediate-scale maps may, and usually do, apply to secondary traverses for large-scale, and to tertiary or quaternary control for small-scale maps.

453a. Example of Secondary Traverse Using Stadia and Compass. In the survey illustrated in Fig. 452*b*, it is assumed that for a permissible error of $1/400$ the method used is that of a compass-stadia traverse, and that the traverse is run before the location of details. The route is chosen in such a manner that the details and ground points may be located along a strip of such width as to overlap slightly the strip of topography to be located from the next adjacent parallel traverse. The traverse closes on a primary station at *O*. The procedure has been described in Art. 184*b*, p. 242.

453b. Secondary Traverse Using Plane Table.—A traverse having the same degree of accuracy as that assumed in the previous article may be run with the plane-table instrument. The method is described in Art. 413, and as in the case above, the instrumentman occupies alternate stations only and the instrument is oriented by use of the compass needle.

This method may also be used for higher degrees of accuracy up to the limit which can be secured with stadia measurements, perhaps $\frac{1}{1,000}$, and with taped distances a precision of perhaps $\frac{1}{2,500}$ can be obtained. For these degrees of precision the table is oriented by backsights. Great care is used in drawing the rays and scaling the distances, and all precautions are observed to insure accuracy (Arts. 421 and 422).

By this method it is more desirable to run the secondary traverse before and separately from the location of details, for the reasons stated in Art. 453. The peculiar advantages of this instrument do not appear in traverse work; therefore, the method is ordinarily of doubtful value. However, the method saves the time of plotting in the office, and it offers a ready means of adjusting the traverse to the primary control stations.

453c. Secondary Traverse Using Transit.—If the accuracy required in the secondary traverse is $\frac{1}{1,000}$ or greater, a transit-tape traverse is most commonly used. The method is fully described for the various degrees of precision in Chap. XIII. If details are being located as the traverse proceeds the method is adapted to the particular conditions, and special care is observed in checking the measurements. For example, even though the consideration of accuracy might not require it, the stadia distances may be read both on foresights and on backsights, both the *A* and the *B* verniers may be read, all distances may be taped in duplicate, etc.

454. Primary Triangulation.—The various field conditions favorable to the use of triangulation to establish the horizontal control are (1) a fairly extended area in an open, hilly region, (2) a city where accurate traversing is difficult because of street traffic, or (3) a rugged mountainous region where traverse work would be slow and laborious. Wooded regions seriously lessen the usefulness of this method; observation towers are required to establish lines of vision between stations, and the necessary expense of time and money is not justified, except for surveys of considerable magnitude.

A general description of the personnel, instruments, and methods for this work is given in Chap. XXVIII.

The method indicated in the following example may be adapted to surveys of greater or smaller extent, and for map scales either larger or smaller, by properly modifying the accuracy of the field measurements.

454a. Example of Primary Triangulation.—The method of establishing primary triangulation with the transit is illustrated in Figs. 454*a* and 454*b*. The conditions assumed for this survey are the same as those assumed in Art. 452*a*, and also it is assumed that the region is suitable for a system of triangulation.

A general layout of the scheme is planned on an existing small-scale map, to serve as a guide in establishing the field stations. The region is then reconnoitered, the positions of the stations are selected on summits where visibility is good, and signals are erected.

Sites for two base lines, "West Base" and "East Base," are selected, one along a highway near one end of the tract and the other along a railroad near the other end of the tract. The base lines for this survey have lengths of about 2,500 and 4,000 ft. respectively, and are measured with sufficient care so that the probable error does not exceed $\frac{1}{10,000}$.

The angles are measured with such accuracy that each station closure, *i.e.*, the sum of the angles measured about each station, does not differ from 360° by more than 30″; and each triangle closure, *i.e.*, the sum of the angles in each triangle, does not differ from 180° by more than 1′30″.

Observations are taken on two kinds of stations, *i.e.*, those marked by signals which are later to be occupied by the instrument, shown by small triangles in the figure; and those marked by such objects as trees, flagstaffs, chimneys, church spires, etc., which are not designed to be used as instrument stations but which will be of great aid in subsequent work. These stations are shown by symbols and names.

A stellar or a solar observation is made at each base line to determine its true azimuth, and to the south end of the West Base are given the arbitrary values of its coordinates as latitude 11,000 ft. north, and longitude 11,500 ft. east, thus placing the tract entirely in the northeast quadrant of the coordinate system.

The field measurements having been completed, the necessary computations and adjustments are made, from which are determined the coordinates of each station in the system. These computations are fully explained in Chap. XXVIII.

The system of coordinates is now projected on a series of map sheets and the positions of all observed objects and stations are plotted by the method of coordinates (Fig. 454*a*).

In Fig. 454*b* are shown to a larger scale the results of the primary triangulation as it exists on and near sheet number 5 of Fig. 454*a*. The instrument stations are shown by small triangles; other objects are indicated by symbols and names.

The checks afforded in this work are (1) the duplicate measurements of the base lines, (2) the station and the triangle closures of the angle measurements, (3) the comparison of the length of East Base as measured directly and as calculated from the measured length of West Base, and (4) the comparison of the azimuth of East Base as observed directly and as computed from West Base.

454b. Triangulation with Plane Table.—For map scales smaller than 1 in. = 500 ft., it is usually possible to obtain the required

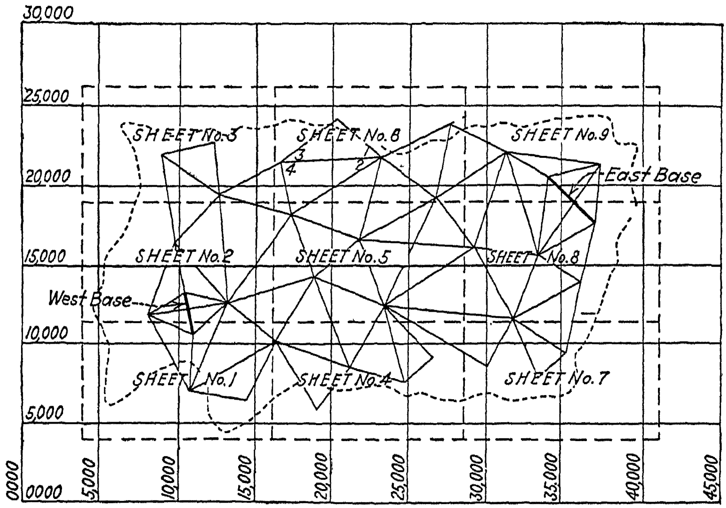


FIG. 454a.

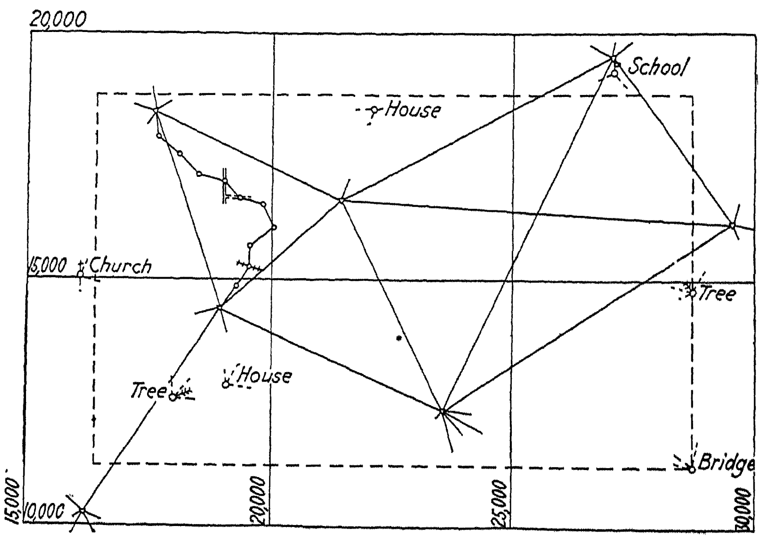


FIG. 454b.—Sheet 5 of Fig. 454a enlarged.

accuracy by use of the plane-table instrument and the method of graphical triangulation (Art. 415, p. 616). Where this method can

properly be used it has the advantage that no computations are required, since the positions and coordinates of all stations are determined graphically on the plane-table sheet.

455. Secondary Triangulation.—The primary triangulation has resulted in the location of a number of transit stations, hilltops, chimneys, trees, and other prominent objects, whose positions have been plotted on the field sheets (Fig. 454*b*). The secondary control stations may now be located either by one of the methods of traversing explained in Chap. XIII, or, where field conditions are suitable, by resection and intersection methods explained in Chap. XXIII. If such a secondary triangulation system is to be established within and under the control of the primary triangulation, thus rendering a secondary traverse unnecessary, this work may be done with either the transit or the plane table. The advantages of the latter instrument are important, however, because it provides a ready means of solving the three-point and two-point problems. On the other hand, it is true that three-point determinations made with the transit can be conveniently plotted in the office by the use of a three-arm protractor; or, if this instrument is not available, a piece of tracing cloth on which the angles have been laid off can be used instead, as explained in Art. 417*c*.

This method of secondary triangulation has the advantage that instrument stations can be chosen at strategic points, unaffected by the cumulative errors inherent in traversing.

456. Small Areas.—In the case of topographic surveys for intermediate-scale maps of small areas, *i.e.*, from a few acres up to a hundred acres or more, the horizontal control can be established either by a single traverse or by a simple triangulation system. If the area is moderately extended, two base lines can be measured, one at each end of the tract, to serve as checks on the work.

The instruments and methods are those which apply to secondary control, and are described in the preceding articles.

457. Vertical Control; General.—The purpose of the vertical control for a topographic survey is to establish bench marks at convenient intervals over the area to serve (1) as points of departure and closure for the leveling operations of the topographic parties when locating ground points, and (2) as reference marks during subsequent construction work.

Primary and secondary level routes are required in very much the same amount, and bear the same relation to each other, as do the primary and secondary traverses or triangulation systems. The level routes often follow the traverse lines, the traverse stations being used as bench marks.

The datum may be assumed for a given survey; but the results of governmental precise levels, referred to sea-level datum, are now available for all but the most isolated regions of the United States.

458. Primary Leveling; Accuracy Required.—Four degrees of accuracy are commonly used in establishing the primary vertical control for topographic surveys for intermediate-scale maps: (1) a maximum error expressed by the coefficient of 0.05 ft.$\sqrt{\text{distance in miles}}$, (2) 0.1 ft.$\sqrt{\text{distance in miles}}$, (3) 0.3 ft.$\sqrt{\text{distance in miles}}$, and (4) 0.5 ft.$\sqrt{\text{distance in miles}}$. The first applies to very flat regions where a contour interval of 1 ft. or less is used, and also in surveys which require the determination of gradients of streams, or which are to establish the grades of proposed drainage and irrigation systems. The second, third, and fourth coefficients apply to surveys in which no more exact use is made of the results than to determine the elevation of ground points for contours having 2, 5, and 10-ft. intervals respectively.

458a. Methods.—The leveling methods applicable to the first two degrees of accuracy stated above have been fully explained in Chap. VIII. For the last two degrees of accuracy mentioned, the method is much the same as for the two former cases, except that the larger permissible errors allow longer sights and greater speed in manipulation of the instrument. The last degree of accuracy listed can be reached by careful stadia leveling (Art. 246, p. 335), which method has important advantages in hilly country.

In leveling work of low precision, it is often desirable to keep the instrument in good adjustment and to manipulate it with care to the end that longer sights may properly be taken than would otherwise be possible.

459. Secondary Leveling.—Since the lengths of the secondary level circuits are, in general, roughly one fourth of the lengths of the primary circuits, and since the errors vary approximately as the square root of the distance, the coefficients of permissible errors (for the same error of closure) are about twice the corresponding coefficients used on the primary circuits. Thus, if the coefficient of error permissible for the primary circuit is 0.05 ft. $\sqrt{\text{miles}}$, the coefficient permissible for the secondary circuit is 0.1 ft. $\sqrt{\text{miles}}$.

The methods are indicated in the previous article.

460. Trigonometric Leveling.—The height of the instrument, either plane table or transit, which has been oriented by the two- or three-point problem, is usually determined by trigonometric leveling, *i.e.*, by a vertical angle and by a horizontal distance, either scaled from the map or measured directly by stadia or tape (see Art. 99,

p. 115). Usually two or more stations are observed, to increase the accuracy of the determination. The method is also applicable to the field conditions where the horizontal control is established by triangulation and where a high degree of accuracy in the measured elevations is not required.

The accuracy required in the vertical angle bears a direct relation to that in the horizontal distance, and the permissible error in each depends upon the contour interval and scale of map. The following values computed for an assumed error of 1′ in the vertical angle at a distance of 1,000 ft., for different values of the measured vertical angle up to 10°, will enable the topographer to determine the precision required both in the vertical and in the horizontal distances: An error of 1′ in any vertical angle up to 10°, at a distance of 1,000 ft., produces an error in elevation of 0.3 ft. (approximately). Errors in the determination of the horizontal distances which likewise cause an error of 0.3 ft. in elevation are tabulated below:

Vertical angle	Error in horizontal distance, feet
1°	18
3	6
6	3
9	2

For example, at a distance of 1,000 ft. and at a vertical angle of 9°, the error in elevation due to an error in the vertical angle of 01′ is 0.3 ft.; likewise, an equal error in elevation is caused by an error of 2 ft. in the horizontal distance. Therefore, if this vertical angle is measured with a maximum error of 1′ the horizontal distance should be measured with a maximum error of 2 ft.

For distances greater than about 1,000 ft., corrections for curvature of the earth and refraction should be made; or preferably, reciprocal measurements to eliminate natural and instrumental errors should be taken (see Arts. 99 and 99*a*, p. 115).

461. Location of Details; General.—It is assumed in the articles which follow that the necessary horizontal and vertical control measurements have been made and that the field party is concerned with the location of details only. If the plane-table method is to be used, the horizontal control is plotted on the plane-table sheet.

The adequacy with which the resultant map meets the purpose of the survey depends largely upon the work of locating details, and the topographer should be completely informed as to the uses to be made of the map, to the end that he may give the right emphasis to each part of the work.

The five typical systems of ground points used in map construction have been described in the previous chapter. The trace-point

and coordinate-point systems yield accurate results if a sufficient number of points are observed, hence, if the purpose of the map requires a comparatively high degree of accuracy, the expense of observing many points is warranted and one of these systems is chosen. But, if the purpose of the survey does not warrant the location of as many points as are required by the two systems mentioned, better results are secured by the use of controlling points for the survey of wide areas, or of cross-section points for route surveys. Of course, these different systems are sometimes combined in the same survey, to meet the various field conditions. Since the requirements for accuracy are not as high in the case of surveys for intermediate-scale maps as for large-scale maps, the controlling-point system is most commonly used in the former case, and either the coordinate- or the trace-point system in the latter case. Most route surveys are drawn to an intermediate scale, and the cross-section-point system is especially well adapted to that purpose and scale of map. The methods described in this chapter apply, for the most part, to the field work of locating ground points according to the controlling-point or the cross-section-point systems.

The comparative merits of the transit and plane-table instruments in locating details have been stated in Art. 424. The field conditions more favorable to the use of the transit as compared with the plane table are many definite points to be located, or ground covered with brush, timber, or vegetation such that at a given station the area of visibility is small. The conditions more favorable to the plane table are open country and many irregular lines and areas to be mapped. Conditions are sometimes such that the advantages of both instruments can be secured by their use in combination. But since these conditions more often occur in the case of surveys for large-scale maps, the description of such use is given in that connection (Art. 471).

462. Accuracy Required in Field Measurements; Objects Other than Contours.—The accuracy required in locating details depends not only upon the scale of the map but also upon the definiteness of the objects sighted. Such definite objects as buildings, bridges, boundary lines, etc., should be located with an accuracy consistent with that with which they can be plotted on the map, which may be assumed to be a map distance of about $\frac{1}{60}$ of an inch. Such less definite objects as shore lines, streams, edges of woods, etc., are located with an accuracy corresponding to a map distance of perhaps $\frac{1}{20}$ or $\frac{1}{30}$ of an inch.

462a. Location of Contours.—The veracity with which contour lines represent the terrain depends upon (1) the accuracy of the

observations, (2) the number of observations, and (3) the distribution of the points located.

Ground points are definite, but since the contour lines must necessarily be generalized to a large extent, it would be inappropriate to locate the points with refined measurements. The error in position should be consistent with the error in elevation which, in general, should not exceed one fifth of a contour interval, *i.e.*, the error in position should not exceed one fifth of the map distance between contours, and the error in elevation should not exceed one fifth of

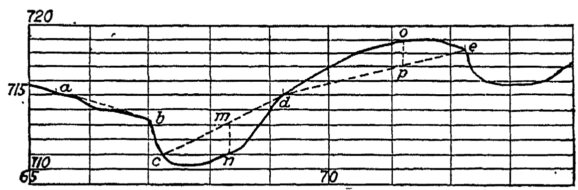


FIG. 462*a*.

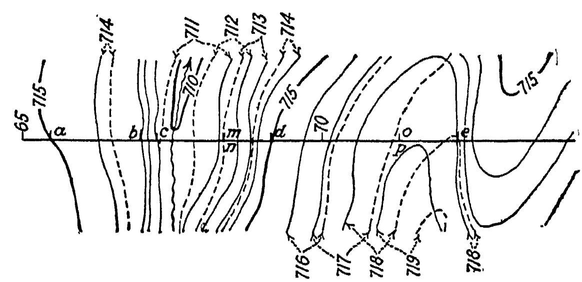


FIG. 462*b*.

the vertical distance between contours. It is true, of course, that the purpose of a given survey may impose greater or less accuracy than that assumed here, but the principle will apply in any case.

The purpose of a topographic survey will be better served by locating a greater number of points with less accuracy, within reasonable limits, than by locating fewer points with greater accuracy. Thus, if for a given survey the contour interval is 5 ft., a better map will be secured by locating with respect to each instrument station perhaps 50 points whose average error in elevation is 1 ft. than by locating 25 points whose average error is only 0.5 ft.

A general principle which should serve as a guide in the selection of ground points may be noted. As an example, let it be supposed that a given survey is to provide a map which shall be accurate to the extent

that if a number of well-distributed points are chosen at random on the map, the average difference between the map elevations and ground elevations of identical points shall not exceed one half of a contour interval. Under this requirement, the attempt is made in the field to choose ground points such that a straight line between any two adjacent points will in no case pass above or below the ground by more than one contour interval. Thus, in Fig. 462*a*, if the ground points were taken only at *a*, *b*, *c*, *d*, and *e* as shown, the resulting map would indicate the straight slopes *cd* and *de*, with the consequent errors in elevation of *mn* and *op* on the profile, showing that additional readings should have been taken at the points *n* and *o*. The corresponding displacement of the contours on the map is shown by dotted and full lines in Fig. 462*b*.

462b. Angular Measurements.—The precision needed in the field measurements of angles may be readily determined by reference to Table 462, which is a tabulation of the angles which correspond to various permissible linear errors either in azimuth (measured along the arc from the true point) or in elevation. For example, for a permissible error of 0.5 ft. in azimuth and for a sight 800 ft. in length, the corresponding permissible error in the horizontal angle is 2.1′. Similarly, for a permissible error of 0.5 ft. in elevation and for a sight of 600 ft., the permissible error in the vertical angle is 2.9′.

TABLE 462.—VALUES OF ANGLES CORRESPONDING TO PERMISSIBLE LINEAR ERRORS IN OBSERVED AZIMUTH OR ELEVATION
(Values in Minutes of Arc)

Length of sight, feet	Permissible error, feet									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
100	3.4′	6.9′	10.3′	13.7′	17.2′	20.6′	24.1′	27.5′	30.9′	34.4′
200	1.7	3.4	5.1	6.9	8.6	10.3	12.0	13.7	15.5	17.2
300	1.1	2.3	3.4	4.6	5.7	6.9	8.0	9.1	10.3	11.4
400	0.9	1.6	2.6	3.4	4.3	5.1	6.0	6.9	7.7	8.6
500	0.7	1.4	2.1	2.7	3.4	4.1	4.8	5.5	6.1	6.9
600	0.6	1.1	1.7	2.3	2.9	3.4	4.0	4.6	5.1	5.7
700	0.5	0.9	1.5	2.0	2.5	3.0	3.4	3.9	4.4	4.9
800	0.4	0.8	1.3	1.7	2.1	2.6	3.0	3.4	3.8	4.3
900	0.4	0.8	1.1	1.5	1.9	2.3	2.7	3.1	3.4	3.8
1,000	0.3	0.7	1.0	1.4	1.7	2.1	2.4	2.8	3.1	3.4

Obviously this table can be used for any desired precision by applying a direct proportion (assuming that the tangents of small angles vary directly with the angle). Thus if it is desired to locate a point to the nearest 2 ft. in azimuth (or elevation) and the length of the sight is 500 ft., the corresponding permissible error in the angle is 14′.

463. Details by Transit-stadia.—The personnel of the topography party using the transit usually consists of the transitman, recorder, and one or two rodmen. In wooded country one or more axemen are usually needed to clear the lines of vision. The organization should be such as to allow not only the instrumentman but also all other members of the party to work as rapidly as possible. Thus, if distances between points are great, a party of four men may be organized as a transitman and three rodmen; or if points are very close together, the party may be organized as a transitman, recorder, computer, and one rodman.

In locating ground points, the vertical angles are usually observed more precisely than are the horizontal angles. Accordingly, the vertical circle of the instrument assumes greater importance than the horizontal circle; but because all vertical angles are measured with respect to a horizontal plane it is important that the horizontal plate should be truly horizontal and remain so without the need of constant releveling. An instrument with a 4½- to 5¼-in. horizontal circle will be satisfactory, provided the field conditions do not require unusual stability and provided the vertical circle has a diameter of at least 5 in. The advantages of lightness and ease of manipulation of a small-size transit are offset to a degree by the greater stability of a larger instrument (5½- to 6¼-in. circle), and the more powerful telescope of the latter permits greater speed and range of observations. If vertical angles are to be measured to the nearest minute of arc, a level tube of about 30″ sensitiveness should be attached to the vernier arm of the vertical arc. The plate levels, the vernier of the vertical arc, and the attached level should be carefully adjusted, and the value of the stadia constant should be carefully determined. Many topographers prefer a stadia circle or Beaman arc for purposes of stadia leveling (Art. 250, p. 343).

It is often advantageous to set the instrument at some isolated but strategic point which has not been located by the horizontal control surveys. This may be done and the position of the instrument located by means of the three-point or two-point problem if definite objects suitably located are visible and if these objects have been, or can be, observed from other stations (Arts. 417 and 418). The elevation of the station may be determined by a stadia- or trigonometric-leveling observation on one or more points within the range of vision from the occupied station.

Relatively inaccessible or far distant points may at times be located by the principle of intersection (Art. 414).

463a. Transit-stadia Method on Hilly Ground.—For the conditions as represented in Art. 452*a*, Art. 454*a*, and Fig. 463, and assuming a

horizontal scale of 1 in. = 500 ft., the procedure of locating details by the transit-stadia method will now be described.

The transit is set up at a station on either the primary or the secondary control, as station B (Fig. 452b). The instrumentman

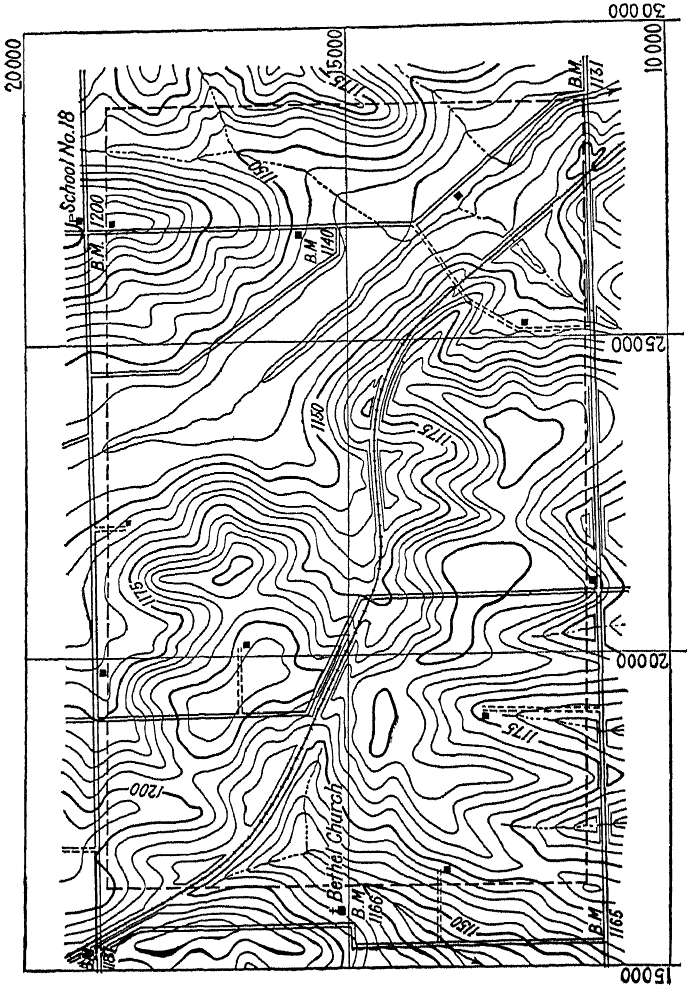


Fig. 463.

orients the transit by sighting on the nearest adjacent station. The elevation of station B, it is assumed, was determined by the level party which ran the secondary levels; therefore, the elevation of

adjacent points can be determined by the methods explained in Art. 248. If a direct rod reading for elevation is possible, or if the stepping method or Beaman arc is used, the procedure of reading the rod is modified as described in Arts. 249 and 250, p. 342.

The recorder keeps the record of all values given him by the instrumentman, and describes all points by remarks or sketches such that the draftsman can interpret all data correctly and draw all features properly on the map. Notes are kept in the form shown by Fig. 248*a*.

The rodmen choose ground points according to the principles stated in Art. 462*a* and illustrated in Fig. 435*a*, at summits, depressions, along ridge and valley lines, and at important changes in slope. Objects other than ground points are also located, as the purpose of the survey may require. The matter of selecting points is important and the rodmen should be carefully instructed and properly trained for their work. They must follow a systematic arrangement of routes such that the entire area is covered and that no important objects are overlooked.

The rodmen are instructed to observe the terrain carefully (often with the aid of a hand level), and to report any important features which can not be well discerned from the transit station. This condition makes it necessary to identify all points occupied by each rodman; and the recorder therefore indicates by some symbol (usually the rodman's initial) whose rod is being sighted. This procedure often enables the draftsman to detect mistakes in the recorded data.

Another method of identification of "side shots," as observations on detail points are called, is that in which each man in the party carries a watch, all of which are set to keep time together within a few seconds. As the instrumentman motions the rodman to a new point, both the rodman and the recorder observe and record the time, thus making it possible to identify any reading made upon either rod.

463b. Transit-stadia Method on Flat Ground.—The procedure in the preceding article is suitable for most surveys for intermediate-scale and for many small-scale maps where the terrain is rolling, hilly, or rough. But if the ground is flat, such that direct rod readings over large areas are possible, a more expeditious method is that of cross-sections as described in Art. 466, except that the transit or plane table may be used instead of the hand level, and that points may be located at changes of slope only. Another method is that of tracing out the contours on the ground, as explained in Art. 471*b* in connection with large-scale surveys where it is more often used.

464. Field Sketches.—Field sketches are valuable aids to supplement the observed data, especially where the ground exhibits many irregular features and where many details are to be mapped. They vary in character from freehand sketches and notes entered in a cross-ruled book, to elaborate field drawings for which an assistant is employed and which amount to an execution of the office procedure in the field. Where details are numerous, a drawing board is employed near the transit, and as the salient points are located by the transit they are plotted on the drawing, usually to a smaller scale than that of the map. The more complex and detailed topographic features are then sketched while the terrain is in view.

465. Details by Plane Table.—The personnel of the plane-table party for mapping details consists of the plane-table man, computer, and one or more rodmen.

The equipment usually includes a plane table, a telescopic alidade with a control level on the vernier arm, a peep-sight alidade, scale, small triangles, 8H pencil, and a stadia slide rule or stadia tables. The alidade should be in good adjustment and the stadia constant should be carefully determined.

For the conditions indicated in Art. 452*a*, Art. 454*a*, and Fig. 463, this method may be described as follows: Prior to going into the field, the horizontal control, the coordinate system, and the outline of the sheet have been adjusted and plotted on the plane-table sheet as shown in Fig. 452*b*. The elevations of all bench marks are a matter of record in the hands of the computer, or sometimes are recorded on the sheet itself. A cover sheet of tough paper is placed over the plane-table sheet to prevent its becoming soiled during the field work. A small portion of the cover is torn away to expose the sheet as the work progresses.

The plane-table man sets up his table at a convenient station as at *B* (Fig. 452*b*), and orients by backsighting on the nearest adjacent station. He then directs the rodmen to the controlling points of the terrain, as mentioned in the case of the transit. When a rodman presents his rod, the instrumentman pivots the peep-sight alidade about the plotted position of the station, draws a short portion of the ray near the end of the alidade farthest from the station point, directs the telescopic alidade on the rod, reads the stadia intercept, sets the cross-hair on the H.I. point, and motions the rodman forward. He next levels the control bubble on the vernier arm and reads the vertical angle. He then plots the point by scaling the horizontal distance (corrected for slope, if necessary, by the computer). The computer has now calculated the elevation of the point and the instrumentman records it on the map near the plotted point. As

rapidly as sufficient data are secured, the plane-table man sketches the contour lines. Other objects of the terrain are located and are drawn either in their finished form or with sufficient detail so that they may be completed in the office.

The utmost skill of the topographer is used in judging the features of the terrain and in representing these on the map with the required precision and with the least expenditure of time.

Many objects are located by the method of intersections, the elevations being determined by trigonometric leveling.

The computer makes whatever calculations are necessary, thus leaving the topographer free to observe data and to draw the map. The calculations for inclined sights are made most rapidly by the use of a slide rule, and the results are sufficiently accurate for most purposes. There is no need to identify the rod readings, or to use special precautions to cover the ground, as is necessary with the transit method, for all plotting is done in the field and mistakes or omissions are at once apparent, and any information brought in by the rodmen can readily be incorporated in the map.

The same principles in choosing points apply to the plane-table method as to the transit method.

465a. Because the plane table permits a ready solution of the three-point and two-point problems, use is made of these methods to enable the topographer to utilize advantageous locations for the instrument which may not have been included in the control surveys. This will be true especially where the control has been established by triangulation.

If the scale of the map is relatively small, perhaps 1 in. = 800 ft. or more, the table may be oriented by use of the magnetic needle.

If the table is set up at a point which has not been included in the control surveys, its elevation is determined either by stadia or by trigonometric levelings.

466. Details by Cross-section-point Method. *Hand Level.*—The personnel of the topography party using the cross-section-point method consists of the topographer, one or two chainmen, and one or more axemen as needed.

The equipment consists of either a cross-ruled, wide-page notebook, or sketch sheets mounted on a board perhaps 12 by 15 in. in size; a topographer's rod having plainly marked divisions usually equal to the contour interval; a 100-ft. steel or metallic tape; hand level or clinometer; and a 5-ft. rod, or "Jacob's staff," at the top of which the hand level or clinometer is mounted.

The procedure, using the hand level, may be described by reference to Figs. 466*a* and 466*b*, which represent respectively the field notes

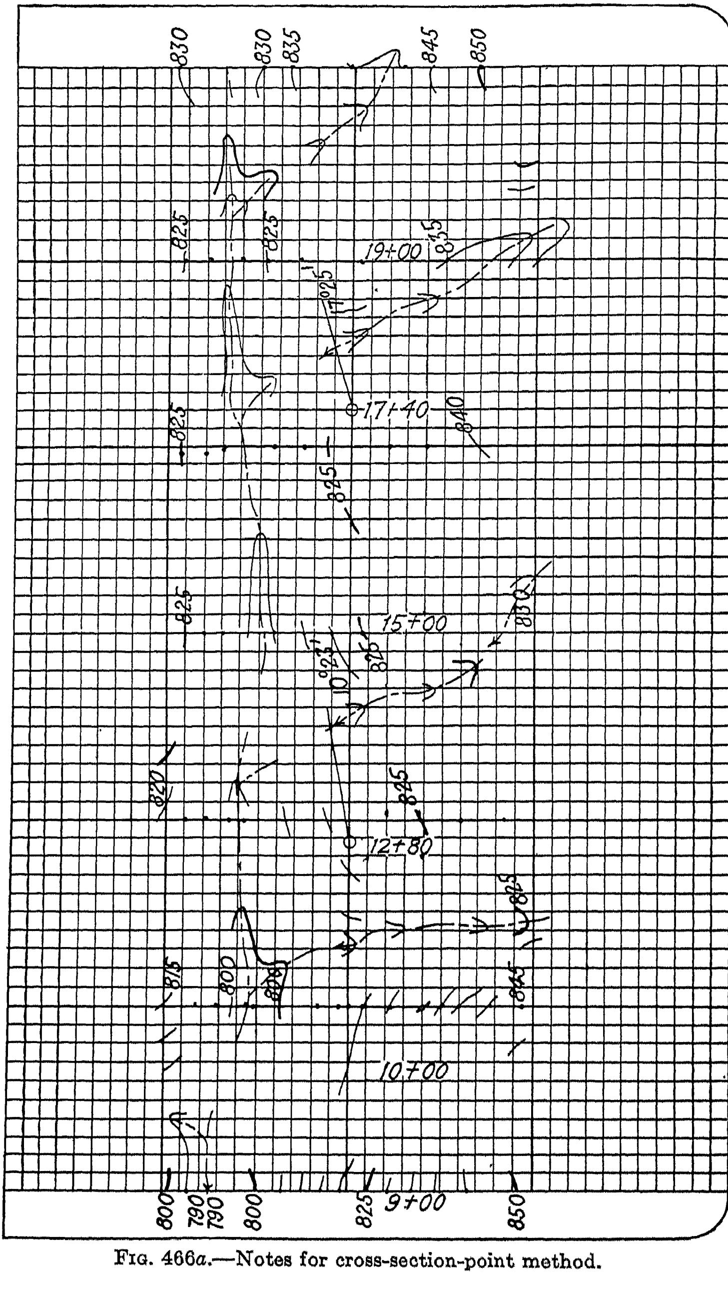


FIG. 466*a*.—Notes for cross-section-point method.

and the finished map. The results of the transit-tape traverse are shown on the ground by stakes set at each 100-ft. station and by hubs set at the transit stations. The level notes giving the elevations of these points are in the hands of the topographer. It is assumed that the contour interval is 5 ft., and that the party, consisting of the topographer and two chainmen, has reached station 9 + 00, where the topographer notifies the chainmen that the elevation of that station is 821.1 ft. The head chainman then moves out to the left side along a line estimated to be at right angles with the traverse line, until the rear chainman by the use of the hand level reads 6.1 ft. on the rod and hence finds that the head chainman has reached a point 1.1 ft. below the traverse station. The head chainman is now on the 820-ft. contour and the distance out from the center, 9.0 ft., is read on the tape and called out to the topographer who plots its position on the cross-ruled sheet of his book (Fig. 466*a*). Usually the directions or trends of the contours are shown at each cross line and along ridge and valley lines, but on the field sheets the contour lines are not sketched for their full length.

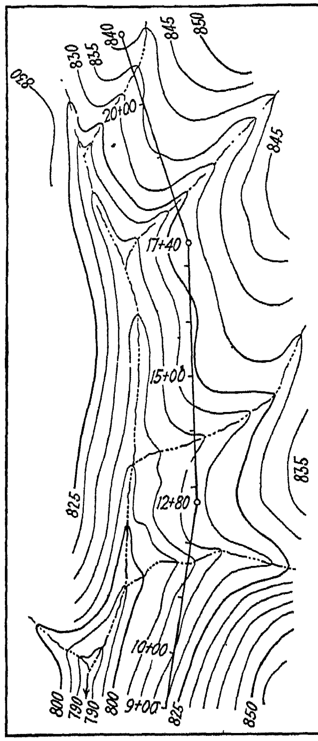


Fig. 466*b*.

The rear chainman now moves down to the position occupied by the head chainman on the 820-ft. contour, and the head chainman moves out until he reaches a point on the 815-ft. contour. The distance is read from the tape, 26 ft., and the point is plotted in the sketch book. This process is repeated until all contour points are located, out to the edge of the strip being surveyed, the distance depending upon the character of the topography. A similar procedure is followed on the right side of the center line except that in going uphill, either the chainmen exchange positions or they exchange duties, the head chainman carrying the hand level and the rear chainman the rod.

Where the traverse follows a valley or a ridge, the width of the strip may be narrow because the range of any feasible location of the route is thereby restricted. Where the topography is comparatively flat, observations may be taken over a width of 500 ft. or more on either side of the traverse line.

Sometimes, when the topography is regular, sketches are omitted and the distance from traverse to contour point is recorded numerically.

Instead of measuring distances from the traverse line out to each contour point, as just described, the cross-section-controlling-point system is sometimes employed (Art. 435). Distances are measured out to all important changes in slope, and the intermediate contour points are located by interpolation.

Clinometer.—For relatively small-scale maps, the purpose of the survey is sometimes met by measuring the distances to governing ground points (not on contours) by pacing, and the inclination of the ground slopes may be determined by the use of a clinometer (Art. 108*a*, p. 133). The topographer stations himself at a chaining station on the traverse line. The rodman moves out along a cross-section line, pacing the distance from the center line as he goes. When he reaches an important change in slope he halts and presents his rod. The topographer sights at a point on the rod at the same height above the ground as his eye and reads the angle of inclination. The rodman calls out the distance, and the topographer either records the angle and distance for later calculation of the difference in elevation, or by means of a table of values he immediately calculates the difference in elevation between the traverse station and the point indicated by the rod. By adding (or subtracting) this difference to the elevation of the center stake, the elevation of the point sighted is determined. The position of the point is then plotted and the elevation (if calculated at this time) is recorded in the sketch book. The topographer proceeds to the point thus located, and the rodman moves forward to the next important change in slope. This process is repeated until the limit of the strip for which the topography is being taken is reached. A similar procedure is followed on the other side of the traverse line. This method yields results less accurate than those secured by the use of the hand level and tape, but it materially increases the speed and reduces the cost.

466a. For the common conditions of a 5-ft. contour interval and a scale of 1 in. = 400 ft., maximum lengths of hand-level or clinometer sights should be limited to about 100 ft. But for smaller scales or larger intervals, longer sights up to perhaps 300 to 500 ft. may be used. If the lengths of sights are limited to 50 ft. or less, the errors

in elevation may be kept below a few tenths of a foot in a distance of 500 ft., thus making the method suitable to surveys using a 1 or 2-ft. interval.

The method here described, making use of either the hand level or the clinometer, has a definite advantage over other methods of locating ground points if the ground cover is dense, because along the cross-section lines the lengths of sight required are not great and the hand level or clinometer is suitable for sighting through very small openings in the undergrowth. The chainmen are thus able to penetrate rapidly thickets which would be prohibitive to stadia sights. Therefore, this method is sometimes used in forested areas of wide extent, the entire tract being surveyed by means of a series of strips having slightly overlapping edges.

466b. Either the transit or the plane table may be used for mapping route surveys, and the practice is advantageous if many details and complex features are to be mapped. If the transit is used, the method is that of controlling points as described in Art. 463*a*. The instrument is set up over any traverse station and is oriented by sighting at an adjacent stake or a hub along the center line; from this position the ground points and other details are observed. If the plane table is used, the ground points may be located either by means of the telescopic alidade and the controlling-point method (Art. 465) or by the cross-section-point method as described in Art. 466. In the latter case, the plane table takes the place of the sketch book and the topographer draws the map in the field.

467. Preliminary Route Surveys.—Preliminary surveys for locating railroads, highways, canals, etc., are topographic surveys made for the purpose of securing data for an intermediate-scale topographic map of the belt of country through which the route will finally pass.

Usually the horizontal control is a continuous transit traverse with distances measured with the tape and stakes set every 100 ft.; and the vertical control is established by a line of profile levels following the transit, the elevations of the ground at the 100-ft. stations being determined, and bench marks being established. The topographic details on either side of the traverse are then obtained by the cross-section method, contours being directly located usually with the hand level, but sometimes with the engineer's level.

Where the country is open, the transit-stadia method is well adapted to preliminary route surveys. For short lines, traverse distances are often determined by stadia, and elevations of traverse stations by vertical angle and stadia distance. Often under these

conditions, details are observed at each transit station as the traverse is run.

Where the country is open, with many details, and the topography is complex, the plane-table method is advantageous. Side shots are taken to controlling points and the map is drawn in the field.

LARGE-SCALE SURVEYS

468. General.—The two principal purposes which large-scale maps serve are (1) the location of structures such as buildings, bridges, dams, and tunnel portals, and (2) the estimation of earthwork for borrow pits, spoil banks, levee construction, and landscape grading.

One of the principal factors which affects the accuracy of map construction is the number of located points per square inch of map area. Since the map area which represents a given unit of ground area varies as the square of the scale to which it is drawn, it is plain that as the scale increases, many more located points are required per acre of ground. Therefore, in the case of large-scale maps, the location of details becomes a much more important part of the survey as a whole than in the case of surveys for maps to smaller scales. Accordingly, the variety of party organizations, instruments, and methods is greater in this class of surveys than in others.

As in the case of intermediate-scale maps, large-scale maps, if the areas are large, are often unified by a system of rectangular coordinates (Art. 452*a*).

The accuracy required in many large-scale surveys is secured best by use of the trace-point or coordinate-point system of ground points, though some surveys are advantageously made by the use of the controlling-point or the cross-section-point method, and by various combinations of methods.

On large-scale maps it is possible to scale the distances between definite points with errors not exceeding 1 or 2 ft.; hence, if the field work is consistent with the map, the important definite points in the survey, *e.g.*, existing structures, should be located with errors in position not greater than 1 ft. If this accuracy is not required by the purposes of the survey, the map may be plotted to a smaller scale. The engineer is accustomed to the large-size sheets of detail drawings and is likely to specify and use a large-scale map where a smaller scale would be more nearly consistent with the purposes of the survey. In this chapter the discussion is based upon the assumption that consistency exists between the accuracy of field measurements and the scale of the map.

469. Horizontal Control.—The degree of the required accuracy of the horizontal control, whether traverse or triangulation system, depends upon the area of the tract and upon the scale of the map. For small areas a precision of $\frac{1}{1,000}$ may be sufficient; on more extensive surveys the primary horizontal control often needs to be established with a precision of $\frac{1}{5,000}$. In precise surveys of cities, and in some surveys of unusual extent, far higher degrees of accuracy may be required in the primary control.

For most surveys of limited extent, the traverse system of control possesses definite advantages over triangulation, and traversing is generally employed; but for larger areas and in open or rolling country where field conditions are favorable, triangulation ordinarily has the advantage over traversing. The triangulation system for city surveys overcomes the difficulty of executing precise transit-tape traverses in the traffic of busy streets. It has the additional advantages that the roofs of many structures provide suitable supports for the instrument stations, and that many objects (such as spires, flagstaffs, and stacks) provide excellent monuments for observations.

The personnel, instruments, and methods for traversing have been described in Chap. XIII. Ordinarily traverses which form the horizontal control are run with transit and tape. Methods of triangulation are described fully in Chap. XXVIII.

470. Vertical Control.—Vertical control for large-scale maps is established by direct leveling. The contour interval is usually 1 or 2 ft., for which the coefficients of precision for the level circuits to establish the bench marks are 0.05 ft.$\sqrt{\text{distance in miles}}$ and 0.1 ft. $\sqrt{\text{distance in miles}}$, respectively.

The methods for this work have been described in Chap. VIII.

471. Details.—It has been stated that the field work of locating ground points and other details for large-scale maps is accomplished by a variety of methods, instruments, and party organizations. Accordingly, an example typical of the use of each system of ground points will be given, and in each case the application of the method to other conditions will be indicated.

471a. Details by Coordinate-point Method. *Transit and Tape.*—This method may be applied in the field in many ways, but the two most common methods of establishing the control and locating the interior points are illustrated in Fig. 471. An example of a finished map is shown in Fig. 434*a*.

By one method, a rectangular traverse, *A*, *B*, *C*, etc. (Fig. 471), is run near the perimeter of the tract. Stakes are set at every 100-ft.

station as the traverse proceeds. In this case the error of closure is apparent in the field, for the point of ending should fall on the point of beginning. If this error is more than the permissible error, the hubs and stakes are reset in such manner as to reduce the error within allowable limits.

A line of profile levels is run around this traverse to close within a permissible error, thus establishing the elevation of each stake and hub. These readings, except on turning points, are taken on the ground, just in front of the numbered side of each stake. The

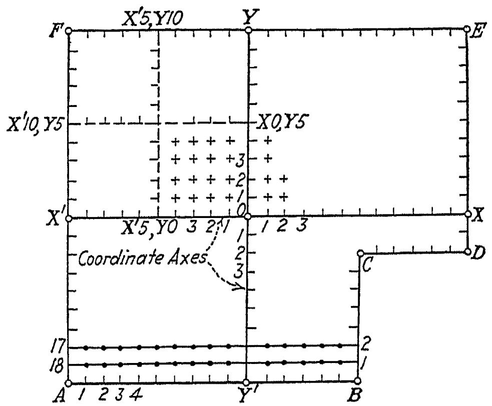


Fig. 471.

elevations thus determined should be correct to the nearest 0.1 ft. which is sufficiently exact to control the lines of levels run within the tract to determine the elevations of the coordinate points.

The interior stakes are set by transit-tape lines beginning at a stake on one side and closing on the corresponding stake on the opposite side, as from *F*-18 to *B*-1, *F*-17 to *B*-2, etc.

The elevations of these interior stakes are then determined with the engineer's level or with a hand level, depending upon the lengths of the lines and the accuracy required.

Irregularities which can not be properly interpreted from observations on the corners of the 100-ft. squares are now observed by means of a hand level and tape, and the features are sketched in a cross-ruled book or on a sketch sheet. Other details such as fences, roads, and buildings are located by measurements taken from adjacent coordinate points. With all this information recorded, the map is constructed in the office as explained in Art. 435.

All bench marks and transit-line hubs are permanently referenced for future use, and in some cases, substantial stakes bearing identifica-

tion marks are left at all 100-ft. station corners to serve as reference monuments during construction work.

By another method, coordinate axes are established by transit-tape lines as OX, OX', OY, OY', hubs being set at the extremities, and from these hubs coordinate lines are run parallel to the axes to intersect at the corners of the tract, thus affording checks on the work. Stakes are set at each 100-ft. station along these lines, and bench-mark levels are run to establish the vertical control. The interior points are set, their elevations are determined, other details are located, and the map is drawn.

Interior Points by Tape Only.—Where lines of vision are obstructed by woods, vegetation, or buildings, the interior coordinate points may be located by tape measurements only. Thus, beginning at station O (Fig. 471) three chainmen, A, B, and C with two 100-ft. tapes proceed as follows: Chainman A takes his position and holds the zero end of his tape at the point X_0Y_1; chainman B takes his position and holds the zero end of his tape at the point X_1Y_0; chainman C then stretches the two tapes, bringing the 100-ft. ends of the tapes to meet at the point X_1Y_1, and a stake is set. Next they proceed in a similar manner to set point $X_1\ Y_2$, etc. In this way all interior points can be established without the aid of transit sights.

If, however, the area is somewhat extended, and if it is desired to gain the advantage of the hand level in penetrating dense undergrowth, then the accuracy required does not permit the errors which would accumulate over long distances, and it is necessary to establish auxiliary or secondary control within the limits of the area to be surveyed. Thus, as shown in Fig. 471, transit-tape lines are run to divide the tract into 500-ft. squares, and lines of profile levels are run to determine the elevations of 100-ft. stations along the sides of these squares. Such lines are $X_0Y_5 - X_{10}'Y_5$ and $X_5'Y_0 - X_5'Y_{10}$, etc. The positions of interior points within the 500-ft. squares are now located by measurements with the tape only, as described above. Checks, both on distances and on the hand-level elevations, are afforded in all directions at distances not greater than 500 ft. If hand-level lines are run over each side of each 100-ft. square, then checks are afforded at the corners of each of these squares.

This method, as applied to a rough wooded area, is described in detail in Ref. 6, p. 697.

Plane Table.—The use of the plane-table instrument is a great aid to mapping by the coordinate method, if many irregular features are to be mapped. The methods of establishing the stations on the ground and of determining the elevations of these points are the same as when using the transit, as described earlier in this article. The details are drawn, however, on a plane-table sheet in the field.

The board is oriented by sighting along a traverse line on the ground, and details are sighted with a peep-sight alidade, the points being located by taped distances from adjacent coordinate points.

471b. Details by Trace-point Method.—If visibility is good, greater accuracy and speed can sometimes be obtained by use of the trace-point system than by the coordinate system of locating ground points. The plane table has important advantages over the transit in this work because fewer points need be observed and the time required in plotting the large number of points sighted with the transit is greatly reduced.

The engineer's level may often be used to advantage with the plane table. The level is readily moved from point to point, and direct rod readings are quickly made to place the rodman on a given contour. The level also permits greater lengths of sight, thus increasing the size of the area which can be mapped from a given plane-table station.

The personnel of the party consists of a topographer at the plane table, a levelman, one or more rodmen, and axemen as needed. A computer is sometimes added to the party.

The levelman, having set up the level at a convenient station, directs the rodman about until he locates a point on a given contour. This point is immediately sighted by the plane-table man and is plotted on the plane-table sheet. The rodman then moves to the next point which may be along the same contour or, on hilly ground, it may be a point on the next higher or lower contour. The distances from plane table to contour points are measured by the stadia method, but definite objects are usually located by taped distances.

It is, of course, possible to use the transit alone in this method, but as in the case of the plane table, it is usually more advantageous to use the engineer's level in conjunction with the transit. The relative merits of the transit and plane-table instruments have been discussed in Art. 424.

471c. Details by Controlling-point Method.—If the terrain has regular features, *i.e.*, gentle slopes and rolling ground, and if the accuracy required is not too great, the controlling-point method may be used. The personnel and procedure for the transit-stadia method have been described in Art. 463*a*, but in some cases where many details are to be sighted the transit may be used to advantage in combination with the plane-table.

If this method is used, the transit is set up on the control station, the position of which is plotted on the plane-table sheet. The plane table is then set up nearby and its position is plotted on the map in its

correct relation to the transit station. In some cases the map distance between the transit and plane-table stations is negligible and the two points are regarded as identical. The plane table is oriented in the usual manner. When a rodman has selected a ground point, the transitman observes the distance and vertical angle to it; the plane-table man observes the direction with the peep-sight alidade, draws a ray towards it from the plotted position of the plane-table station, and plots its position at the correct distance scaled from the plotted position from the transit station. The transitman, or computer, if one is present, computes the elevation of the point, which is recorded for immediate use on the map.

This combination of instruments utilizes all the advantages of both and provides an extremely rapid and efficient method of mapping where conditions are favorable, *i.e.*, where many points are to be located, a large scale is used, and the visibility is good. As compared with the use of the plane table alone, the combined use of the two instruments will save time in the field, but may not reduce the total cost because of the expense incurred by the addition of a transitman to the field party.

471d. Details by Cross-section-point Method.—As in the case of intermediate-scale maps, so in the case of large-scale maps, where a comparatively narrow strip of topography is desired the cross-section-point method may be used.

The method as applied to surveys for intermediate-scale maps has been described in Art. 466. The principal difference in the method in the case of large-scale maps is the use of the engineer's level or the use of the method of stadia leveling instead of hand leveling.

The transit is set up on a chaining station on the traverse, a right angle is turned, and the position of the contour next higher, or lower, than the instrument is located by a direct level reading and a stadia distance. By similar readings the successive contours are located out to the edge of the strip, on both sides of the traverse line.

If the differences in elevation are greater than can be observed directly on the rod, stadia-level readings may be taken at important changes in slope along the cross-section lines.

If hand-level readings are limited in length to 50 ft. or less, the errors can be limited to a few tenths of a foot in 500 ft. If this accuracy is sufficient, the hand-level method has the important advantage of permitting levels to be taken through dense undergrowth which would prevent the use of telescopic sights.

The plane table can, of course, be used in the cross-section-point method and the relative advantages and disadvantages previously mentioned will apply.

The cross-section-point method may often be used to expedite the survey of areas for which coordinate-point control (Art. 471*a*) has been established.

472. Location of Details Summarized.—The preceding descriptions of the methods of locating details for both intermediate- and large-scale maps may be summarized somewhat as follows:

As regards the use of instruments, (1) either the plane table or the transit may be used under practically all conditions, but under given conditions the advantages of one over the other are important, as explained in Art. 424; (2) for vision through dense woods or other vegetation, the hand level possesses large advantages over any other instrument for determining elevations.

As regards the use of the various systems of ground points, (1) for small-scale maps the controlling-point system is almost universally used; (2) for intermediate-scale maps of hilly or rolling ground the controlling-point system, and for intermediate-scale maps of flat ground or for route surveys the cross-section-point system is generally used; (3) for large-scale maps the coordinate-point or the trace-point system is commonly used. When high precision is the principal consideration, as for example in earthwork estimates, it can be secured more economically by the trace-point than by the coordinate-point system, provided that the ground surface is somewhat irregular in form; but if the ground is regular, permitting a certain amount of generalization in drawing the contour lines, the coordinate-point system will usually be preferred.

473. Building-site Surveys.—In the preparation of his plans for a building, the architect requires a map of the site to show the information necessary to the proper location of the building both in plan and in elevation. Such maps are usually drawn to the scale of 1 in. = 10 ft., or 1 in. = 20 ft.

The party consists of an instrumentman and one or two chainmen, and the equipment includes that of a transit party. Frequently, because of the large number of elevations to be determined, an engineer's level is also used. The notes may be kept in a transit or a topography notebook, or a sketch-board may be used to good advantage since it provides more space for sketching and recording details than does the single page of a notebook.

First the lot corners are located and permanent markers are set at these points. Then, using the property lines as reference lines, all objects are located, usually by tape measurements only, the instrumentman recording the data and drawing such sketches as are necessary. On extensive sites, or where it is not convenient for

the chainmen to locate objects by coordinate measurements, transit angles and taped distances may be used.

The details should be shown and described as follows: (1) lot corners, state kind; (2) property lines, give dimensions and the distances from the walks; (3) street lines, show widths; (4) sidewalks and drives, give kind and widths; (5) pavements, give kind and widths; (6) gas and water mains, state size and show exact

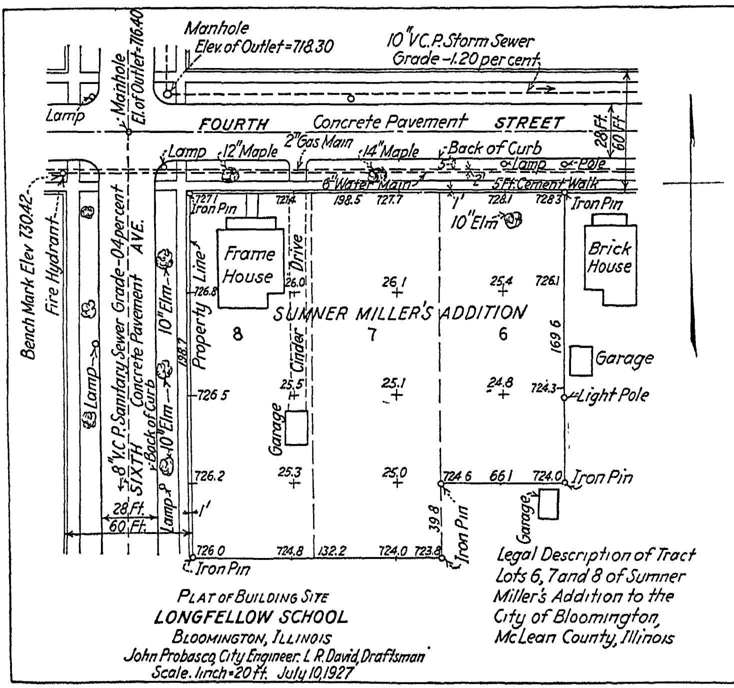


FIG. 473.—Map of building site.

location; (7) manholes and storm and sanitary sewers, give size and kind of pipe; (8) trees, state kind and size; (9) poles of all kinds; (10) fire hydrants; and (11) existing structures on or near the site, state materials of construction. Also, give the elevations of: (1) inverts of sewer outlets from manholes and the gradients of the sewers, (2) the reference bench mark, with description, (3) points along the sidewalks, curbs, and lot lines at intervals of 50 ft., and (4) ground points at the corners of 50-ft. squares. The elevations of the sewer inverts, sidewalks, and curbs should be taken to hun-

dredths, and all ground points to tenths of a foot; also contour lines are shown if the ground is irregular.

The map should give the legal description of the tract, and the other information which is ordinarily shown on the plat of an urban land survey. The drawing is made on tracing cloth, the size being the same as the other sheets of the architect's plans so that a print may be bound with each set of plans. A typical plat of this kind is shown in Fig. 473.

Methods of setting grades for buildings are described in Art. 141, p. 195.

473a. Establishing Points by Intersection.—In the construction of bridge piers, dams, and similar structures, it often happens that points must be established under conditions which render the use of the tape difficult or impossible. Such points are usually established at the intersection of two transit lines, employing two transits in known positions. The problem is the inverse of that in which the position of a ground point is determined by the method of intersection, as described for the transit in Art. 224, p. 305 and for the plane table in Art. 414, p. 615. The degree of precision with which measurements are made is commensurate with the requirements of the survey.

For example, it is desired to locate the central point C of a bridge pier in the middle of a stream of moderate width, on a tangent line of a roadway. From a transit station A on shore, on the center line of the roadway, a measured base line AB is laid off along the shore, of such length and azimuth that favorable intersection angles of a triangle ABC will be secured. The angle ABC is computed from the known angle CAB and the sides AB and AC of known length. Transits are set up at A and B, and at B the angle ABC is set off. The point C is then established, by simultaneous sighting, at the intersection of the line of sight BC with the line of sight along the roadway, prolonged from A. The location is checked by similar sights taken either on the other side of the roadway or on the other side of the stream. To establish the corners of the pier, a similar procedure is followed but with correspondingly different values of the angles at A and B, which must be computed in each case.

For long sights or for work of high precision, various transit stations on shore are established by a system of triangulation, such that favorable intersection angles and checks will be obtained for all parts of the work.

Prior to the construction of a dam, a number of transit stations, or reference points, are permanently established upstream and downstream from the dam, at advantageous locations and elevations for sighting on the various parts of the structure as work proceeds.

These reference points are usually established by triangulation from a measured base line on one side of the valley, and all points are referred to a rectangular system of coordinates, both in plan and in elevation. To establish the horizontal position of a point on the dam, as for the purpose of setting concrete forms, simultaneous sights are taken from two transits set on known reference points, each transit being sighted in a direction previously computed from the coordinates of the reference point and of the point to be established. The elevation of the point is usually established by direct leveling. However, it may be established by setting off on one (or, as a check, both) of the transits the computed vertical angle, the height of the instrument being known; this method is the inverse of indirect leveling, described in Art. 99, p. 115.

Where the character of the topography renders chaining difficult, railroad curves are occasionally laid out by the method of intersection, with one transit at, say, the P.C. or an intermediate point, and another transit at the P.T. However, some of the angles of intersection in this case will be such that the precision is relatively low.

References

1. Adams, C. R., "Mid-winter River Surveys," *Eng. News*, Vol. 64, No. 4, p. 98, July 1910.
2. (Anonymous), "Topographic Surveys on Irrigation Work in British Columbia," *Eng. Cont.*, Vol. 43, No. 17, p. 384, April 1915.
3. Birket, W. A., "Topographic Surveys for Drainage Districts," *Proc. Ill. Soc. Eng.*, 1913.
4. Bishop, L. E., "Speed on Topographic Surveys," *Eng. Rec.*, Vol. 70, No. 12, p. 319, September 1914.
5. Brown, W. N., "Accuracy of Stadia Measurements," *Proc. Am. Soc. Civ. Eng.*, Vol. 39, p. 2267, 1913.
6. Davis, R. E., and Rayner, W. H., "Topographic Survey of a Rough, Wooded Area," *Eng. Rec.*, Vol. 73, No. 10, p. 316, Mar. 4, 1916.
7. Elicker, C. S., "Accuracy of Topographic Surveys," *Eng. News-Record*, Vol. 84, p. 702, April 8, 1920.
8. Follet, W. W., "Topographic Survey of Mexican Boundary," *Proc. Am. Soc. Civ. Eng.*, Vol. 39, p. 1821, October 1913.
9. Gale, L. E., "Topographic Survey of Truckee-Carson Project," *Eng. Cont.*, Vol. 41, No. 8, p. 254, Feb. 25, 1914.
10. Johnson, H. T., "Combined Use of Transit and Plane Table," *Eng. News*, Vol. 76, No. 11, p. 488, September 1916.
11. Jones, V. K., "Topographic Survey of a Land Grant in New Mexico," *Eng. Cont.*, Vol. 38, No. 5, p. 133, July 1912.
12. Landreth, W. B., "Stadia Topographic Surveys," *Trans. Am. Soc. Civ. Eng.*, Vol. 44, p. 92, 1900.

13. Levitt, Boris, "Topographic Surveys on the Catskill Aqueduct," *Eng. News*, Vol. 60, No. 10, p. 249, September 1908.
14. Mitchell, H. C., "Topographic Survey of Cincinnati," *Eng. News*, Vol. 69, No. 14, p. 653, April 1913.
15. Newlands, J. E., "Surveys on Canadian Pacific Railway Irrigation Projects," *Eng. Rec.*, Vol. 65, No. 2, p. 41, Jan. 13, 1912.
16. Shirely, J. W., "Topographic Survey of Baltimore," *Eng. Rec.*, Vol. 59, No. 1, p. 18, January 1909.
17. Taylor, A. D., "Cost of Topographic Surveys," *Landscape Arch.*, Vol. 13, No. 3, p. 198, April 1923.
18. Tidd, A. W., "Four Stadia Surveys for the Catskill Aqueduct," *Eng. News*, Vol. 62, No. 17, p. 428, October 1909.
19. Van Ornum, J. L., "Topographic Survey of the Mexican Boundary," *Trans. Am. Soc. Civ. Eng.*, Vol. 34, p. 259, 1895.
20. Whittier, W. E., "Topographic Survey of the Yakima Project," *Eng. Cont.*, Vol. 37, No. 24, p. 669, June 1912.

CHAPTER XXVI

HYDROGRAPHIC SURVEYING AND FLOW MEASUREMENT

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HYDROGRAPHIC SURVEYS

474. General.—Hydrographic surveys are those which are made in relation to any considerable body of water, such as a bay, harbor, lake, or river. These surveys are made for the purposes of (1) determination of channel depths for navigation, (2) determination of quantities of subaqueous excavation, (3) location of rocks, sand bars, lights, and buoys for navigation purposes, and (4) measurement of areas subject to scour or silting. In the case of rivers, surveys are made for flood control, power development, navigation, water supply, and water storage.

Since a certain amount of shore line location is included in most hydrographic surveys, a single control survey is located on shore and serves both for soundings and for shore line details.

475. Horizontal Control.—As in topographic surveying the horizontal control is a series of connected lines whose azimuths and lengths have been determined. For rough work the control may be a stadia or plane-table traverse. More accurate work requires a control run with transit and tape; and for extended surveys where great accuracy is required the control is based upon a system of triangulation executed within the required limits of error. For planning a system of traverses or triangulation points no definite rules can be given. Topography, relief, wooded areas, highways, and railroads are all features which fix the character of the control. Long, narrow rivers or inlets, with shore conditions favorable to traversing, are usually surveyed from a single traverse line on one shore. If the body of water is more than one-half mile wide it is more economical to traverse both shore lines and connect the two traverses both in azimuth and distance by frequent ties.

Where the shore lines of rivers and lakes are obscured by woods and can not be traversed economically, a system of triangulation is used.

In addition to a measured base line at the beginning and end of the survey, check base lines are measured every ten or fifteen miles as the work progresses. The control for large lakes and ocean shore lines consists of a network of connected triangles on shore. These are supplemented where necessary by traverse lines along the shore, connecting two or more triangulation stations.

476. Vertical Control.—A chain of bench marks is established to serve as a vertical control. These bench marks are near the shore line and are located at frequent intervals so that gages may be set conveniently.

477. Shore Line Details.—Most hydrographic surveys require the location of all irregularities in shore line, all prominent features of topography and culture, and all lighthouses, buoys, etc., in order that these points may be used for references in range line and sounding work. The above details are best located by stadia or plane-table methods which have been fully described in Chapters XIV and XXIII. The shore line is located by a level party in much the same manner as any contour is traced. Points are marked only at changes in direction and are subsequently located by the stadia party.

478. Establishing Datum.—On some tidewater surveys it is necessary to establish a datum from tidal observations. To obtain most accurate results the observations must extend over a period of several years. However, observations extending over one lunar month will give results satisfactory for all but the more precise surveys. The procedure is as follows:

1. The gage is set where it is protected from rough wave action and where the water level is not influenced by local conditions.
2. The gage is located in sufficient depth of water to give a definite gage reading at low tide.
3. The zero of the gage is referred to a permanent bench mark on shore.
4. The elevations of high and low water are read daily for one lunar month.
5. The mean of an equal number of high and low readings gives the approximate value of mean sea level.
6. When the gage reading for mean sea level is obtained, the proper elevation for the bench mark on shore is calculated.

479. Location of Soundings.—The determination of the relief of the bottom of a body of water is made by soundings. The depth of the sounding is referred to water level at the time it is made and is corrected to the datum determined by the gage. Before the corrected soundings can be plotted on the map their position with

reference to the shore traverse is determined by one of the following methods:

1. By taking soundings on a known range line and reading one angle either from a boat or from a fixed point on shore.
2. By rowing at a uniform rate along a known range line and taking soundings at equal intervals of time.
3. By taking soundings from a boat at the intersections of known range lines.
4. By reading two angles simultaneously from two fixed points on shore.
5. By taking readings with the transit and stadia.
6. By taking soundings at known distances along a wire stretched between stations.
7. By reading two angles from a boat to three fixed points on shore.

The U. S. Coast and Geodetic Survey has recently developed a radio acoustic method of position finding (Ref. 24, p. 753).

479a. Range Line and Angle Read from Shore.—The range may be fixed by two flags or signals on shore, or one flag on shore and a buoy set in position at some distance offshore. If buoys are used they are located from the shore survey. The positions of all range signals must be accurately known and plotted on the map before the positions of soundings can be plotted. Signals may be located either by stadia, by transit and tape, or by triangulation.

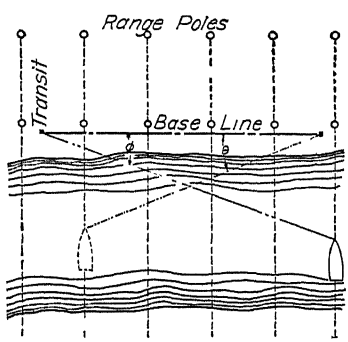

FIG. 479*a*.—Ranges and one angle read from shore.

Signals defining a range should be far enough apart to allow easy projection of the range across the water. The intersecting ray from the transit to the boat should cross the range line at an angle as near ninety degrees as practicable. Figure 479*a* shows one method of laying off the ranges adapted to a regular shore line. Irregular shore lines leave much to the ingenuity and experience of the engineer in selecting range lines to fit the particular work at hand. Bends in rivers and curved coast lines are more conveniently laid out by range lines radiating from some fixed point on shore, such as a flagstaff, church spire, or chimney. The fixed point should be sharply defined and plainly visible and also should be at a sufficient distance from the shore line to make each range approximately

parallel to neighboring ranges. This method of radiating ranges reduces by one half the number of flags needed on shore. If the ranges diverge too much before reaching the opposite shore, a few ranges crossing the radial ranges are run to fill the existing gaps.

A modification of this method is that in which an observer in the boat measures with a sextant the angle between the range line and a control station on shore. This method increases the amount of office work and generally has no advantage over reading the angle from the shore.

479b. Known Range and Time Intervals.—This method is generally used where extreme accuracy in locating the position of a sounding is not required. If the ends of the ranges are not marked with buoys whose positions have been located, the first and last sounding are located by angle readings and the intermediate readings are interpolated according to the time intervals. In still water where it is possible to row at a uniform speed, the time and space intervals will closely correspond. The boat should start at a sufficient distance back of the initial sounding to be traveling at a uniform rate when it reaches the beginning of the range. The speed of the boat is then kept uniform and the soundings are taken and are plotted under the assumption that a distance along the range is proportional to the time consumed in traveling that distance. This method is applicable only where the water is relatively still, the distance short, and the required accuracy low.

479c. Intersecting Ranges.—When the object of the survey is to determine changes in the bottom due to scour or silt, or to determine the quantity of material removed by dredging, it is necessary to repeat the soundings at the same points. Fixed ranges are located on shore so that they intersect at approximately a right angle, and are permanently marked. The boat proceeds to the intersections and takes soundings as desired. The accuracy of the method depends upon the distance between intersecting ranges and upon the precision with which the points of intersection are located as soundings are taken. The system of ranges used must be adapted by the engineer to the topography of the shore line and to the shape of the body of water. Rougher methods, such as a fixed range and time intervals, are not sufficiently precise for work of this character.

479d. Two Angles Read from Shore.—Two transits are set up at previously determined points on the shore traverse, or more frequently at instrument points selected to give good visibility and good intersections, which points are later tied to the shore traverse. The lines from the two transits to the sounding should intersect as nearly at a right angle as the location will permit. Each transit-

man orients his transit on a known azimuth line, unclamps the upper motion, and follows the sounding rod with the vertical cross-hair. When the sounding signal is given both transitmen simultaneously observe and record the horizontal angle and time. The time of sounding is also taken by the recorder in the boat.

Both transitmen should compare watches with the recorder twice daily, as sometimes this is the only means of identifying soundings when the transitman misses an angle or incorrectly numbers a sounding. The transitmen should check the orientation of their instruments at frequent intervals during the day.

This method is applicable where it is impossible to keep the boat on a fixed range, or where the shore topography is unsuitable to the laying out of a system of intersecting range lines. The method has the disadvantage of requiring two shore observers who frequently must move to new positions to secure good intersections. Work must be suspended while new positions are being occupied, and much time is lost in this way.

479e. Transit and Stadia.—The stadia method is well adapted to smooth and shallow waters where the survey is made in connection with the topographic mapping of shore lines. Using a heavy flat-bottom boat in quiet water it is possible to read the stadia interval with the foot of the rod resting on the bottom of the boat; however, if the water is but slightly rough, reading the stadia interval becomes both slow and uncertain. If the rod is long and sufficiently weighted to remain upright in the water, the same rod may be used both for sounding and for reading the stadia interval. The transit is set up near the water level to avoid reading vertical angles.

The work proceeds in much the same manner as described under stadia surveying (Art. 248*a*, p. 339). At the instant the sounding is taken the transitman reads the rod interval and turns the vertical cross-hair on the sounding pole. The azimuth is read and recorded while the boat moves to the position of the next sounding. The main advantage of the stadia method is the ease and rapidity with which the soundings can be plotted with the polar protractor. This method is not suitable where soundings are taken far from shore.

479f. Distances Along a Wire Stretched between Stations.—On narrow channels which can be navigated by a boat, and where successive soundings on the same section are desirable, a single wire or wire cable is stretched across the channel as shown in Fig. 479*b* and is marked by metal tags at appropriate known distances along the wire from a reference point or zero station on shore, such as the plumb bob and stake shown at 0 + 00. When the wire has been disturbed and it is desired to repeat the soundings, the plumb

bob is again suspended at zero on the wire and the wire is adjusted until the bob is over the zero station. This is a very accurate method but is much more expensive than locating the soundings by intersecting ranges.

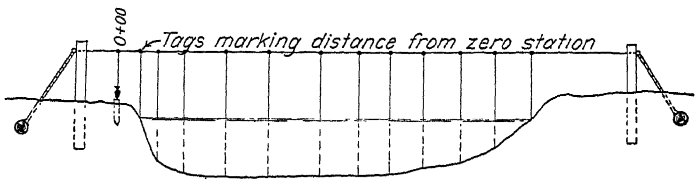

FIG. 479*b*.—Soundings from a marked wire or cable.

479g. Two Angles Read from Boat.—As each sounding is taken two angles are simultaneously observed from the boat to three fixed points on shore whose relative positions are known, as illustrated by Fig. 479*c*. This is an application of the "three-point" problem (see Arts. 417 and 564*d*). In Fig. 479*c*, θ and ϕ are the angles read by the observer in the boat to the known points A, B, and C on shore. Since a boat is too unstable to support a transit, the angles are read with the sextant. Two angles are sufficient to locate a sounding unless the boat happens to be on the circumference of a circle passing through A, B, and C, as shown in Fig. 417*c*, p. 620. The location of the sounding is then indeterminate.

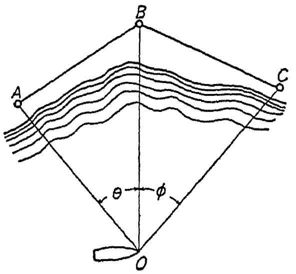

FIG. 479*c*.—Two angles read from boat.

The accuracy of the location will vary with the relative position of the known points A, B, and C, as follows:

1. If A, B, and C are in a straight line or if B is nearer the boat than A and C, the position is strong unless one of the angles θ or ϕ is small.
2. Extremely long sights will give small values for θ or ϕ and the location will be weak.
3. On long or short sights small angles should be avoided as they are difficult to plot and may give weak locations.
4. The error in the plotted position of the point, due to errors in plotting the angles, increases with the length of sight and better results can be obtained with shorter sights.
5. The accuracy of location is poor when the point occupied approaches the circle through the three fixed points.

This method may be used in combination with the time-interval method with satisfactory results. The boat is rowed at a uniform

rate and is kept approximately on a range line. About one third to one half of the soundings are located by two angles read with the sextant, depending upon the uniformity of the bottom. Soundings taken between sextant readings are plotted in proportion to the time intervals. This reduces the labor of plotting and speeds up the work of observing in the field.

480. The Sextant.—The transit and other instruments used in land surveys are not adapted for use in a boat where the support is unstable. The sextant, shown in Fig. 480*a*, is well suited to hydrographic work and has the added advantage of measuring angles in any plane. It is called a "sextant" because its limb includes but one sixth of a circle. Although the arc is limited to 60 degrees the instrument will measure angles to 120 degrees. It is the most accurate hand instrument yet devised for measuring angles. It is used principally by navigators and surveyors for measuring angles from a boat, but it is also employed on exploratory, reconnaissance, and preliminary surveys on land.

The theory of the sextant is based upon the optical principle that if a ray of light undergoes two successive reflections in the same plane by two plane mirrors, the angle between the first and last direction of the ray is twice the angle between the mirrors.

The essential features of the sextant are illustrated in Fig. 480*b*, with the instrument in position for measuring a horizontal angle *FEL*. An index mirror *I* is rigidly attached to a movable arm *ID*, which is fitted with a vernier, clamp, and tangent-screw, and which moves along the graduated arc *AB*. A second mirror *H*, called the *horizon glass*, having the lower half of the glass silvered and the upper half clear, is rigidly attached to the frame. A telescope *E*, also rigidly attached to the frame, points into the mirror *H*.

With signals at *L* and *F* and the eye at *E* it is desired to measure the angle *FEL*. The ray of light from signal *L* passes through the clear portion of glass *H* on through the telescope to the eye at *E*. The ray of light from signal *F* strikes the index mirror at *I* and is reflected to *h* and then through the telescope to *E*. Each set of rays forms its own image on its respective half of the objective. By moving the arm *ID*, these images may be made to move over one another and there will be one position in which they coincide. An observation with the sextant consists in bringing the two images into exact coincidence and reading the vernier on limb *AB*.

To prove angle *FEL* equals two times angle *IDh*, *i.e.*, the angle between the signals is equal to twice the angle between the mirrors: Draw *IP* and *hp* normal to the two mirrors, then the angles of incidence and reflection of the two mirrors are i and i' respectively. By trigonometry

$$
\begin{aligned}
FEL &= FIh - IhE \\
&= 2i - 2i' \\
&= 2(i - i')
\end{aligned}
$$

Also

$$
\begin{aligned}
IDh &= HhI - hID \\
&= (90^\circ - i') - (90^\circ - i) \\
&= i - i'
\end{aligned}
$$

Therefore FEL (the angle between the objects) equals twice the angle IDh (the angle between the mirrors).

480a. Adjustments of the Sextant.—Following are the common adjustments of the sextant:

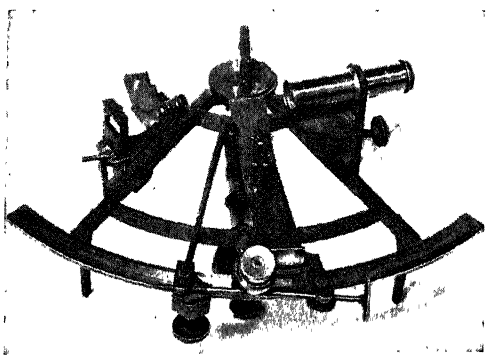

FIG. 480*a*.—The sextant.

1. *To Make the Index Mirror Perpendicular to the Plane of the Sextant.* Set the vernier arm at a reading near 30 degrees, then observe the image of the arc in the index mirror. If the index mirror is perpendicular to the plane of the sextant the image in the mirror will appear to form a continuous curve with the visible portion of the arc appearing outside the glass. If the image appears above the arc the mirror leans forward, if below it leans backward. Correct by means of adjusting screws provided for this purpose. Some instruments have no provision for this adjustment, in which case adjust by loosening the back screws and inserting thin paper shims between the mirror and the frame.

2. *To Make the Horizon Glass Perpendicular to the Plane of the Sextant.* Sight the instrument on some clearly defined horizontal line such as the roof of a building. If the reflected image of this line as seen in the silvered portion of the horizon glass does not coincide with the image as viewed through the clear portion, the horizon glass must be adjusted by tipping backward or forward as for the index mirror. This adjustment should be made after the adjustment of the index mirror.

3. *To Make the Horizon Glass Parallel to the Index Mirror When the Vernier Reads Zero.*—After the first two adjustments are made, set the vernier to read zero. Sight through the telescope and the transparent

portion of the horizon glass at a well defined distant point. If the direct and reflected images coincide the mirrors are parallel. If not, adjust the horizon glass until these images coincide.

Modern instruments are fitted with an adjusting screw at the base of the horizon glass, but some of the older instruments have no provision for this adjustment and an index error for the instrument must be determined. Sight on the distant point with the vernier clamped at zero, then bring the two images into coincidence by moving the index arm. The vernier will now read the index error which is to be applied to all observed angles.

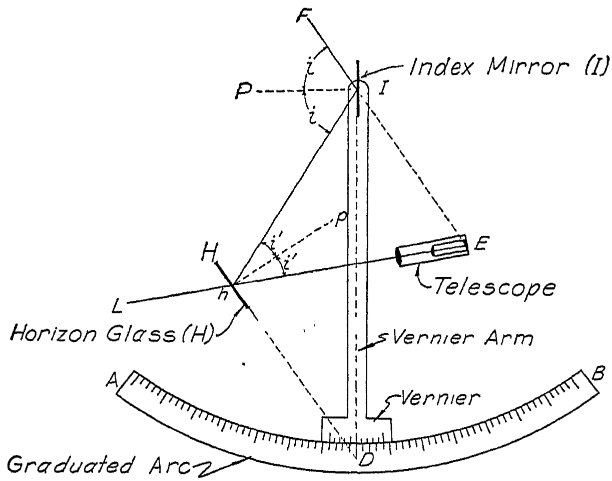

Fig. 480*b*.

4. *To Make the Line of Sight of the Telescope Parallel to the Plane of the Sextant.*—The reticule of the telescope contains four wires which form a square near the center of the telescope. Set the instrument on a solid horizontal surface (as a table) about 20 ft. from a wall, with the plane of the graduated arc in a horizontal position. Sight through the telescope and on the wall set a mark that appears to be in the center of the square. Select two small blocks of wood of equal height such that when placed near the ends of the graduated limb their tops are very nearly in the same horizontal plane as the telescopic line of sight. Sight over these blocks and on the wall set a mark in the same vertical plane with the first mark. If the two marks thus established are within one-half inch of each other the angular error is less than two seconds and can be neglected. When this error is large enough to require adjustment it is done by means of the screws on the telescope collar. Some sextants have no provision for this adjustment.

480b. Measuring Angles with the Sextant.—The handle of the sextant is held in the right hand and the plane of the arc is made

to coincide with the plane of the two objects between which the angle is to be measured. The sextant is turned in the plane of the objects until the left-hand object may be viewed through the telescope and the clear portion of the horizon glass. Holding the instrument in this position, the index arm is moved with the left hand until the images of the two objects coincide. The final setting is made with the tangent-screw, and a test for coincidence is made by twisting the sextant slightly in the hand to make the reflected image move back and forth across the position of coincidence. When the setting is thus verified, the vernier is clamped and read.

The possible accuracy of angle measurement with the sextant depends upon the size of the angle and upon the length of sight. It is evident from Fig. 480*b* that the angle *FEL* actually measured has its vertex *E* not at the eye but at the intersection of the sight rays *FE* and *LE* from the flags. The distance to this intersection will increase as the angle decreases, and for small angles the vertex may be at a considerable distance back of the observer. Hence the sextant is not an instrument of precision for small angles, say less than 15°, and short distances, say less than 1000 ft. If objects sighted are a great distance away, the angular error is usually small.

481. Equipment Used in Making Soundings.—The speed and accuracy of making soundings depends much upon the character of the equipment used. The selection and testing of the instruments used is a matter of importance.

Sounding Rods.—Up to depths of about 16 ft. and with low current velocities, rods can be used to advantage. The rod is usually made in 4-ft. sections for convenience in carrying, and must be of sufficient thickness to withstand the pressure of the current. The edges are generally rounded to give minimum resistance to the flowing water. The lower end is fitted with a metal shoe of sufficient weight to hold it upright in the water and having area enough to prevent its sinking into the mud or sand. Rods are ordinarily graduated on both sides to feet and tenths, the zero being at the bottom of the shoe. Since it is difficult to hold the rod vertical in flowing water, often a long wire anchored upstream is attached to the lower end of the rod.

Sounding Lines.—Sounding lines may be of cotton, hemp cord, sash chain, piano wire, or small linked steel chain. To one end of the sounding line is attached a lead and at intervals along the line are markers by means of which depths may be read.

Cotton or hemp lines must be stretched before using. This is done by drawing the rope tightly between two posts, wetting it,

and allowing it to dry. This operation is repeated several times. Finally the rope when wet is stretched taut and is then graduated, starting with zero at the bottom of the lead and marking each foot with a piece of cloth drawn through the strands of the rope. The 5 and 10-ft. intervals are best marked with leather tags similar in shape to the notched brass tags used on the surveyor's chain. The rope should be kept dry when not in use, and should be soaked in water for at least an hour before being used. This will allow the rope to assume its tested length.

Although brass sash chains and small linked steel chains do not stretch, the wear on the link surfaces is appreciable and it is necessary frequently to compare them with a steel tape and to reset the 5 and 10-ft. markings.

Another sounding line much used in government work is made of braided cotton with a phosphor-bronze stranded wire core. It is reliable, does not stretch appreciably, and does not require initial stretching as do cotton and hemp lines.

Sounding Leads.—The weights used with sounding lines vary from 3 to 25 lb., depending upon the depth of water and the velocity of current. For streams of moderate depth a 10-lb. weight is usually heavy enough.

Leads are usually made similar in shape to a window weight with a slight taper toward the top or "eye" end. They are circular in cross-section and 3 to 4 times as long as their average diameter.

Signals and Ranges.—Signal masts are usually made of 4 by 4-in. timber painted white. They are firmly braced in an upright position and are fitted with flags of distinctive marking. Range poles may be either of 1 by 2-in. cross-section or may be round poles, fitted with an iron shoe. Marking and identification depend upon the particular work. If only a few ranges are used, colored flags may serve; otherwise the ranges should be marked by Roman numerals reading up or down the pole.

Where points marking the range are in shallow water, the range markers are usually 1 by 2-in. wooden poles, weighted at the bottom and held in position vertically by means of guy wires.

In deep water the range points are marked by buoys. Wooden buoys are made about 10 in. in diameter, tapered slightly toward the bottom. The length should be 2 to 3 ft. in tideless water and longer where tides are encountered. A hole is bored through the vertical axis of the buoy to accommodate a flag pole. To the lower end is fixed the anchor or weight line to hold the buoy in correct position. Where no tide exists the buoy may be guyed in position, otherwise due allowance must be made for tides, wind, and current.

482. Making the Soundings.—If the depth is not more than 75 ft. the sounding is made without stopping the boat. The leadsman casts the lead forward far enough to allow it to reach bottom as the line comes into a vertical position. Where the current is so swift as to make this method impracticable a line of soundings is taken by allowing the boat to drift with the current, the leadsman lifting the lead between soundings only enough to clear obstructions. Then the boat is rowed upstream and a second line of soundings is taken paralleling the first, and so on. Soundings in deep, still water are taken by stopping the boat for each sounding.

The field record should show the locality, the date, the names of observers, the designation of range or line, the serial number of sounding in range, the time, the two angles if sextants are used, the points sighted on shore, the depth of each sounding, the gage reading for water level, and the error or correction of the sounding line. If the angles are read from shore, each transitman records the azimuth of the sounding, the time, the range and serial designation, and any other information which might be useful in identifying his notes with those of the other observers.

Tide gage readings are taken from the gage reader's records and entered in the field notes at the end of each day's work. The tide gage should be located as near the soundings as convenient and must be in the same tidal basin.

Soundings are often taken to advantage when a lake or river is frozen over. Holes are bored in the ice with an ice auger. A marked sounding line to which is suspended a long narrow weight is lowered through the holes and depths are recorded. If the weather is not too severe, the soundings are best located by the transit-stadia method. In very severe weather the soundings are best located by intersecting ranges.

483. Reducing Soundings to Datum.—Before soundings can be plotted they must be reduced to datum by subtracting (algebraically) from the sounding the corresponding gage correction. If it is necessary to make corrections for wind, current, or erroneous length of sounding line they should be made at this time.

484. Plotting the Soundings.—With any combination of ranges and angles read from the shore the plotting is relatively simple. Stations marked by range poles and buoys as well as those occupied by the transit are tied to the shore traverse and their positions are plotted on the map. This forms the control from which soundings are plotted. The lines connecting the range markers are drawn and the intersecting transit lines are plotted with a polar protractor.

Many ingenious methods, of which (1) and (2) below are examples, have been used in plotting soundings located by two angles read from the opposite ends of a base line on shore.

1. *Two Polar Protractors.*—Two polar protractors, 6 to 10 in. in diameter, are oriented over the instrument stations in position to plot true azimuths. One end of a silk thread is glued to the center of each protractor circle. Two operators are used. Each operator draws the thread taut over the azimuth reading on his protractor representing the transit reading for that sounding. The two threads intersect at the plotted position of the sounding.

Another variation of the two-protractor method is to have the traverse plotted on transparent paper or cloth. One paper protractor has its radial lines marked in black and the other in colored ink, preferably red. The two protractors are placed under the tracing cloth and oriented in azimuth with their centers directly under the instrument stations. The protractor graduated in black may be white paper or cloth; it is placed under the protractor graduated in red which must be made of transparent material, preferably thin celluloid. The intersection of the black and red lines will locate the sounding when the two protractors are set at the azimuth readings of the two instruments.

2. *Two Tangent Protractors.*—If the angles are read directly by setting the transit circle at zero when sighted along the base line the soundings may be plotted by the use of two tangent protractors. The natural tangent of each observed angle is recorded in the field notes opposite the observed angle. Two tangent protractors are placed over the plotted position of the transit points with their zeros along the base line. When the movable arms of the protractors are set at the tangents of the angles read from the respective stations their intersection will be the plotted position of the sounding.

3. *Tracing-cloth Method.*—When the sounding is to be located by sextant angles read from the boat the tracing-cloth method is one of the best. Lay off the angles θ and ϕ of Fig. 479c, extending the rays OA, OB, and OC an indefinite length. Place the tracing cloth over the drawing and shift until rays OA, OB, and OC pass through the plotted positions of A, B, and C. The point O will then be directly over its position for plotting.

4. *Three-armed Protractor.*—Another method involves the use of the three-armed protractor. In this instrument the circle is graduated in both directions from 0 to 360 degrees. The middle or fixed arm is fastened permanently in position with its ruling edge fixed at zero on the protractor circle. The two movable arms are fitted with verniers, clamps, and tangent-screws and are placed one on each side of the fixed arm. The ruling edge passes through the setting of the vernier and the center of the protractor circle. The angles θ and ϕ are set on the movable arms and the ruling edges of the three arms are adjusted to pass through points A, B, and C. The center of the protractor is now over the correct position of the point, which may be marked through a small hole in the protractor plate.

If the position of the sounding is on or near the circumference of a circle passing through *A*, *B*, and *C*, neither of the last two methods apply (Art. 417*a*, p. 619).

485. Hydrographic Maps.—A hydrographic map is similar to the ordinary topographic map but has its own particular symbols. These may be found in almost any book on topographic drawing or in the manual issued by the U. S. Coast and Geodetic Survey (Ref. 23, p. 753). See also Art. 269, p. 384, and Arts. 437–437*d*, p. 650. The amount and kind of information shown in a hydrographic map varies with the use of the map. A harbor map should show enough shore-line topography to locate and plan wharves, docks, warehouses, roads, and streets along the water front. A navigation chart should show only shore details which are useful aids to navigation, such as church spires, smokestacks, towers, and similar landmarks. Maps of rivers should show both low- and high-water marks and all topography within the zone between these marks. A hydrographic map should contain the following information:

1. Datum used for elevations.
2. High- and low-water lines.
3. Soundings in feet and tenths, the decimal point occupying the exact plotted location of the point.
4. Lines of equal depth interpolated from soundings. On navigation charts the interval for lines of equal depth is one fathom or 6 feet. These are shown by dot-and-dash lines, the number of dots between dashes representing the number of fathoms of depth. For dredging rivers or harbors the interval is 1, 2, or 3 feet.
5. Conventional signs for land features as on topographic maps.
6. Lighthouses, navigation lights, buoys, etc., shown either by conventional signs or lettered on the map.

SPECIAL HYDROGRAPHIC SURVEYS

486. Sweep or Wire Drag.—In harbors and inlets where the mean depth is only slightly greater than navigation requirements, or where coral reefs and pinnacle rocks are likely to occur, there is no certainty that any system of sounding previously described will develop all of the small areas dangerous to navigation. This defect has led to the introduction of the *sweep* or *wire drag*, which came into general use about the year 1900. Beginning with sweeps 200 to 1000 ft. long, this method and its application have grown to the use of sweeps 10,000 to 15,000 ft. long with which it is possible to cover several square miles of area in a working day. The present wire drag used by the U. S. Coast and Geodetic Survey consists of a wire of suitable length which may be set at any required depth.

The wire is supported at this depth by means of buoys placed at intervals and connected to the drag wire by vertical wires of adjustable length. A sinker is attached to the lower end of each of these vertical wires to maintain the drag wire at even depth. Wooden subsurface floats are attached to the drag wire at 100-ft. intervals, to prevent it from sagging between buoys. The drag is pulled through the water by two power launches steering slightly divergent courses to keep the drag taut. The resistance of the water causes the drag wire and the buoys to assume a parabolic curve while the wire meets with no obstruction. When an obstruction is met the buoys assume the position of two straight lines intersecting over the obstruction. This spot when found is located by sextant observations to reference points on shore, and soundings are taken for the minimum depth. The point may then be plotted, and the dragging is resumed. This method makes it possible to mark and chart coast-line navigation lanes far in advance of more detailed surveys.

487. Determination of Stream Slope.—In all natural channels the slope for limited portions of the stream is a variable quantity. It not only varies at different stages but also varies for the same stage at different times if local channel conditions have been changed. Because of these variations great care must be taken in the determination of stream slope and in its use in discharge formulas. For example, a slope determination for low stages would give results grossly in error for flood stages. For reliable results the slope determination should be made at or near the stage for which the discharge is desired.

To determine surface slope, say for a section 1000 ft. long, a gage of the stilling-box type is installed on each side of the stream at each end of the section. The zeros of the gages are connected to permanent bench marks on shore. The gages are then read simultaneously every 10 to 15 minutes for 6 to 8 hours. The mean of these readings at each end of the section determines the water elevation at that point in the stream. The difference in elevation between the ends of the section divided by the distance is the slope, usually expressed as a fraction. A slope of 2 ft. per mile would be expressed as $^{2}/_{5280}$.

The surface slope indicates the true slope only when the velocity of flow is uniform, a rare condition even in artificial channels. For accurate measurement the *energy slope*, which takes into account the velocity head at both ends of the reach, should be used. The energy slope is the rate of fall of the line joining the points at an elevation above the water surface equal to the velocity head. The

area of the water prism at each end of the reach is developed, from which the velocities and the corresponding velocity heads are calculated and applied to the surface slope to determine the energy slope.

Stream slopes for ordinary surveys are often taken by holding the stadia rod at water level for various points as the survey is in progress. Such a determination is useful for mapping purposes but is unreliable for purposes of computing discharge.

488. Measurement of Surface Currents.—In harbors and inlets it is often desirable to know the direction and velocity of currents at all tidal stages. This is done by locating the path and computing the velocity of floats. The float is designed to give minimum wind resistance and to extend under water a sufficient depth to measure the current in question (see Art. 498). A length of 2 ft. is ordinarily used for surface currents while 20 ft. is about the maximum for deep currents. For greater depths a current meter is used. The float may be made of 2 by 4-in. wood, weighted at the lower end. The top is approximately flush with the water surface and to it is fitted a small red flag. As soon as the float attains the velocity of the current it is located at regular time intervals with reference to fixed points on shore. This may be done by two angles read either from the shore or from a boat following the float, as previously described. From the time and distance measurements the velocity is calculated.

Another method of direct measurement is used by the U. S. Coast and Geodetic Survey. A boat is anchored and its position is determined by sextant angles read to three known points on shore. A line marked every 10.13 ft. to represent tenths of knots in current velocity is attached to the float. Time is taken at 60-second intervals, using a stop watch. The float is set adrift and the number of tag intervals of 10.13 ft. run out in 60 seconds is recorded. One tenth of this number is the current velocity in knots, or nautical miles per hour. The nautical mile equals 6080.2 feet.

The direction of the current may be determined by sextant angles taken from the boat between known shore signals and the float. This information combined with known distance from the anchored boat to the float will definitely fix its location.

A rougher method sometimes used is to time the passage of a free float from one fixed range to another. The point at which the float crosses the range may be located by one angle read from the shore and the distance may be scaled from the plotted position of the points.

489. Measurement of Dredged Material.—Subaqueous surveys to be followed by dredging should be made by one of the methods

by which soundings may be repeated at exactly the same location, *i.e.*, intersecting ranges, or distances along a wire stretched on a fixed cross-section. Dredged material may be measured either in place or in scows. In the first method, soundings on a fixed section are taken both before and after dredging and the change in cross-sectional area is determined by calculation or by use of the planimeter. When this quantity has been determined for each section, the volume of excavation may be computed by the average-end-area method (see Art. 161, p. 220).

If the contract calls for payment by scow measurement, each scow is numbered and the capacity of each pocket of the scow is carefully determined by mensuration. When the scow is filled to capacity the inspector records a full measurement. If a pocket is not filled to capacity the inspector measures the outage and deducts it from a scow measurement. When deck scows are used the material is deposited on the deck in such shape that it can be easily measured and its volume calculated.

Excavated material in scows is sometimes measured by the amount of water displaced in loading. It is necessary to know the dimensions of the scow, the weight per unit volume of the water in which it floats, and the weight per unit volume of the dredged material. The length of the water line and the depth of immersion must be measured both empty and loaded. When these have been determined the yardage may be calculated for a rectangular scow by the formula

$$C = \frac{\frac{l + l'}{2}(d - d')bn}{W} \qquad (1)$$

in which

C = load in cubic yards
l' = length of longitudinal water line (empty) in feet
l = length of longitudinal water line (loaded) in feet
b = width of scow in feet
d' = depth of immersion (empty) in feet
d = depth of immersion (loaded) in feet
n = weight of one cubic foot of water
W = weight of one cubic yard of dredged material.

For fresh water n is taken as 62.4 lb. per cu. ft.
For salt water n is taken as 64.0 lb. per cu. ft.

490. Capacity of Existing Lakes or Reservoirs.—Two general methods are used in determining the capacity of existing lakes or reservoirs, the *contour* method and the *cross-section* method.

1. *Contour Method.*—The contour method gives more reliable results. A shore traverse is run from which the water line and the desired shore topography are located by stadia. A sufficient number of soundings is then taken by methods suited to the particular conditions surrounding the survey. From the sounding elevations covering the immersed area, the subaqueous contours are plotted. The area enclosed by the water line and by each contour is determined by planimeter. The average of the enclosed areas at two consecutive contours multiplied by the contour interval or vertical distance between them gives the volume of water lying between the two contours. The volume between the bottom contour and the deepest part is generally small and is estimated or neglected. A summation of these partial volumes gives the capacity of the lake or reservoir. This volume is usually expressed in acre-feet, one acre-foot being 43,560 cubic feet.

2. *Cross-section Method.*—When only a fair degree of accuracy is required the cross-section method is used. The outline of the water surface is found as in the contour method. The water outline is then plotted and divided into approximate trapezoids and triangles. The boundary lines between trapezoids or between trapezoids and triangles are on the sections which it is desired to measure. Soundings are taken on these sections by any suitable method of location. The perpendicular distances between sections and the altitudes of all triangles are determined by field measurement. The sections are plotted on cross-section paper and the end areas are determined by planimeter. The approximate volume is figured by average end areas.

491. Snow Surveys.—In areas which for their water supply depend to a considerable degree upon melted snow from mountainous regions, the determination of the amount and distribution of the snowfall is of aid in forecasting the run-off of streams. This information permits proper regulation and distribution of water by irrigation and storage districts, public utilities, municipal districts, etc. The assumption is usually made that the spring and summer run-off will be approximately proportional to the winter's accumulation of snow, but more refined forecasts are made possible by comparing the data for a given year with the cumulative data of previous years, which are taken as a normal (Ref. 1*a*, p. 753). Snow surveys are made annually in most of the Rocky Mountain and Pacific Coast states.

Snow courses are established at key locations which are considered representative of the entire area. They are preferably located in the early winter when some snow has fallen. A typical course

forming part of the survey for a given watershed is perhaps 1000 feet in length, with provision for measurement of the depth of snow at 50-foot intervals. It is not necessarily straight throughout the length. The location is marked by poles or by boards nailed to trees, and a detailed record of the location and markings is kept.

The depth and density of the snow are determined by means of a special sampling tube of metal, 1½ to 3 inches in diameter, having at the lower end a toothed cutter for drilling through hard crusts or ice layers. The cutter and contents are weighed by means of a spring scale which is calibrated to read directly in inches of water.

FLOW MEASUREMENT

492. General.—Discharge measurements of a stream are usually made in connection with problems of water supply, power development, and flood flow. The determination of flow in canals is an important part of irrigation work.

The discharge of a natural stream is a function of the rainfall upon its drainage area and the characteristics of that area, and may vary from zero flow to violent and destructive floods. To procure an accurate knowledge of stream flow requires regular observations extending over a period of years. Much of this work has been done by the Water Resources Branch of the U. S. Geological Survey, cooperating with the individual states in a comprehensive study of our inland waters. The information thus collected is available to the public in the Water Supply Papers of the U. S. Geological Survey, Washington, D. C.

Discharge measurements are made

1. To determine a particular flow without regard to stage of stream.
2. To determine flows for several definite gage readings throughout the range of stage, in order to plot a rating curve for the station. From this curve the discharge for any subsequent period is computed from the curve of water stage developed in the recording gage.
3. To obtain a formula or coefficient for dams, weirs, or rating flumes.

The three classes of studies are made with the same instruments and by the same methods except that observations extending over a long period of time require the installation of some form of permanent gage.

492a. Discharge and Volume Units.—*Discharge* is the rate at which the water in a stream flows past a given section. The units of discharge commonly employed are as follows:

Second-foot.—A rate of flow which produces 1 cu. ft. of water per second. It may be represented by 1 cu. ft. of water flowing with a velocity of 1 ft. per sec.

Second-feet per Square Mile.—The ratio of the discharge in second-feet at a particular section to the area in square miles of the drainage area above that section.

Gallons per Minute or *Gallons per Day.*—The common units for expressing flow or pumping duty for domestic water supply. For municipal supplies the unit is usually expressed in millions of gallons per day.

Miner's Inch.—Formerly a common unit in mining and irrigation work. It is the quantity of water that will flow through an orifice 1 in. square under a head of 4 to 12 in , the head varying in the several Western states. Aside from this variation in the legal value for head, the unit is based upon a false assumption that for a given head the discharge will be proportional to the area of the opening. For these reasons this unit of measurement should be discarded in favor of the more accurate and clearly defined units given above.

The units of volume commonly employed are as follows:

Acre-foot.—The quantity of water required to cover an acre one foot deep, equal to 43,560 cubic feet.

Run-off in Inches.—For any drainage area, the depth in inches to which the area would be covered if all the water flowing from it in a given time were uniformly distributed over the area.

493. Factors Controlling Discharge.—The determination of the amount of water flowing past a given section in a given time is called a discharge measurement. The discharge unit is usually the second-foot, and the discharge rate is the product of two factors, the cross-sectional area and the mean forward velocity of the water in the section where the area is measured.

The area of a given cross-section can be determined accurately by methods described earlier in this chapter. Sufficient depth measurements should be taken to make the portion of stream-bed profile between any two measurements practically a straight line.

Mean velocity is difficult to determine accurately because it is a function of slope, shape and regularity of stream bed, straightness of channel, and many other factors which tend to cause cross and eddy currents in the water. As a rule, the more single measurements taken, the better the determination of mean velocity.

Another factor affecting the value and accuracy of stream measurement is the choice of a gaging station. In locating a permanent gaging station the engineer should select a site for, and secure as many as possible of, the following conditions:

1. The general course of the stream for several hundred feet above and for some distance below the section should be straight.

2. The section should have a definite stream *control* or permanent obstruction to insure that the relation between gage reading and flow remains constant, and to maintain a pool of water under the gage at low stages.

3. The velocity of the anticipated minimum flow should be great enough to be recorded accurately by the type of meter it is intended to use.

4. The stream bed should be smooth and permanent, but preferably not stony.

5. Banks should be permanent, high enough to contain floods, and clear of brush.

6. The station should be so located that bridges, dams, or other works will not affect the reliability of gage readings.

7. The section should not be near the junction with another stream.

8. Conditions should be such that a permanent gage may easily be installed.

494. Selecting the Control for Gaging.—One of the most important conditions of a permanent gaging station is the control for gaging. There are two controls, one for low water and one for high water.

The low-water control is usually an outcropping rock ledge, a bar of loose rock, or a gravel bar extending entirely across the stream. The purpose of the control is to produce, for any given discharge, gage readings which are always the same. There is thus maintained a pool of water under the gage at all stages, and the gage reading for zero discharge is always the same. Where a natural low-water control does not exist, an artificial control is constructed of concrete or masonry.

The high-water control is usually the stretch of channel below the gage. At flood stages the water rises high enough to overtop small obstructions and the channel serves the same purpose as the ledge or bar.

A tight dam gives the best control for all stages, and gaging stations are frequently located at such points. After a number of high and low readings have been taken, the dam may be rated as a weir (Art. 516).

The gage height at which the last trickle of water is flowing over the control, known as the *height of zero flow*, is found by wading over the control and taking a level reading on the lowest pass through it. Knowing the gage height of water level and the lowest elevation on the control, the gage reading for zero flow is calculated.

495. Water-stage Registers.—A water-stage register is a gage which will indicate the elevation of the water surface with respect

to a known datum. The zero of the gage should be so set that the reading will never be negative. There are three general types of gages; *staff, chain,* and *automatic* or recording gages.

495a. Staff Gages.—Direct staff gages may be either vertical or inclined, and are made of a wooden post or board fastened solidly in position. They are graduated to read in feet and tenths. Either the graduations are painted directly on the wood, or metal strips previously graduated are fastened to the face of the post. Staff gages used in lake or harbor work are generally enclosed in a stilling box (Fig. 495*a*). The stilling box should be approximately 4 by

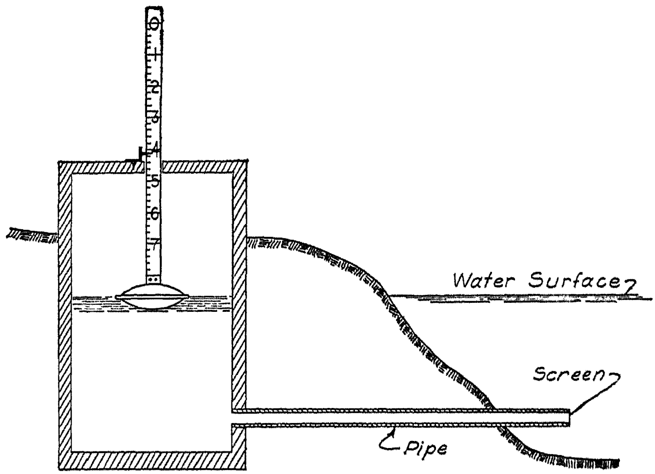

FIG. 495*a*.—Floating staff gage with stilling box.

4 ft. in plan and of sufficient depth to be operative at all stages. The water enters the stilling box through a pipe (usually 4 or 6 in. in diameter) fitted with a screen at the outer end.

It is often desirable to have the zero of the gage placed at some convenient distance above the water. This is accomplished by fitting a float to the bottom of a graduated staff which is free to move up or down as the water rises or falls in the well or stilling box. The rod is so graduated that an index at the top of the box will read zero at minimum flow and will increase as the water rises in the well. This type of gage, known as the *indirect* staff gage, is often used in power houses or similar locations where reading an exposed staff gage is impracticable. Figure 495*a* illustrates its use.

495b. Chain Gage.—The chain gage consists of a steel or brass chain with a weight (usually 12 lb.) suspended at one end (see Fig. 495*b*). The chain is marked with rivets at intervals (usually 5 ft.)

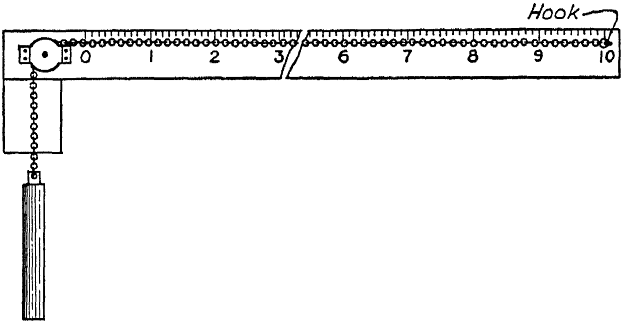

FIG. 495*b*.—Chain gage.

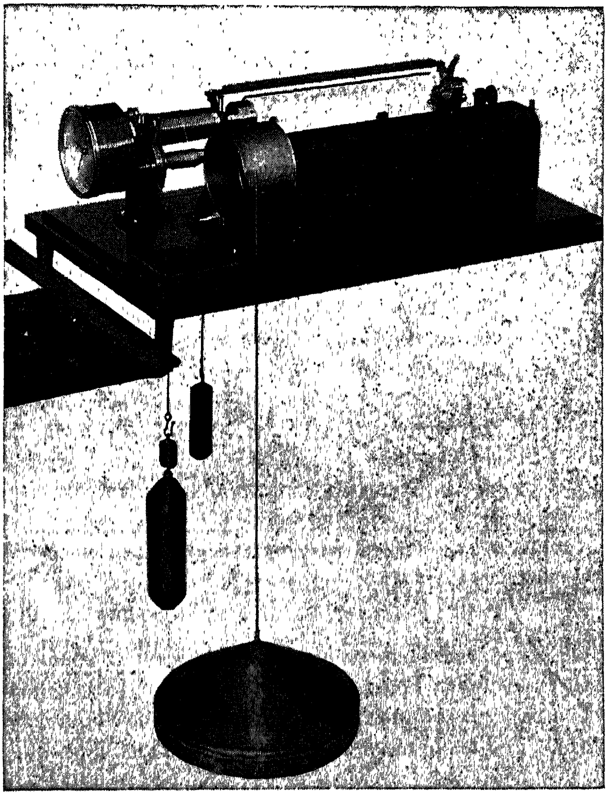

FIG. 495*c*.—Recording gage.

and runs over a pulley at one end of the gage box in which a horizontal board forms a scale graduated in feet and tenths. The weight is let down until it just touches the water surface, and the end of the chain is read on the scale. A chain gage is usually mounted on a bridge, but may be set up on the stream bank with the pulley end over the water.

495c. Recording Tide and River Gages.—A recording gage is placed in a gage house where it is protected from the elements, insects, and the public. Connection with the water is established by a pipe into the gage pit. A copper float attached to the gage

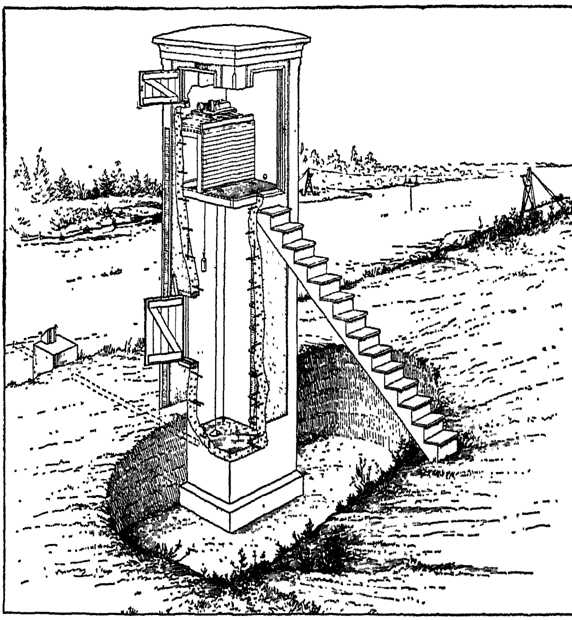

FIG. 495*d*.—Typical current-meter gaging station.

rests on the water and moves up or down as the water level rises or falls.

With one type the float record is marked on a specially ruled coordinate paper by a pencil suitably connected to the float (Fig. 495*c*). The paper is mounted on a revolving drum which is actuated by clockwork, and is changed weekly when the clock is wound. The recorded graph shows the relation between water-level elevation and time. Some recording gages print the gage reading directly on a sheet of paper at regular intervals of time by an intermittent printing device. Their main features of operation are essentially

the same. Fig. 495*c* shows a type of recording gage which is in common use. Fig. 495*d* shows the assembly and method of installation.

495d. Hook Gage.—The hook gage shown in Fig. 495*e* is used for precise measurement of the head of water flowing over a weir (Art. 510), generally for refined work of short duration. The gage is installed in a stilling box at any convenient point near the weir, the water being conveyed to the box by a pipe. The water in the box being at rest, its surface indicates the precise water level above the weir.

To measure the depth of water flowing over the weir, the level of the crest is determined with a leveling instrument and this elevation is transferred to a mark on the inside of the gage box. The gage scale is set to read zero, and the gage is fastened to the side of the box by two screws through the slots shown in the illustration, so that the point of the hook is at the same elevation as the mark. The point of the hook will now be under water and level with the crest of the weir. The depth of water flowing over the weir is the distance from the point of the hook in this position to the exact surface of the water.

To read the gage, the hook is raised until it pierces the surface; it is clamped in position, and by means of the slow-motion screw its height is adjusted until the water surface shows no distortion. This position, which gives the exact elevation of the water surface, is then read on the vernier to thousands of feet. An advantage of this gage is that after the zero positions have been found it can be carried from one weir to another, and thus duplication of installations may be avoided.

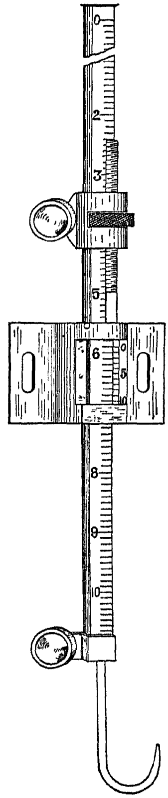

FIG. 495*e*. Standard hook gage.

496. Measuring the Cross-section.—The cross-section of the stream is preferably measured at low water. Starting above high-water level a profile of both banks and of the water section as far as wading is possible, is secured by leveling. The remaining submerged section is taken by soundings referred to water level. The distance between soundings depends upon the width of stream, the shape of stream bed, and the accuracy desired. As a general rule, no less than 15 soundings should be taken, with a minimum distance between soundings of 1 foot. Enough care must be taken to assure the observer that the profile between soundings is approximately a straight line.

The precision with which measurements should be made varies with the depth, the material of the stream bed, and the method used to determine current velocity. Soundings are usually observed to the nearest tenth of a foot. It is obvious that a given error of measurement is proportionally much larger in a 2- than in a 20-ft. sounding. Since the percentage of error is likely to be greater in shallow streams, the observer in fixing the closest reading unit should keep in mind the error in cross-sectional area rather than the single error in depth.

497. Instruments for Measuring Current Velocity.—Current velocities are commonly determined either by the use of floats or by the use of a current meter, of which there are several types. These instruments and methods will now be described in detail.

498. Floats.—The three common types of floats used in measuring stream velocity are *surface*, *subsurface*, and *rod* floats.

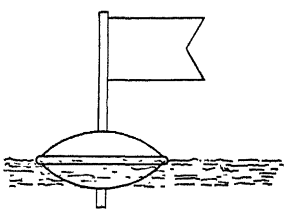

Fig. 498*a*.—Surface float.

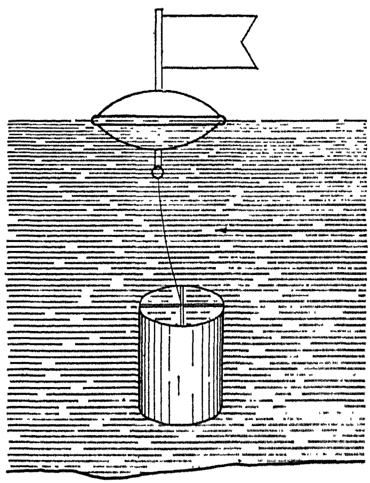

Fig. 498*b*.—Subsurface float.

1. *Surface Float.*—The surface float is designed to measure surface velocities and should be made light in weight and of such a shape as to offer the least resistance to floating debris, ripples, eddy currents, wind, and other extraneous forces. Fig. 498*a* illustrates a type of surface float which is easily made in any sheet metal shop and which gives reliable results. Improvised floats of jugs, bottles, rounded blocks of wood, etc. are often used where nothing better is available.

The use of surface floats is the quickest and most economical method of measuring stream velocity, but due to the effects of wind

and cross currents the results may be considerably in error. Since the surface float measures the velocity of water filaments close to the surface, the observed velocity must be multiplied by a coefficient to reduce it to the mean velocity for any particular stream. Unless this coefficient is carefully determined by current meter, accurate results can not be obtained. Since the value of the coefficient varies greatly, the use of a general coefficient is likely to lead to large errors. The surface float is therefore used principally in reconnaissance work, in locating gaging stations, or in measuring flood velocities.

2. *Subsurface Float.*—This is sometimes called the *double float* (see Fig. 498*b*). It consists of a small surface float from which is suspended a second float slightly heavier than water. The connecting cord is light, strong, and adjustable to any desired depth. The submerged float is a hollow cylinder, thus offering the same lateral resistance in all directions and the minimum vertical resistance to rising currents. It should have stability of flotation in an upright position and should be weighted just enough to keep the cord taut and to resist upward eddies.

This float gives more reliable results than the surface float because it is less affected by wind and eddy currents. It has the same disadvantage as the surface float in that it measures the velocity of a definite filament of water only, but it can be set at a depth near which the average or mean velocity is likely to occur. Wind resistance, cross currents, and the drag of the cord through the water affect the accuracy of results, but to a less degree than in measurements made by surface floats. The two main objections to subsurface floats are the uncertainty as to whether the cord is vertical and the modifying effect of the surface float.

3. *Rod Float.*—The rod float is usually a cylindrical tube of tin, copper, or brass, one or two inches in diameter. The tube is sealed at the bottom and weighted with shot until it will float in an upright position with 2 to 6 in. projecting above the surface of the water. A short section of bamboo fishing rod weighted with mercury is also used; this can be made to float with but ½ in. showing above water. Its glazed surface prevents absorption of water. A wooden rod is sometimes used but is inconvenient to adjust with the proper weight.

The length of the rod should be adjusted to just clear obstructions in the stream bed. This length is usually slightly greater than 0.9 of the depth. The rod integrates the velocity in a vertical plane, and were it possible to extend the rod to the full depth of the stream a very close value for the mean velocity in the vertical plane would

be obtained. The velocity at the bottom of the stream is considerably less than the mean velocity. Since the lower 0.1 of the current does not act on the rod and its retarding effect is lost, velocities secured by rod floats are slightly greater than the actual mean velocities. The percentage of error varies with the depth of immersion and with the shape of the channel. (For more detailed information see Refs. 15 and 16, p. 753.)

499. Method of Making Float Measurements.—Figure 499 illustrates the usual method of measuring stream velocities by means of floats. Two parallel sections *AB* and *CD* are established 200 to

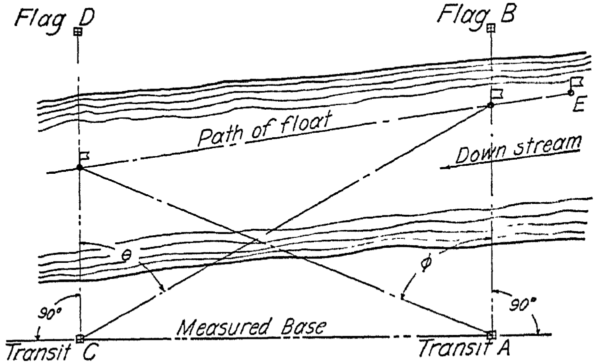

FIG. 499.—Measuring velocity by means of floats.

300 ft. apart and the base line *AC* is measured. The party consists of two transitmen, a timekeeper, and two men to release and recover the floats. A float is released at *E*, 50 to 100 ft. above section *AB*. The transitman at *A*, with vernier set at zero, sights at flag *B*. The transitman at *C*, with vernier clamped at zero, follows float *E*. As the float approaches section *AB* the timekeeper calls "get ready" and the transitman at *C* keeps the vertical cross-hair on the float by means of the lower tangent-screw until the transitman at *A* calls "tick" as the float passes section *AB*.

The transitman at *C* now clamps the lower plate, turns the line of sight to flag *D*, and reads the angle θ. The transitman at *A* now follows the float until the timekeeper again gives the "get ready" signal, and by means of the upper tangent-screw keeps the vertical cross-hair on the float until the transitman at *C* calls "tick." The angle ϕ is read and the time of float between sections is recorded.

The sections, base line, and angles are now plotted and the path of the float is either scaled or calculated. The distance divided by the time gives the mean velocity of the float.

In some cases, the passage of the float is timed over a measured reach and the distance of the float from the shore is measured at the mid-point of the reach; this value is taken as the average.

500. Current Meters; General.—Stream velocity may be measured indirectly by means of a current meter. The essential parts of a current meter are:

1. A wheel fitted with cups or vanes so that the impact of the flowing water causes the wheel to revolve.
2. A counting device to indicate or record the number of revolutions of the wheel.

There are two general classes of revolving current meters. The first class, represented by the Price meter (see Figs. 500*a* and 505*a*), has the axis of rotation normal to the direction of stream flow, and the wheel is fitted with conical cup-shaped vanes. The rotation is due to the difference in pressure on the opposite sides of the cups. The second class has the axis of rotation parallel to the direction of stream flow, and a wheel with spiral or helicoidal-shaped vanes, and the rotation of the wheel is due to the direct impulse and pressure of the water upon the vanes. Two examples of this type, the Haskell and the Fteley meters, are shown in Figs. 500*b* and 500*c*.

A recording or indicating device is necessary for determining the number of revolutions of the wheel. Various devices operated either on the mechanical, electrical, or acoustical principle are used for this purpose. The telephone receiver and the acoustical indicator are the most satisfactory in general practice because they enable the operator to detect any irregularities caused by trouble with the meter or the electrical circuit. A stop watch is necessary for the proper timing of the observations.

500a. Price Meter.—This meter, developed largely by the U. S. Geological Survey, is used in the major part of the work of the Survey. The meter consists of a wheel made with six conical cups fastened to a vertical shaft as shown in Figs. 500*a* and 505*a*. The upper end of the cup shaft is fitted with either a worm gear or an eccentric that passes into the cylindrical contact chamber. This chamber contains a mechanism for making mechanical or, more commonly, electrical contact which indicates by a click either each revolution or each fifth revolution. The mechanism for indicating each fifth revolution, called the *penta-count*, is used for velocities above about 6 feet per second, since for higher velocities the ear is unable to distinguish the separate clicks for each single revolution.

To the yoke which holds the cup wheel in place is attached a vane or tail to hold the meter heading into the current. A vertical stem to support the weight and to supply a connection to the cable

by which the meter is suspended is also attached to the yoke (Fig. 500*a*). This type of meter is also equipped for use with a graduated wading rod which is held in the hands of the observer, in which case the weight is not used (Fig. 505*a*).

500b. Ellis Meter.—In its principle of construction the Ellis meter is similar to the Price meter. It has a cupped wheel mounted on a vertical shaft with an acoustical chamber at the top of the vertical shaft. The wheel is surrounded by a shield or cage to prevent weeds or debris from damaging the cups. The meter is supported by a rod and swings in a gimbal mount. The weight is fastened to the lower end of the rod. The tail is composed of four vanes fastened to the end of the frame opposite the wheel.

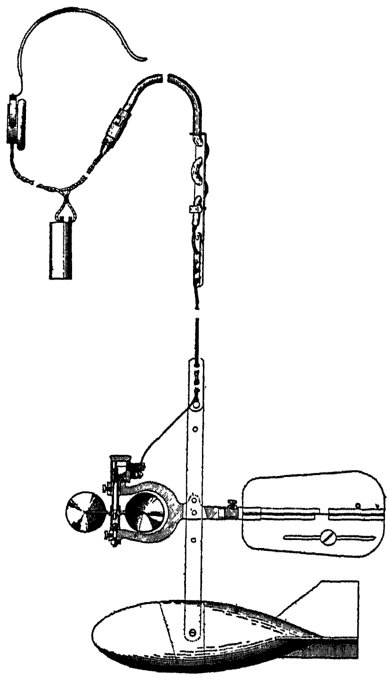

Fig. 500*a*.—Price meter mounted with cable and weight.

500c. Haskell Meter.—The Haskell meter (Fig. 500*b*) has its axis of revolution horizontal (parallel to the direction of stream flow) and is fitted with a conical-shaped screw propeller wheel designed to operate by direct pressure of the current. The meter is supported by a cable and is mounted on gimbals. It is fitted with a recording device and four exceptionally large vanes. It has been used extensively in the gaging of large, deep rivers.

500d. Fteley Meter.—The Fteley meter (Fig. 500*c*) consists of a wheel having a number of helicoidal-shaped blades mounted on a horizontal axis (parallel to the direction of stream flow). The periphery of the wheel is protected by a thin rim of width equal to that of the blades. The rim strengthens the blades and protects them from grass and floating debris, and is intended to reduce the errors due to cross currents; however, its value with regard to the last feature has been questioned. The bearings of the axis are of a non-corrosive metal having a low coefficient of friction. One end of the axis is connected by gears to the counting device which may be either acoustical or electrical. This meter is manufactured to be used in connection with a measuring rod in the hands of the observer and is not equipped for cable measurements. It is therefore not suitable for deep streams or high velocities.

500e. Hoff Meter.—This meter (Fig. 500*d*) has its axis of revolution horizontal (parallel to the direction of stream flow) and is fitted

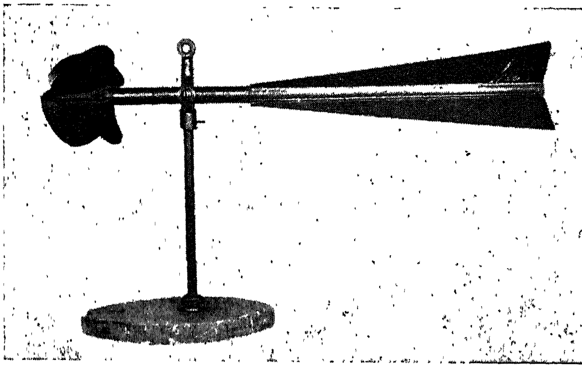

FIG. 500*b*.—Haskell current meter.

with a rubber propeller having either 3 or 4 blades, according to the type of meter desired. The propeller with 3 inclined blades is used for measurements at high velocities, and the propeller with 4 straight blades is used for low velocities. The Hoff meter has been used extensively, especially in California and neighboring states, and has given satisfactory results. Its chief characteristics are as follows:

FIG. 500*c*.—Fteley current meter.

1. The rubber propeller is but little heavier than water, and should give less bearing friction and respond more readily to changes in the velocity of the water than all-metal propellers.

2. The flexible rubber propeller is not so liable to injury from floating debris as a propeller fitted with metal blades. Grass and moss do not wind around the shaft as on cup meters.

3. The blades are so designed that the forces which cause the propeller to revolve are derived solely from the axial components of the downstream currents.

4. It is adapted to low, medium, and high velocities as shown by its rating curve (Ref. 4, p. 753).

5. By shifting a gear in the contact head, the operator may cause the meter to indicate either each single, each fifth, or each tenth revolution. For a given velocity of water, the propeller of the Hoff meter turns more than twice as fast as that of the Price meter.

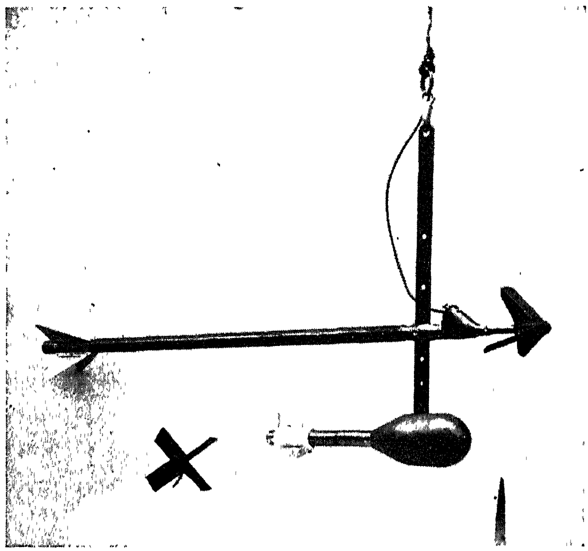

Fig. 500*d*.—Hoff current meter.

501. Meter Supports.—The meter may be supported on a rod either (1) by suspending the meter from the end of the rod and holding the rod in the hands or clamping the rod to some support, or (2) by clamping the meter to an upright graduated rod, with the meter fitted to slide up or down the rod. The U. S. Geological Survey uses the latter type fitted with an auxiliary rod to hold the meter in place. This type can be used without lifting the supporting rod out of the water.

The requirements for a cable support are (1) sufficient strength to support the meter and weight, (2) small cross-section to minimize water resistance, (3) insulation against short circuits in the indicating device, (4) toughness and flexibility to withstand hard usage. The cable may be graduated, but because of cable stretch the measurements are usually referred to an index point and the distance to this point is measured by scale or tape.

502. Rating Current Meters.—The object of rating a current meter is to make possible the calculation of the rate of flow from the observed number of revolutions of the wheel in a known interval of time. The current meter does not measure velocities directly but indicates the number of wheel revolutions per unit of time. The standard method of rating a meter consists in towing it through a body of still water for a known distance at various known velocities and counting the number of revolutions of the wheel. Usually the meter is attached to a small car which is driven along a level track beside the rating flume or channel. It may be suspended from a suitably propelled boat, although the car is to be preferred because of water disturbance caused by the boat.

The velocities at which the meter is moved through the water should cover a range from the lowest velocity at which the wheel will rotate to the maximum flood velocity. The length and cross-section of the flume or channel of still water in which the meter is rated is an important factor in the accuracy of the rating. The measured course over which the timing is done should be at least 100 ft. long with about 30 ft. allowed at each end for bringing the car to a uniform speed at the start and to avoid bringing the meter wheel to an abrupt stop at the end. Longer courses permitting a timed run of 200 to 400 ft. are to be preferred. The water channel should be of a depth to allow complete immersion of the meter assembly and wide enough to prevent disturbance due to wave action caused by the meter. General practice has fixed a channel 5 ft. wide by 5 ft. deep as about the minimum size that will assure a correct rating.

Other factors which influence the rating curve are the method of supporting the meter (see Ref. 16, p. 753) and the size and position of the weight (Ref. 5). Meters usually indicate a higher velocity when supported by a cable than when supported by a rod. The percentage of this difference is a maximum at low velocities and decreases as the velocity increases. Observations indicate that variations in position and size of weight are productive of slight variations in rating coefficient and that the value of the coefficient is more nearly uniform when the weight is suspended below the meter than when placed above it.

Meter ratings should be made at a properly equipped rating station.[1] When in constant use a meter should have a check rating

[1] The following is a partial list of well-equipped rating stations: (1) National Bureau of Standards, Chevy Chase, Md., (2) Colorado Agricultural College, Fort Collins, Colo., (3) Cornell University, Ithaca, N. Y., (4) Irrigation Branch, Department of Interior, Calgary, Alberta, Canada, (5) Rensselaer Polytechnic Institute, Troy, N. Y., (6) Univer-

yearly. The complete details of rating a meter and preparing the rating curve are given in Ref. 8, p. 753.

The rating station of the National Bureau of Standards at Chevy Chase, Md., near Washington, D. C., has a 6 by 6-ft. channel 400 ft. long made of reinforced concrete. The rating car is driven by a constant-speed motor; however, the velocity of the car is computed from the time and the distance traveled. Eight to ten double runs are made at velocities of 0.5 to 7.5 ft. per second. The average of the two values obtained by each double run is used to determine each of the individual points on the rating curve. Both lower and higher velocities than the above may be run but 0.5 to 7.5 ft. per second is the desirable range for average conditions of current-meter measurement. The number of revolutions of the meter wheel is recorded electrically. An electrical distance recorder is placed in circuit with the meter wheel so that the exact distance for a given number of revolutions of the wheel is obtained. The time is taken by a stop watch which is also started and stopped by an electrical control.

502a. Meter Rating Curves.—Several factors such as bearing friction, slip of the blades, inertia, retarding effect of the water, position of the weight, etc. influence the relation between the speed of the wheel and the velocity of the rating car. Were it not for the above factors this relation would be a constant for all velocities and the rating curve would be a straight line. However, the effect of these factors diminishes proportionately as the velocities increase so that the resulting curve is essentially a straight line, except for low velocities. For this reason two rating curves are made, one for low and one for high velocities (Fig. 502). In either case the curve will not pass exactly through the origin of coordinates but will cross the axis of velocities at the point where the wheel has overcome the retarding factors and begins to revolve. This is usually in the neighborhood of 0.1 to 0.2 ft. per second. However, the straight portion of the curve when prolonged may pass through the origin (Fig. 502).

The velocities in feet per second are plotted as abscissas with the revolutions of the wheel per second as ordinates, and a mean curve is drawn through the plotted points. The scale should be relatively large for a close determination. Fig. 502 shows a rating curve for a Price meter. A velocity table can be prepared from such a curve or the curve can be used directly to reduce wheel revolutions per second to velocity in feet per second.

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sity of California, Berkeley, Calif., (7) University of Michigan, Ann Arbor, Mich., (8) University of Toronto, Toronto, Ontario, Canada, and (9) Worcester Polytechnic Institute, Worcester, Mass.

Because of its simplicity and the small error introduced by its use, the equation of the rating curve is taken as that of a straight line for all ordinary meter work. The form of the equation is

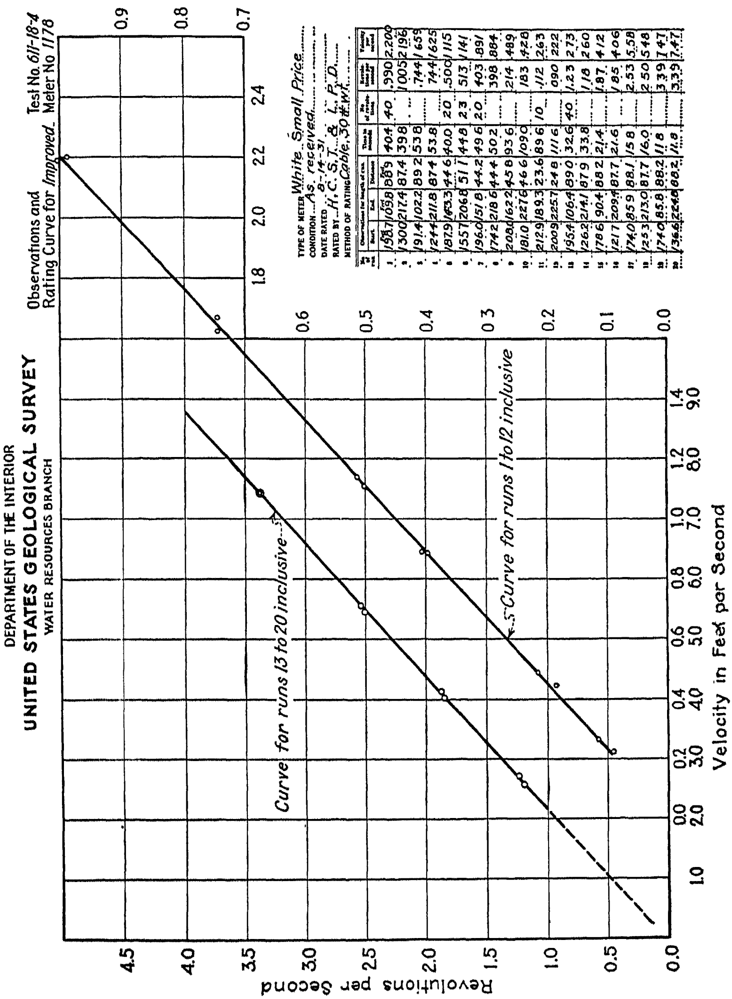


FIG. 502.—Meter rating curve.

$V = aR + b$ where V is the velocity of the water and R the revolutions of the wheel per second. The coefficient a is the ratio of the revolutions per second to the velocity in feet per second. The constant b represents the velocity that will just overcome the retarding effect of the factors mentioned above. This is sometimes called the *observational equation* because the constants a and b may be

determined by substituting values of V and R taken from the observer's field notes and solving simultaneously any two equations that may be set up.

503. Velocity Measurements.—The velocity desired in discharge measurements is the mean horizontal velocity in a vertical line at the measuring point. The methods commonly used for determining this value are (1) vertical-velocity-curve method, (2) two-tenths and eight-tenths method, (3) six-tenths method, (4) integration method, and (5) subsurface method.

503a. Vertical-velocity-curve Method.—Measurements of horizontal velocity are made at 0.5 ft. beneath the surface and at each tenth of the depth from the surface to as near the bed of the stream as the meter will operate. If the stream is relatively shallow, measurements at each one-fifth of the depth are taken. These measured velocities are plotted as abscissas and the respective depths as ordinates. A smooth curve drawn through the plotted points defines for a given vertical line the *vertical velocity curve*, which shows the velocity at each point in the vertical. The area under the curve (bounded by the velocity curve, the top and bottom ordinates, and the vertical axis) is equal to the product of the mean velocity and the total depth in that vertical line. The area may be determined either by planimeter or by Simpson's One-third Rule (Art. 278*b*, p. 404), and may be multiplied by the interval between measurements to determine the flow for that vertical strip of the cross-section. The sum of the flows for the individual strips is the total quantity Q for the stream at that cross-section.

The vertical-velocity-curve method is the most accurate means of determining mean velocities but requires too much time for general use. It is valuable as a basis of comparison with other methods, for measurements under ice, for determining a coefficient for the subsurface method, and for unusual conditions of flow.

503b. Two-tenths and Eight-tenths Method.—Observations are made in the vertical at two points only, at 0.2 and 0.8 the total depth measured downward from the water surface. The mean of these two velocities is taken as the mean horizontal velocity for that particular vertical. This method (see Ref. 8, p. 753) is based upon the theory that the vertical velocity curve is a parabola and that the mean of the ordinates at 0.2114 and 0.7886 depth below surface gives the mean ordinate. A study of various vertical velocity curves indicates that this relation holds substantially true for many conditions of flow, and experience proves that this method gives results of an accuracy consistent with the other uncertainties of most stream gaging work. The Water Resources Branch of the

U. S. Geological Survey uses this method almost exclusively in stream discharge measurements.

503c. Six-tenths Method.—A single observation is taken at a distance below the surface equal to 0.6 the total depth of the stream at that particular vertical. The velocity at 0.6 the depth is taken as the mean horizontal velocity of the vertical. The method is based upon the same theory as the 0.2 and 0.8 method. It has the advantage of requiring fewer readings, and in general it gives satisfactory results in natural streams, although when tested in artificial channels it runs about 5 per cent high.

503d. Integration Method.—The meter is slowly lowered in the vertical at a uniform rate to the bed of the stream and is then raised at the same rate to the surface. The total time and the number of revolutions during this interval constitute a measurement. From these data the average revolutions per second can be found and the mean velocity taken from the meter rating curve. This method is based upon the theory that all horizontal velocities in the vertical have acted equally upon the meter wheel and that their averages should be the mean velocity. Meters of the cup type are not as well suited to this method as those of the propeller type.

503e. Subsurface Method.—In this method the meter is held at just sufficient depth below the surface to avoid the surface disturbance, usually 6 to 8 inches. The subsurface velocity found must be multiplied by a coefficient to reduce it to mean horizontal velocity. This coefficient varies with the depth and velocity of the stream; the deeper and swifter the stream, the higher the coefficient. This method is less accurate than those described in the preceding articles, and it is used principally in measuring flood discharges where time and changing water stage are important. The coefficient most frequently used for flood measurements is 0.9 although for large floods it may be as great as 0.95. If the method is used for ordinary stages a good value of the coefficient is 0.85.

504. Recording Field Measurements.—Field observations are recorded as they are made, usually on forms specially prepared for discharge measurements. The forms shown in Fig. 504*a* and Fig. 504*b* are in common use. The following values should be recorded:

1. The distance of each vertical from the initial point.
2. Depth of the vertical.
3. Depth from surface to point where the observation is made.
4. Duration in seconds of velocity observation.
5. Number of revolutions of wheel during this time interval.
6. Gage reading at beginning and end of measurements.

505. Measurements with Current Meter.—Current-meter measurements are commonly divided into three classes: (1) wading, (2)

DISCHARGE MEASUREMENT NOTES

Date.................................., 19........ No. of Meas..........................

.... River at / Creek near ...., State of............................

Width........................Area........................Mean Vel........................Cor. M. G. H.________

Party.. Disch.________________

Staff gage, checked with level and found..

Chain length, checked with steel tape, 12-lb. pull, found..............................ft.

" " changed to..................ft. at............o'clock. Correct length...................ft.

" " corrected on basis of levels to........................ft. at........................o'clock.

Gage reading	Time	Station	
....	....	....	Meter No..
....	....	....	Date rated ....
....	....	....	Meas. began....................; ended..................
....	....	....	Time of meas. (hrs) ....Method..........
....	....	....	No. meas. sec's..................Coef..................
....	....	....	Av. width sec...............Av. depth.............
Weighted mean G. Ht.......................ft.			G. Ht. change (total.)........................
Correct " " " ....ft.			...............% diff. by..................rating table.

Meas. from cable, bridge, boat, wading. Meas. at....................ft. above, below gage.

If not at regular section note location and conditions...

....Area from soundings (date)..............................

Method of suspension........................Stay wire...................Approx. dist. to W. S....................

Arrangement of weights and meter; top hole........; middle hole........; bottom hole........

Gage inspected, found.................................... Cable inspected, found...................................

Distance apart of measuring points verified with steel tape and found..............................

Wind...............................upstr., downstr., across. Angle of current...

Observer seen.. G. Ht. book inspected..

Examine station locality and report any abnormal conditions which might change relation of G. Ht. to disch., e. g., change of control; ice or debris on control; backwater from; condition of station equipment..

....

Sheet No. 1 of..............sheets. If insufficient space, use back of sheet, with reference letters.

FIG. 504*a*.—Form for discharge measurement notes.

bridge, and (3) cable-car measurements. Most hydrographic engineers prefer the wading method where it is at all possible to secure good measurements. Bridge piers and abutments interfere with the free flow of the water, and cable sections are expensive to

install. Measurements are sometimes taken from a boat, but uncertainty always exists as to whether the influence of the boat upon the current has been entirely eliminated.

Knowledge of the stage is an important item in measuring discharge. When the stage is rapidly changing, speed is essential and the gage should be read several times during the measurement. No general rule can be given for the number of vertical sections to be read on a given cross-section, but it is better to err on the side of too many rather than too few. Thirty verticals make a good

Date..............................19........ No. of Meas...............................

DISCHARGE MEASUREMENT NOTES

....River, at...

Dist. from initial point	Depth	Depth of observat.	Rev-olu-tions	Time in sec-onds	VELOCITY			Area	Mean Depth	Width	Discharge
					At point	Mean in ver-tical	Mean in sec-tion				
Totals											

No. of Sheets. Comp. by Chk. by

FIG. 504*b*.—Form for discharge measurement notes.

determination for a single channel of uniform cross-section, 300 to 400 ft. wide. To simplify calculations, it is desirable to take the readings at uniform distances apart.

505a. Wading Method.—This method, illustrated by Fig. 505*a*, is used wherever the depth and velocity of the stream will permit. The reference point is first fixed and a wading line is stretched across the stream. The line may be a light cable marked with lead pellets at 5-ft. intervals or may be a well-stretched cord with the 5-ft. intervals marked with black paint. The accuracy of the intervals should be checked frequently by tape measurement. The meter is set up, all moving parts are checked, and the screws, electrical connections, and the adjustment of the meter head are examined. The meter is mounted on a jointed wading rod on which it may be slid up or down. When the observer is ready to start measurements,

the time and the gage reading to the nearest hundredth of a foot are recorded.

To take a reading, the total depth of the vertical section is measured with the sounding rod and the depths (as 0.6 depth or 0.2 and 0.8 depth) at which the meter is to be set on the sounding rod are calculated. When more than one reading is to be taken, most observers prefer to take the bottom reading first and to work toward the top. Care must be taken to operate the meter far enough and upstream from the body of the observer to prevent disturbing the meter by cross-currents.

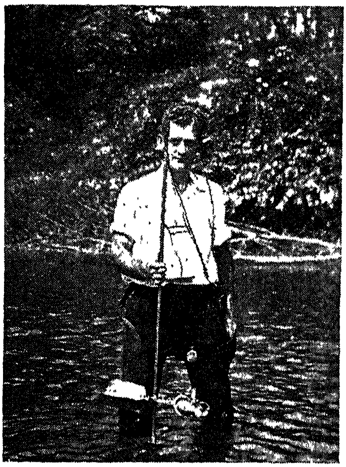


FIG. 505*a*.—Current-meter measurements by wading.

In making the velocity measurements, it is important that they are taken at the proper point in the vertical, that the flow is uniform, that the meter is working freely, and that the time of observation is of sufficient length to assure average conditions of flow. Observations of velocity are timed with a stop watch to the nearest one-fifth second over a period of 40 to 70 seconds; the longer the time interval, the better. The number of revolutions of the meter should be checked by noting the number at the half-time and doubling this number to compare with the final reading.

On small streams of uniform cross-section, minimum velocities of 0.2 ft. per sec. may be read with good results, but in most cases sections should not be located where velocities are below 0.5 ft. per sec. Maximum flood velocities seldom exceed the rating of the meter but care must always be taken to see that the meter is working freely and that fine grass or other fibrous material has not collected about the spindle, meter cups, or bearings. When all readings at a station have been taken, the meter is dismantled, dried, oiled, and repacked in its case. This is an essential practice in keeping the meter in good working order.

505b. Bridge Method.—When taking measurements from a bridge the verticals are located by measuring the desired distances along the guard rail and by marking the points with keel or paint. The reference or zero point is usually taken as the face of one abutment. The meter assembly is suspended by a small insulated wire cable

which is often marked at 3-ft. intervals by tags of different colors. The lower end of the cable is fastened to a metal bar or strap about a foot long, having holes drilled at each end and at the middle (Fig. 500*a*). The meter is usually attached to the bar at the second hole, and a 15 or 30-lb. weight is attached to the bar at the bottom hole. The size of the weight depends upon the depth and velocity of the stream.

When the station is to be gaged, the meter is examined as described in the preceding article. The observer measures the distances from

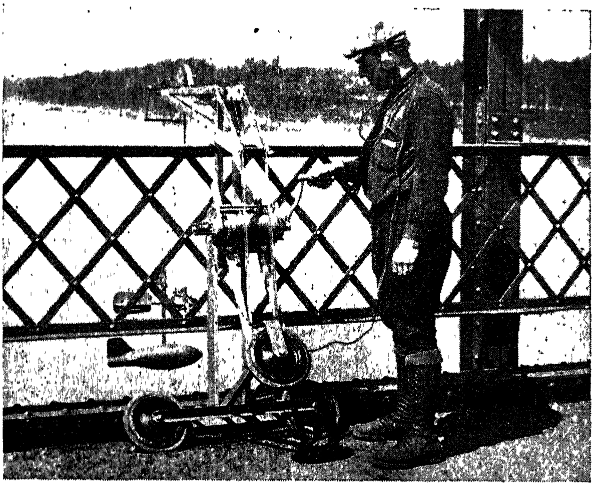


FIG. 505*b*.—Equipment for measuring current from a bridge.

the initial point to the water's edge at both banks, reads the water-stage register (usually a chain gage) and proceeds from vertical to vertical across the bridge. At each vertical he first measures the depth of the stream, then calculates the depths (as 0.2 and 0.8 depth) at which velocity observations are to be taken, and finally suspends the meter at each of these depths and observes the velocity of the stream filament as indicated by the number of revolutions of the meter during a given number of seconds. At the conclusion of the current-meter measurement at the last vertical, the water-stage recorder is again read and the distances from the initial point to the water's edge at both banks are again measured.

If the water's edge is on the face of an abutment, the depth and velocity should be taken; if on a sloping bank, the depth is of course zero and the velocity is assumed to be zero.

When measuring the depth of a vertical, the meter is lowered until the weight touches the bottom; the observer then marks with

his thumb the point on the cable where it goes over the rail. He then raises the meter slowly until the first colored tag attached to the cable appears at the surface of the water. This tag reading plus the measurement from the thumb to the guard rail gives the depth of the vertical. By a similar procedure the meter is lowered to the required calculated depths.

Where the river is deep and the current swift the arrangement shown in Fig. 505*b* is used. It consists of a framework of steel which is mounted on wheels and which extends out over the guard rail. The meter cable is wound on a drum fitted with a friction clutch and with a depth-recording device graduated in tenths of feet.

In calculating positions for the desired depth readings, allowance must be made for the distance between the center line of the meter wheel and the bottom of the weight.

FIG. 505*c*.—Stream gaging from a cable car.

505c. Cable-car Method.—Fig. 505*c* shows one of the cable cars used by the U. S. Geological Survey. Previous to erection, the cable is marked at the desired intervals with black paint. The cable is suspended in an advantageous location either between trees or

between towers. The fact that the cable may be erected at a section where the various factors of measurement are favorable, gives this type of station a distinct advantage over the bridge type so far as accuracy is concerned.

Sounding the depth of verticals is done in the same manner as for bridge sections, both for hand and for reel meter cables. For cable stations it is customary to use the 0.2 and 0.8 method, with verticals 5 or 10 ft. apart.

506. Discharge Measurements Under Ice.—When stream measurements are to be continued through the winter months, the reconnaissance is made previous to cold weather and sections suitable to measurement through the ice are located. If this precaution has not been taken, an examination may be made of the long, straight pools above the riffles where the stream is not frozen over. In selecting a suitable section, holes are cut through the ice near the center of the stream and near each shore line, and observations are made to determine if there is a measurable velocity and absence of slush or needle ice.

If conditions are found to be satisfactory, the section is laid off on the ice and is tied to an initial point on shore. Additional holes are cut for the measuring points, and observations of velocity are made as described in preceding articles. The total depth of the vertical is taken from the bottom of the ice to the bed of the stream. This is determined by measuring the depth of the stream bed to the surface of the water in the hole, and then measuring with an ice stick from the bottom of the ice to the water surface. The depth of the water minus the reading of the ice stick is the depth of the vertical.

The methods suitable for measurements under ice are:

1. The 0.2 and 0.8 method.
2. The vertical-velocity-curve method.
3. A method of taking a single reading at 0.5 the depth and multiplying by a coefficient to reduce to mean velocity. This coefficient may be taken as 0.88, or vertical-velocity-curve measurements may be taken to secure a value better suited to the section.

Many studies made of velocity at 0.2 and 0.8 depth indicate that results obtained by this method are reasonably accurate under ice conditions (see Ref. 9, p. 753). Hydrographic engineers generally consider the accuracy of this method to be in keeping with other uncertainties such as the effects of floating ice, ice freezing on meter, and other cold-weather conditions. The method is used by the U. S. Geological Survey except where the stream is so shallow as to compel the use of the mid-depth method.

507. Station Rating Curve.—When successive discharges are plotted as abscissas and their corresponding gage heights as ordinates the resulting graph is known as a *station rating curve*. The accuracy of such a curve will depend upon the stability of the section at the gaging station, the precision of the method used in velocity measurements, the accuracy of determining the cross-section, the care with which gage readings are taken, and the manner of distribution

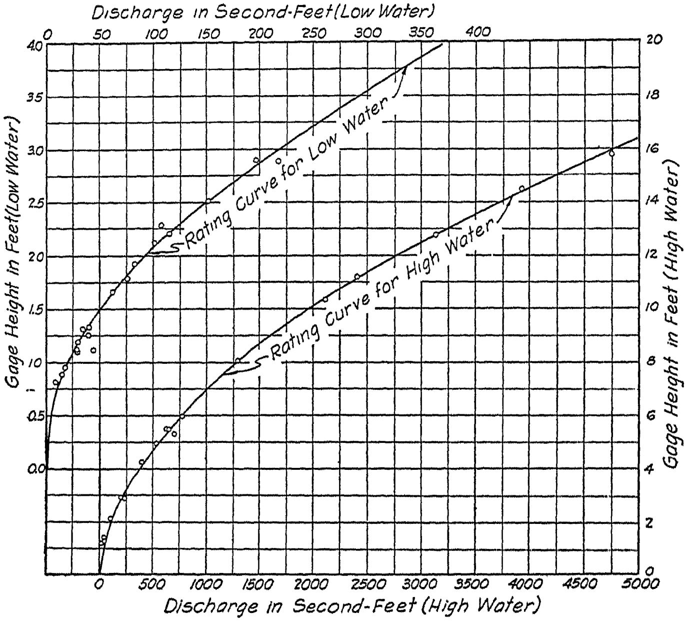


FIG. 507.—Station rating curve for Tiffin River near Brunersburg, Ohio, 1926 to 1931.

of discharge measurements from low to high stages. Under favorable conditions, variations of individual discharges from the mean curve should be slight. If large variations appear in plotting, they are generally due to mistakes in discharge calculations and can be corrected by a second calculation. If this does not locate the discrepancy, a second discharge measurement should be made at or near the same gage reading.

For a particular measurement, it is important that the flow become established so that the observed gage height indicates the true stage of the stream at the time of measurement.

Assuming definite stream controls, irregularities in the discharge curve must be caused either by inaccurate observations, mistakes

in computation, or errors in plotting; a careful check of all three factors is advisable before the discharge curve may be used with confidence.

Fig. 507 shows a station rating curve of the Tiffin River near Brunersburg, Ohio.

508. Discharge Computations.—Discharge is usually calculated from the field observations by means of an expression for the summation of partial discharges each calculated from the observed depth, the mean velocity in the vertical, and the distance between verticals. Let d_0, d_1, d_2 . . . d_n represent the measured depths of verticals, l_1, l_2, l_3 . . . l_n the respective distances between verticals, and v_0, v_1, v_2 . . . v_n the mean velocities in the verticals. The discharge for any partial area is its average depth, times the average mean velocity, times the distance between the two verticals. A summation of all partial discharges is equal to the total discharge Q, which may be expressed as follows:

$$Q = l_1\frac{(d_0 + d_1)}{2}\,\frac{(v_0 + v_1)}{2} + l_2\frac{(d_1 + d_2)}{2}\,\frac{(v_1 + v_2)}{2} \cdots + l_n\frac{(d_{n-1} + d_n)}{2}\,\frac{(v_{n-1} + v_n)}{2} \qquad (2)$$

Simpson's One-third Rule, considering the lower boundary of the partial area as an arc of a parabola, has sometimes been employed for finding partial areas, but considering the precision of observational data the added labor of this refinement is not justified.

An equally accurate method involving less computation is that of using the metered vertical and its measured depth as the mean of a zone extending from that vertical halfway to the verticals on both sides.

Field sheets such as those shown in Figs. 504*a* and 504*b* are usually returned to the office as soon as field measurements have been made and recorded. There the partial discharges and their summation are recorded on each field sheet as soon as calculated and these values are checked. The sheet is then filed as a part of the permanent record.

509. Discharge by the Slope Method.—This method involves an accurate determination of (1) slope of water surface (for non-uniform flow this is corrected for difference in velocity head), (2) mean area of channel cross-section, (3) mean hydraulic radius, and (4) character of stream bed; and choice of a proper roughness factor. With these data the mean velocity is calculated by the Chezy formula $V = C\sqrt{RS}$ in which V equals the mean velocity, C is a coefficient of the roughness of the stream bed, R is the mean hydraulic radius, and S is the slope of the water surface.

The mean area is the mean of the water cross-section in the reach of channel considered. The mean hydraulic radius R is the mean area of the water cross-section divided by the *wetted perimeter*, or that part of the cross-section of the stream wet by the flowing water. For most natural streams the value of R is approximately equal to the mean depth.

In artificial channels the area and wetted perimeter are so nearly constant that only one determination of R at each end of the reach is necessary. For natural channels at least three sets of measurements should be made and the mean of the three used to calculate the mean area, mean wetted perimeter, and mean hydraulic radius. If the areas at the ends of the reach differ, the velocity will differ and the slope of the water surface must be corrected to take care of the change in velocity head.

509a. Kutter's Formula and Coefficients.—The best known and most widely used expression for determining the value of C is "Kutter's Formula," published in 1869. It is based upon experimental data, and is as follows:

$$C = \frac{41.65 + \dfrac{0.00281}{S} + \dfrac{1.811}{n}}{1 + \dfrac{n}{\sqrt{R}}\left[41.65 + \dfrac{0.00281}{S}\right]} \tag{3}$$

where n is a retardation factor depending upon the roughness of channel, R is the mean hydraulic radius, and S is the slope of the water surface. Other formulas of equal merit are discussed in textbooks on hydraulics. Values of n were assigned by Kutter.

A more extensive table of values for n was compiled by Robt. E. Horton (see Ref. 7, p. 753). This table covers a wide range of conditions both for artificial and for natural channels and is a valuable addition to Kutter's work.

F. C. Scobey has published coefficients for Kutter's formula, based upon the results of extensive field tests (Ref. 19, p. 753). These coefficients are given in Table 509. The values are applicable for velocities up to about 5 feet per second and for hydraulic radii up to about 2 feet. For greater velocities or for greater hydraulic radii, slightly lower values of n should be used. It is emphasized that the selection of the value of n to be used in a particular case is largely a matter of judgment, and that the results of two men should not be discredited solely for the reason that they disagree slightly. It is also considered necessary to allow for overload in the design, rather than to follow the common practice of choosing a high value of n to allow for overload.

TABLE 509.—SCOBEY'S COEFFICIENTS FOR KUTTER'S FORMULA

Material of construction	Construction	Alinement	Operating conditions[1]	n
Concrete....	Best	Straight	Clear	0.012
	Good	Tangents and curves	(see note)[2]	0.014
	Average	Average	Average	0.016
	Rough	Irregular	Deposits	0.018
Wood.......	Best (surfaced lumber)	Straight	Clear	0.012
	Average (unplaned lumber)	Average	Average	0.015
	Rough	Sharp bends	Deposits	0.016
Metal (flume)	Countersunk joints	Straight	Clear	0.012
	Projecting joints	Straight	Clear	0.015
	Corrugated	Straight	Clear	0.022
Masonry....	Best	Average	Average	0.016
Earth.......	Best	Straight	Excellent	0.016
	Good	Good	Clear	0.020
	Average	Average	Average	0.0225
	Ordinary (small ditches)	Average	Some growth	0.025
	(Eroded after construction)	Irregular	Heavy growth	0.030
Cobbles.....	Well packed	Average	Average	0.027

[1] Freedom from vegetation, deposits of sand or gravel, and other local obstructions such as repairs.

[2] Design value of U. S. Bureau of Reclamation.

Glazed sewer pipe has about the same coefficient n as good concrete. Yarnell and Woodward have developed definite formulas for flow in drain tile (Ref. 21, p. 753).

509b. Value of the Slope Method.—The Chezy formula presupposes uniform flow, a condition rarely met in natural streams but closely approached in some straight artificial channels. For long flumes, conduits, large sewers, etc., whose cross-section and slope are uniform the formula will give fairly reliable results. Short structures are nearly always under backwater or drop-off conditions.

Use of the formula for natural streams should be confined to stretches where slope, stream bed, and channel approach uniformity and where more precise methods are impracticable. This method is used principally for rough determinations of discharge of streams at flood stages, often long after the flood has passed its crest, but when there are still evidences of the high-water stage left upon the banks.

510. Weirs; General.—For measuring the flow in irrigation and power canals, large sewers, small rocky streams, and other streams not suitable to current-meter measurements, a *weir* provides one of the most accurate methods available. A weir is a notch, as in the top of a vertical plank, for measuring the quantity of flowing water. It is also defined (Ref. 3, p. 753) as any obstruction placed in a channel, over which water must flow.

The measurements necessary to compute the discharge over a weir are as follows:

1. Depth of water flowing over the crest of the weir.
2. Length of crest, if weir is rectangular or trapezoidal.
3. Angle of side slopes, if weir is triangular or trapezoidal.
4. Whether sharp or flat crested.
5. Shape of crest if weir is flat crested.
6. Height of crest above bottom of approach channel.
7. Width and depth of approach channel.
8. Velocity of approach.
9. Number and nature of end contractions.

Having given the above data, a formula is chosen depending upon the type of weir, and a coefficient is selected depending upon the shape of the weir crest and upon the conditions of flow. By proper substitution in the chosen formula the discharge is computed.

511. Definitions.

Head.—The depth of water flowing over the weir, measured from the crest elevation to the pool level of the impounded water some distance upstream from the crest.

Crest.—The lower surface of the notch over which the water flows.

Sharp-crested Weir.—A weir for which the crest is bevelled like a chisel point with the upstream face vertical. (See Figs. 511*a*, 511*b*, and 511*c*.)

Flat-crested Weir.—A weir whose crest has appreciable width (see Fig. 511*d*).

Contraction.—A weir is said to have *end contractions* when the sides of the notch are at some distance from the sides of the channel of approach (Fig. 511*a*). When this distance is equal to or exceeds $3H$ the weir is said to have *full end contractions*.

Suppressed Weir.—A weir for which the sides of the weir coincide with the sides of the channel of approach (Fig. 511*b*).

Submerged Weir.—A weir for which the water level on the downstream side of the weir is higher than the crest of the weir (Fig. 511*c*), or more precisely one for which the downstream water level has been raised to such an extent as to affect the discharge over the weir.

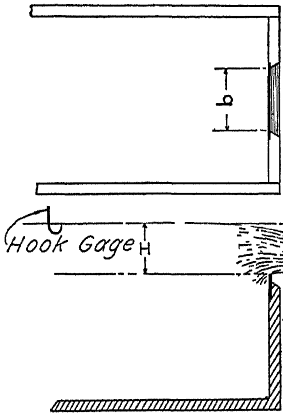


FIG. 511*a*.

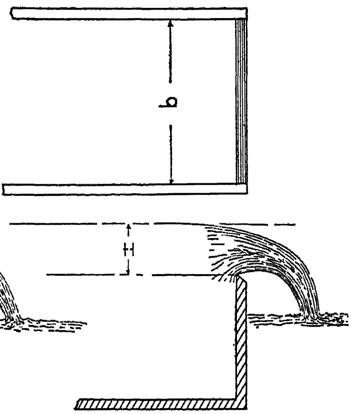


FIG. 511*b*.

FIG. 511*a*.—Sharp-crested rectangular weir with end contractions.
FIG. 511*b*.—Sharp-crested rectangular weir with end contractions suppressed.

Velocity of Approach.—The mean velocity of the water at the point where the head is measured.

Velocity Head.—The loss in head over the weir, due to an appreciable velocity of approach, expressed as follows: $h_0 = \frac{v_0^2}{2g}$, where h_0 equals the velocity head in feet, v_0 equals the velocity of approach in ft. per sec.,

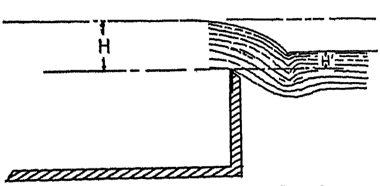


FIG. 511*c*.—Submerged weir.

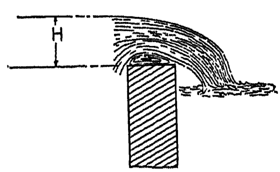


FIG. 511*d*.—Flat-crested weir.

and g is the acceleration due to gravity (usually taken as 32.16 ft. per sec. per sec. in the English system of units).

512. Rectangular Weirs.—If a small rectangular orifice (hole) be cut in the side of a vessel and allowed to discharge water under an appreciable head, the theoretical velocity of the discharge would be $\sqrt{2gh}$, where h equals the head of water in feet on the center of the orifice. Let B equal the width in feet and Z the depth in

feet of the rectangular opening. The theoretical discharge formula would then be

$$Q = BZ\sqrt{2gh} \tag{4}$$

A more exact formula may be derived (see Ref. 13, p. 753) by the use of the calculus, which gives the true theoretical discharge from a rectangular orifice as

$$Q = \tfrac{2}{3}B\sqrt{2g}(h_2^{3/2} - h_1^{3/2}) \tag{5}$$

where h_1 is the head over the top and h_2 the head over the bottom of the orifice. If we let $h_1 = 0$ and $h_2 = H$, the orifice becomes a rectangular weir and the discharge in cu. ft. per sec. is given by the formula

$$Q = \tfrac{2}{3}B\sqrt{2g}\cdot H^{3/2} \tag{6}$$

The formula assumes the velocity at any point over the crest of the weir as due to the head of water above that point; therefore, H is not measured in the vertical plane of the crest but far enough upstream from the weir to miss the downward slope of the surface curve caused by the increased velocity of the water flowing over the weir (see Fig. 511*a*).

Equation (6) gives the theoretical discharge when the velocity at the hook gage is zero. If an appreciable velocity exists at the hook gage the formula must be modified to allow for the velocity of approach (Art. 513). The discharge, allowing for the velocity of approach, is then

$$Q = \tfrac{2}{3}B\sqrt{2g}\cdot(H + h_0)^{3/2} \tag{7}$$

This is the true theoretical discharge when H is measured at the hook gage and h_0 is determined from the mean velocity V. Friction between the water and the edges of the weir, and absence or presence of end contractions make the actual discharge something less than the theoretical. Allowance is made for this discrepancy by use of the coefficient c_d derived by experiment. The velocity head h_0 is also modified by a constant n due to the fact that the velocity of approach is not a constant throughout the cross-section of the channel. The formula for actual discharge is

$$Q = c_d\tfrac{2}{3}B\sqrt{2g}\cdot(H + nh_0)^{3/2} \tag{8}$$

The discharge coefficient c_d is always less than unity. The value of n varies from 1 to 1.5. Hamilton Smith (Ref. 20, p. 753) found the value of $n = 1.4$ suitable for weirs with end contractions and $n = \frac{4}{3}$ suitable for suppressed weirs.

Studies made by Francis, Fteley, and Stearns have been compiled into tables by Hamilton Smith for weirs having end contractions and for weirs with end contractions suppressed (see Tables XIV and XV).

A study of Table XIV shows that the coefficient c_d increases with the length of crest. This is due to the fact that the effect of end contractions is independent of the length of the weir. Both Tables XIV and XV show that the coefficient increases as the head of water over the crest diminishes. Since the greatest variation in coefficients occurs at small heads, a small head should be avoided in accurate discharge measurements.

The weir formulas of Hamilton Smith are simple and convenient to use. Tables XVI and XVII give values for the coefficient c_d to be used in the formula $Q = c_d B H^{3/2}$ in which Q is the discharge in cubic feet per second, c_d a coefficient based upon experiment, B the length of crest in feet, and H the head on crest in feet. When velocity of approach must be considered, H is increased to $(H + 1.4h_0)$ for weirs having end contractions and to $(H + \frac{4}{3}h_0)$ for suppressed weirs.

Weir formulas by Francis, Bazin, Fteley, Stearns, Cone, Lyman, and Schoder and Turner are in common use. For formulas and coefficients, see Refs. 2, 6, 11, 13, and 18, p. 753.

513. Correction for Velocity of Approach.—When the velocity of approach is zero, the head measured by the hook gage is the *effective head* and is substituted for H in the discharge formula. If the water approaches the section of the hook gage with appreciable velocity an addition for *velocity head of approach* must be made to the gage reading to secure accurate discharge results. The amount to be added may be determined as follows:

1. The general discharge formula is solved for Q, using the hook-gage reading as H.

2. $v_0 = \dfrac{Q}{A}$, where A is the cross-sectional area at the hook gage, and v_0 is the mean velocity in this cross-section.

3. Then $h_0 = \dfrac{v_0^2}{2g} = \dfrac{Q^2}{A^2 \cdot 2g}$.

Since h_0 is generally very small as compared with H, little error is introduced into the results by this approximation.

514. Submerged Weirs.—When the water on the downstream side of a weir rises above the level of the crest, the weir is said to be *submerged*, and the formulas given in Arts. 512 and 513 are inapplicable to this condition. In Fig. 511*c* let H be the head above the crest measured on the upstream side and H' the head above the crest

on the downstream side. For small values of H' the contractions are suppressed and the discharge is increased. As H' increases to appreciable values the discharge decreases and becomes zero when $H' = H$. Lack of experimental knowledge regarding submerged weirs makes them unreliable for precise measurements. Their use should be avoided except in cases of standard weirs flowing as submerged weirs during floods. Experiments with submerged weirs have been mostly confined to weirs without end contractions.

Cox's formula (Ref. 3, p. 753) for flow over sharp-crested submerged weirs is

$$Q = c_d B H^{3/2}, \text{ where } c_d = 4.3\sqrt[4]{1 - (S + 0.002)} - 0.822 \qquad (9)$$

in which B is the length of weir in feet, H is the upstream head on weir in feet (corrected for velocity of approach), and S is the per cent submergence $= \dfrac{\text{Downstream head}}{\text{Upstream head}}$. This formula is applicable only when the nappe, or sheet of water, flows above and does not plunge under the surface. The downstream head is measured at a distance from the weir equal to 2.54 times the height of weir, or at the lowest point of the surface.

The chief advantage in the use of the submerged weir is that it requires but little loss in head. Another device used to measure flow without large loss of head is a specially tapered section of flume called a *venturi flume*, which is used in many of the larger canals of the West. The venturi flume has an additional advantage in that it is not subject to silting, as is a weir.

515. Triangular and Trapezoidal Weirs.—Triangular weirs (Fig. 515*a*) are sometimes used where the flow of water is small. The

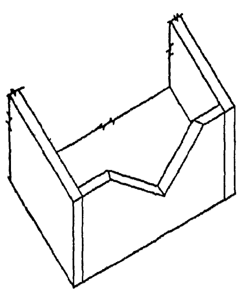


FIG. 515*a*.—Triangular weir.

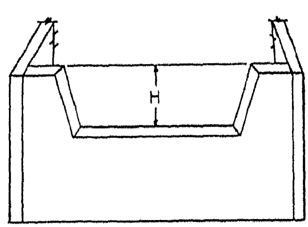


FIG. 515*b*.—Trapezoidal weir.

inner edges should be sharp to insure full contraction and the notch should preferably be cut to a right angle to conform to known coefficients. Trapezoidal weirs (Fig. 515*b*) are favored by some engineers because their coefficients vary less than those for rectangular weirs.

The ends of the notch are sloped outward. When the horizontal component of the slope is equal to one-fourth H, the weir is called a *Cipolletti weir*. The additional discharge at the ends tends to balance the effect of the end contractions. Were this balance perfect, the discharge over a Cipolletti weir with end contractions would be the same as that over a rectangular suppressed weir having the same length of crest. For a full discussion of triangular and trapezoidal weirs see Ref. 13, p. 753.

516. Use of Dams as Weirs.—Where no water is diverted around the dam or where means of measuring the diversion are at hand, dams may be utilized as weirs for measuring discharge. The use of dams as weirs has the advantage of supplying a continuous record for all conditions of flow. Dams on larger streams are expensive to construct and are seldom built for use as weirs alone. In Ref. 12, p. 753, Mead lists the following requirements:

1. Sufficient fall over the weir to prevent interference of backwater during stages of high water.
2. Little or no leakage around or under the dam.
3. Dam high enough to confine the stream flow to the weir section during all stages.
4. Crest level and free from obstructions.
5. Crest and weir must conform to some type whose coefficients are known and can be used in the general formula $Q = c_dBH^{3/2}$.
6. If the crest is adjustable, care must be taken to secure its exact elevation and to guard against leakage.
7. Provision must be made for careful measurement of all water diverted through or around the dam.

Where the cross-section of the dam and the shape of the weir do not conform to an experimental weir whose coefficients are known, it is often practicable in the following manner to determine a coefficient for the dam in question:

1. Establish two velocity-area sections suitable for careful current-meter work, the one above the dam being fitted for measuring the higher heads over the weir and the one below the dam being suitable for measuring low-water flow.
2. Establish a gage to read the water level above the weir.
3. The discharge Q for a given head over the weir is calculated from the current-meter and area measurements. Substitute Q and H in the weir formula and solve for the coefficient c_d.
4. When sufficient determinations ranging from low to high heads have been made, a curve giving values of c_d for all heads over the weir may be constructed by plotting the values of c_d as abscissas and the corresponding value of H as ordinates and drawing a smooth curve through the mean values of the plotted points.

5. The curve should not be extended in either direction beyond the point where discharge measurements were discontinued, as results obtained in this way are likely to be greatly in error.

517. Construction of Weirs.—In selecting a site for the installation of a weir the following items are to be considered:

1. Banks must be high enough to contain the flow for all stages at which measurements are desired.
2. Banks and bottom material should be such that leakage can be prevented. Shale, loose seamy rock, coarse gravel, etc. are undesirable.
3. For the elevation and length of crest and for the proposed type of weir, the rise in water level should be calculated for extreme high and low discharges. This will indicate the possibilities of the weir selected.
4. If the weir selected is suitable it should be noted whether the table of coefficients covers the entire range of possible heads. Assuming coefficients beyond the range of the tables may introduce large errors.

Precautions to be observed in the construction of the weir are equally important:

1. The crest should be exactly level, and the upstream face of the weir should be vertical and sharpened to a width not to exceed ¼ in., with the bevel on the downstream side.
2. End contractions should be at least three times the greatest head over the weir.
3. To insure a low velocity of approach, the depth below the crest on the upstream side should be greater than twice the maximum head on the crest.
4. The fall on the downstream side should be sufficient to insure a freely falling sheet so that the outflowing stream of water will be completely surrounded by air.

Small weirs are best constructed of wooden planks or sheet metal, with wooden sheet piling to prevent subsurface flow.

518. Problems.

1. The zero elevation of an indirect staff gage is 745.41 and the gage reads 10.24 ft. when 3.52 ft. of water is flowing over the lowest point on the control. What is the zero gage reading?
2. A rainfall of 2 in. per hr. falls for a period of 4 hr. on a drainage area of 100 sq. mi. If the estimated run-off is 25 per cent, how many acre-feet would be impounded if the water could be stored?
3. The right and left water's edges of a stream are 10 and 80 ft. respectively from an initial zero point. Verticals are located at distances

of 15, 20, 25, 35, 45, 55, 60, 62, and 65 ft. from the initial point. Depths of verticals are 2.6, 3.8, 4.6, 7.8, 8.4, 8.8, 8.2, 6.1, and 5.4 ft. Velocities measured by the 0.6 method are 0.65, 1.57, 2.40, 2.88, 3.12, 3.80, 4.28, 3.40, and 1.82 ft. per sec. respectively. Considering that this method gives results that are 5 per cent too high, what is the actual discharge of the section in cu. ft. per sec.?

4. A storage dam used as a weir has a discharge equation $Q = 3.16 B H^{1.55}$. If B, the length of weir, equals 500 ft. and H, the head of water over the weir, equals 8 ft., what is the discharge of the weir in cu. ft. per sec.?

5. A semicircular flume 15 ft. in diameter has a grade of -0.15 per cent. The flume is built of well planed timber and has an average depth of 4.5 ft. of water in the center of the flume. What is the discharge in cu. ft. per sec.?

6. The discharge of a sewer is to be measured by a sharp-crested rectangular weir having a length of 2 ft. The weir has full contractions at both ends and the hook gage shows a head of water over the crest of the weir of 5 in. Neglecting the effect of velocity of approach, what is the discharge of the weir in cu. ft. per sec.?

7. Outline a practical method of measuring the exact discharge in gallons per min. of a spring flowing somewhere between 15 and 20 gal. per min.

8. What method would you use in measuring the discharge in cu. ft. per sec. of the following: (1) a small rocky creek 8 to 10 ft. wide; (2) a river 150 ft. wide and 5 to 8 ft. deep; (3) a storage dam operating as a weir, the coefficient of which is not known; and (4) a river one-half mile wide and 20 ft. deep?

References

1. Casler, M. D., "Stream Flow in General Terms," *Trans. Am. Soc. Civ. Eng.*, Vol. 94, pp. 1–61, 1930.

1*a*. Church, J. E., "Snow Surveying; Its Principles and Possibilities," *Geographical Review*, October 1933.

2. Cone, V. M., "Flow Through Weir Notches with Thin Edges and Full Contractions," U. S. Dept. Agr., *Jour. Agr. Research*, Vol. 5, No. 23, pp. 1051–1113, March 6, 1916.

3. Cox, G. N., "The Submerged Weir as a Measuring Device," University of Wisconsin, Engineering Experiment Station Bull. 67, 1928.

4. Fortier, Samuel, and Hoff, E. J., "Defects in Current Meters and a New Design," *Eng. News-Record*, Vol. 85, pp. 923–924, 1920.

5. Grover, N. C., "Meter Ratings with Various Suspension Arrangements, 1928–1929 Ratings," Water Resources Branch of U. S. Geological Survey, Washington, D. C.

6. Horton, R. E., "Weir Experiments, Coefficients, and Formulas," U. S. Geological Survey, Water Supply Paper 200, 1907. Government Printing Office, Washington, D. C.

7. Horton, R. E., "Some Better Kutter's Formula Coefficients," *Eng. News*, Vol. 75, pp. 373–374, Feb. 24, 1916.
8. Hoyt, J. C., and Grover, N. C., "River Discharge," John Wiley and Sons, New York, 1920.
9. Hoyt, W. G., "The Effects of Ice on Stream Flow," U. S. Geological Survey, Water Supply Paper 337, 1913. Government Printing Office, Washington, D. C.
10. Logan, G. W., "Elements of Hydrographic Surveying," U. S. Naval Institute, Annapolis, Md., 1911.
11. Lyman, R. R., "Measurement of Flow of Streams by Approved Forms of Weirs, with New Formulas and Diagrams," *Trans. Am. Soc. Civ. Eng.*, Vol. 77, pp. 1189–1337, 1914.
12. Mead, D. W., "Water Power Engineering," McGraw-Hill Book Co., New York, 1915.
13. Merriman, Mansfield, "Treatise on Hydraulics," John Wiley and Sons, New York, 1916.
14. Messum, S. V. S. C., "Hydrographic Surveying," Chas. Griffin and Co., London, 1910.
15. Murphy, E. C., "Accuracy of Stream Measurements," U. S. Geological Survey, Water Supply Paper 64, 1902. Government Printing Office, Washington, D. C.
16. Murphy, E. C., "Accuracy of Stream Measurements," U. S. Geological Survey, Water Supply Paper 95, 1904. Government Printing Office, Washington, D. C.
17. Ramser, C. E., "The Flow of Water in Drainage Channels," U. S. Dept. Agr., Tech. Bull. 129, November 1929. Government Printing Office, Washington, D. C.
18. Schoder, E. W., and Turner, K. B., "Precise Weir Measurements," *Trans. Am. Soc. Civ. Eng.*, Vol. 93, pp. 999–1190, 1929.
19. Scobey, F. C., "The Flow of Water in Irrigation Channels," U. S. Dept. Agr., Bull. 194, 1924. Government Printing Office, Washington, D. C.
20. Smith, Hamilton, "Hydraulics," John Wiley and Sons, New York, 1886.
21. Yarnell, D. L., and Woodward, S. M., "The Flow of Water in Drain Tile," U. S. Dept. Agr., Bull. 854 (Professional Paper), 1920. Government Printing Office, Washington, D. C.
22. Zeskind, L. M., "Instructions for Tidal Current Surveys," U. S. Coast and Geodetic Survey, Special Publication 124, 1926. Government Printing Office, Washington, D. C.
23. "Hydrographic Manual," U. S. Coast and Geodetic Survey, Special Publication 143, 1928. Government Printing Office, Washington, D. C.
24. "Radio Acoustic Method of Position Finding in Hydrographic Surveys," U. S. Coast and Geodetic Survey, Special Publication 107, 1924. Government Printing Office, Washington, D. C.

CHAPTER XXVII

PHOTOGRAPHIC SURVEYING

TERRESTRIAL SURVEYS

519. General.—The use of photography in surveying and mapping is not of recent origin, having had a wide and extensive application before the end of the last century. The immense advantages of the wealth of detail secured, and the speed of the photographic record, have drawn many eminent surveyors and scientists to deal with the problem of rendering the method practical. The success achieved in terrestrial methods was greatly augmented within more recent years by the application of the stereoscopic principle, which reduced the amount of field work and greatly increased the accuracy of the results, and especially by the invention of autographic devices for minimizing the vast amount of office work which the older methods entailed, thus greatly reducing the total cost of such work.

These achievements have been nearly eclipsed by the introduction of *aerial photography*, which became important during the World War. The great advantage of the aerial photograph is that when the plate is exposed with the optical axis of the camera in the vertical position, it yields a view closely approximating an orthographic plan view, *i.e.*, a map. Unfortunately, however, the unstable conditions incident to photographic exposures from a rapidly moving plane, the expense of airplane flights, and especially the inadequate evidence secured as to the relief of the ground surface, have proved to be real difficulties in the way of a wider application of aerial photography to topographic mapping.

Any photograph, terrestrial or aerial, is a perspective view. Practically all engineering maps are orthographic projections, or they are small areas of other kinds of map projections, ordinarily considered as orthographic. Accordingly, any attempt to use a photograph as though it were a map requires an understanding of the principles of perspective and of translating the data in the photograph into those of the map. Failure in this regard has been responsible, within recent years, for many unfortunate experiences in the use of aerial photographs. In this chapter, therefore, before attempting

the discussion of aerial surveying, there is given a brief description of the instruments and methods used in terrestrial photographic surveying, for the twofold purpose of describing a method which still has important advantages under certain conditions, and of making clear those principles which are fundamental to all photographic surveying work, whether terrestrial or aerial.

520. Definitions.—Following are definitions of the common technical terms used in this chapter. Many of these are illustrated in Figs. 520, 521*a*, and 521*b*, in which are shown the principal planes

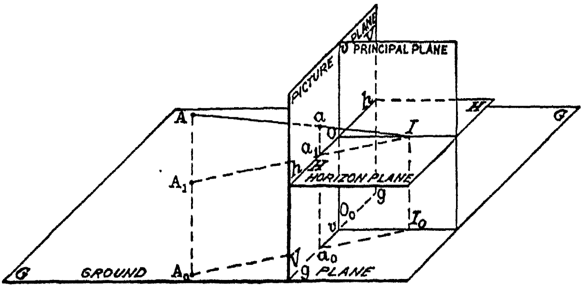


FIG. 520.—The perspective planes.

and lines used in constructing the perspective view. The letters following the terms refer to points, lines, and planes shown in these figures.

Point of View I.—The optical center of the camera lens, or in the field, the camera station.

Picture Plane VV.—The plane of the perspective view, perpendicular to the optical axis of the camera lens, and at a distance f from the point of view.

Horizon Plane HH.—The horizontal plane containing the point of view.

Ground Plane or Datum Plane GG.—Any reference plane below and parallel with the horizon plane; usually taken below the lowest point in the terrain.

Principal Plane.—The vertical plane perpendicular to the picture plane and containing the point of view *I*.

Frontal Plane.—Any plane parallel with the picture plane, between it and the object sighted.

Horizon Line hh.—The trace of the intersection of the picture plane and the horizon plane.

Principal Line vv.—The trace of the intersection of the picture plane and the principal plane.

Ground Line gg.—The trace of the intersection of the picture plane and the ground plane.

Principal Point O.—The point of intersection of the principal line and the horizon line.

Focal Length of Lens f.—(see Art. 104*b*, p. 123).

Orthographic Projection.—The view which results from projecting each point of an object along a line perpendicular to the plane in which the view appears.

Iconometry.—The procedure of constructing the orthographic plan and elevation views from the perspective view.

Nadir N.—In aerial surveying, the point vertically beneath the camera station.

Central Ray.—The line of vision from the point of view I to the principal point O.

521. Principles of Perspective; Ocular Vision.—Let it be supposed that Fig. 521*a* represents an object, a picture plane VV, a camera lens I, and a camera plate $V'V'$. It is evident that the images are formed by rays of light passing along straight lines; that the image on the picture plane is an erected (positive) image and the image on the camera plate is an inverted (negative) image; and that the sizes of the images are equal if their distances from the lens are equal.

It may also be supposed that I represents the eye of an observer, the plane $V'V'$ being the retina of the eye. Thus it is evident that the retina receives inverted images but that it interprets the observed images along the same lines in which they are received, and therefore the observed image appears as shown in the picture plane, erect and in front of the eye.

In the remarks that follow, since it is obvious that the positive and negative views are in every respect similar, the positive view in the picture plane only will be considered.

521a. Radial Projection.—Reference to Figs. 521*a* and 521*b* shows that a perspective view is formed by rays of light passing from the

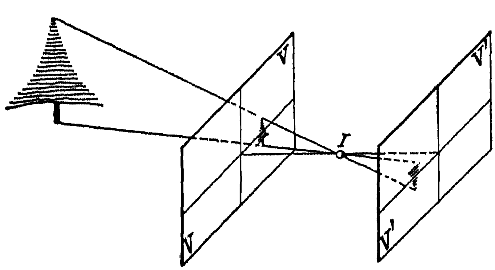


Fig. 521*a*.—Ocular vision.

object to the point of view and intercepted on one or more planes perpendicular to the principal line. Therefore, a perspective view is a radial projection on a plane, the point of radiation being the point of view I. If that plane is considered which is at a distance from I

equal to the focal length of the lens f, we have the perspective in the picture plane, *i.e.*, a photograph.

521b. Constructing the Perspective View.—Figure 521*c* shows the orthographic plan and elevation views of an irregular solid having parallel bases and vertical sides; and also shows the perspective view of this figure as pictured, let us say, by a camera having a focal length f.

The line gg represents, in plan view, the horizontal trace of the picture plane. Hence, each point of the object will appear along the

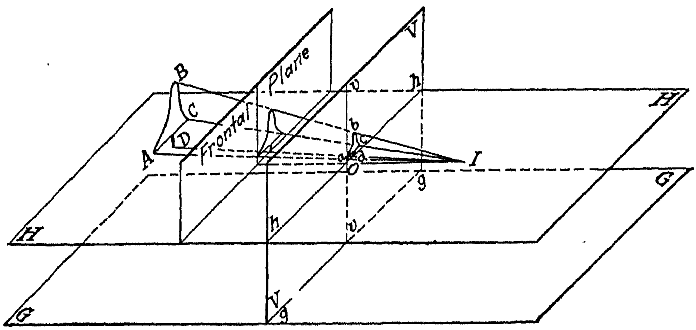


FIG. 521*b*.—Projection planes.

rays drawn to it from I. For the plan view here shown, the rays are intercepted by the trace gg, at the points aa_o, OO_o, bb_o, etc. Therefore, if these points are projected vertically above to the picture plane (perspective view), these projection lines will contain the corresponding points in the perspective view.

The line vv represents, in the elevation view, the vertical trace of the picture plane. The linear distance I-I_o represents, to some scale, the height of the point of view above the ground plane. Accordingly, the radial projection of the points of the object, B, C, B_o, C_o, etc. will appear in the picture plane where these rays from the object to I cut the trace vv, as at b, c, b_o, c_o, etc.

The point a in the perspective view is fixed by the conditions that it lies in the line of projection vertically to the picture plane, of the point aa_o (plan) and at a distance above gg (perspective) equal to the distance O_oa (elevation). Also, the point c_o in the perspective view is located along the vertical projection of the point cc_o (plan) and at a distance O_oc_o (elevation) above the ground line gg (perspective).

Similarly, the other vertices of the figure are located in the picture plane. The lines connecting these points constitute the perspective view of the outlines of the figure.

521c. Principles Stated.—Careful inspection of the perspective view now permits consideration of a few of the important principles of all similar views:

1. All vertical lines of the object will be seen as parallel vertical lines in the perspective; as aa_o, bb_o, etc.

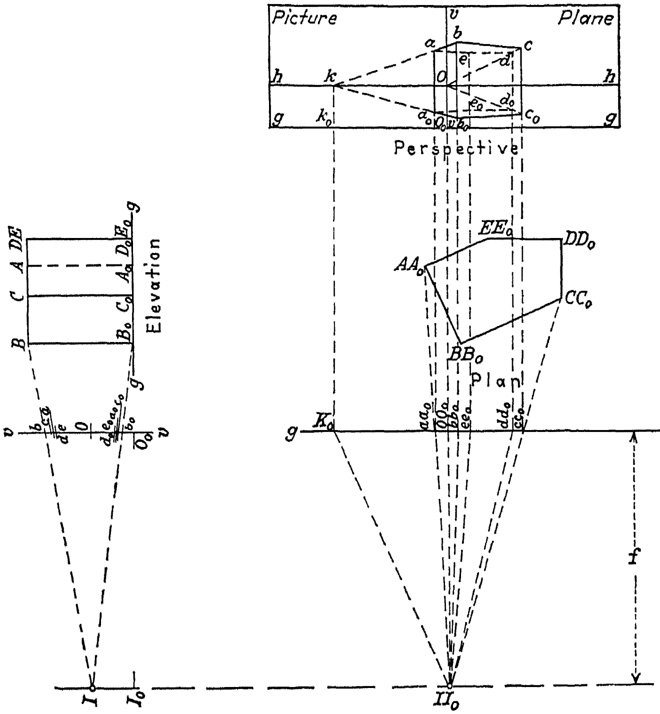


Fig. 521c.—Construction of the perspective view.

2. Parallel horizontal lines of the object not parallel with or perpendicular to the horizon line hh will appear, when prolonged, to meet on the horizon line at a vanishing point, as k. The position of k, for lines AB and A_oB_o (plan), is located by drawing from the point of view II_o (plan) a line II_o-K_o, parallel with the corresponding line AA_o-BB_o in the plan view. This line II_o-K_o meets the trace gg at K_o. This latter point K_o is projected to the horizon line of the picture plane (perspective) to locate the vanishing point at k.

3. Horizontal lines parallel with the picture plane appear as horizontal lines in the perspective; thus the lines ED and E_oD_o (plan) appear as ed and e_od_o (perspective).

4. Horizontal lines perpendicular to the picture plane, when prolonged, appear to pass through the principal point. Thus the lines

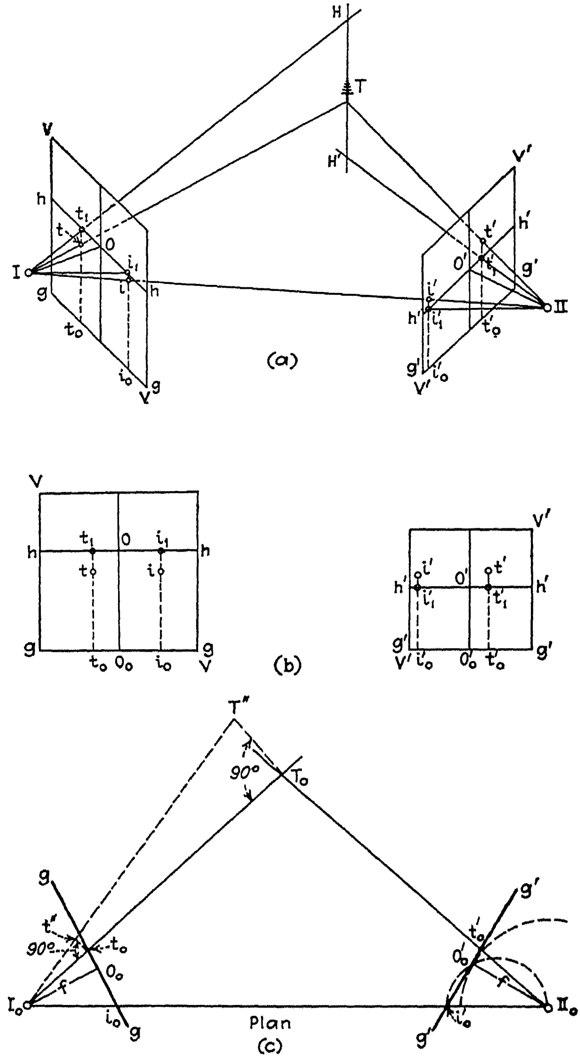


FIG. 523a–c.—Constructing the plan view.

CD and C_oD_o (plan) appear in the perspective as *cd* and c_od_o, which lines when prolonged pass through the point *O*.

5. The data required in the construction of the perspective include: (*a*) the orthographic plan view; (*b*) either the orthographic elevation

view or the known elevations of all points above the ground line gg; (c) the position of the point of view II_o (plan); and (d) the distance f from the point of view II_o to the picture plane.

It is obvious that any number of different perspectives of the same object may be constructed by changing the values of the variable data listed above. In particular, it may be noted that the view in a frontal plane is in every respect similar to that in the picture plane, the most common example being the enlargement of a photograph.

522. Iconometry; Data Required.—The inverse procedure to that just described is termed *iconometry*. Iconometry consists in constructing the orthographic plan from a perspective view. In photographic surveying, it is the office procedure of constructing the map from the photographs of the terrain. The requisite known data include: (1) the focal length of the camera lens; (2) the plotted position of the camera station; (3) the scale of the map; (4) the known direction (azimuth) to some object pictured in the photograph; and (5), if elevations are to be computed, the elevation above the datum (ground plane) of the camera station (horizon plane).

The known direction to a pictured object may consist of either (1) a measured angle at the camera station between a known station and the pictured object,. or (2) the pictured position of the object must appear in two pictures, one taken from one end of a base line and the other taken from the other end.

The elevation of the camera station is determined either directly by field leveling operations, or indirectly from the known elevations of two or more pictured points.

523. Constructing the Plan View.—From Art. 521b it is clear that the position of any point a, in the perspective view of an object A, is determined by the direction of the ray from the camera station to the object, II_oAA_o (plan), and by the elevation O_oa of the object above or below the ground line (elevation). Inversely, to plot the orthographic plan view of any point in the perspective view, there must be known both the direction to the point II_oaa_o (plan), and its distance AA_o, *i.e.*, its elevation, above the ground line (elevation). The procedure is to plot the direction of the ray II_oaa_o (plan) indefinitely, and to extend the ray Ia (elevation) to a distance such that the value of the distance, to scale, of AA_o (elevation) is equal to the known elevation of A above the ground plane. The point A in the orthographic plan is now fixed on the indefinitely extended ray II_oaa_o (plan) at a distance from the picture plane equal to the distance from A to the picture plane (elevation).

In constructing a map from a photograph, the direction of the ray on the map to any point shown in the photograph can ordinarily be determined by methods presently to be explained, but the elevation of the point in the picture plane above or below the ground plane usually is not known. Hence, the point can not be plotted from the data furnished by a single photograph. Advantage is taken, however, of the principle of intersections as used in plane-table operations (Art. 414, p. 615) and the position in plan of each point is located by the intersection of the two rays to it drawn from the plotted positions of two different camera stations.

The general procedure is illustrated in Figs. 523*a*–*c*. Here are shown in Fig. 523*a* the positions of two points of view I and II at different elevations, an object T, the picture planes VV and $V'V'$, and the points at which the connecting rays pierce the planes. Figure 523*b* shows the orthographic view of each picture plane.

It is evident from Fig. 523*a*, that the ray from station I to the object T pierces the picture plane at the point t, which point projected vertically to the ground line gg falls at the point t_o; and similarly the points i, i', and t' are projected to the points i_o, i'_o, and t'_o respectively. These projections are shown in Fig. 523*b*.

In Fig. 523*c* are shown the points I_o and II_o, being the orthographic projections on the ground plane of the stations I and II. The line I_o-II_o represents the plan view of the line I-II. If the ground line (picture trace) gg is constructed so that the point i_o falls on the line I_o-II_o, the point t_o marks the direction of the ray I_ot_o which, extended indefinitely, contains the position of the point T_o, the latter point being the orthographic projection of the object T.

Likewise, if the trace $g'g'$ is plotted so that the point i'_o falls on the line I_o-II_o, the ray II_o-t'_o (the distance $O'_ot'_o$ being made equal to $O'_ot'_o$, Fig. 523*b*), extended indefinitely, also contains the position of the point T_o. The intersection of the rays I_ot_o and $II_ot'_o$ locates the point T_o.

The ground lines gg and $g'g'$ do not appear on the photograph, and accordingly the points t and t' can not be projected to such lines; but the horizon lines hh and $h'h'$ are parallel to the respective ground lines. Hence, the distance $Ot_1 = O_ot_o$, $Oi_1 = O_oi_o$, etc. (Fig. 523*b*). Thus the horizontal distance from the principal line to a pictured point t may be measured along the horizon line as well as along the ground line.

524. Drawing the Picture Trace.—In practice, the ground lines gg and $g'g'$ are termed *picture traces*, and will be so designated in the following remarks. The procedure of drawing, in its proper position on the plan view, the picture trace of any photograph is

termed *orienting* the picture trace. Various methods are used, a common one being as follows: The distance $O'_o i'_o$ (Fig. 523*b*) is scaled from the photograph; the distance f is known and is always equal to the actual focal length of the photograph,[1] regardless of the scale to which the map is drawn. Hence, the distance $i'_o II_o$ (Fig. 523*c*) is given by the relation,

$$i'_o II_o = \sqrt{(i'_o O'_o)^2 + f^2}$$

This distance is laid off from the point II_o along the line II_o-I_o, and on this length as a diameter, a semicircle is constructed. From the point II_o, an arc of radius f is swung to intersect the semicircle at O'_o. The line $g'g'$ drawn through the points i'_o and O'_o fixes the picture trace in its correct position.

In a similar manner, the trace gg is drawn for the station I_o.

We may consider the principal point O of a photograph as the origin of a system of rectangular coordinates, of which the principal line is the Y-axis, and the horizon line is the X-axis; then with the picture traces correctly drawn, the plotting of points proceeds as follows: The abscissa of any point in one photograph is measured along its corresponding picture trace; likewise, the abscissa of the same point in the other photograph is measured along its picture trace, as, for example, $O_o t_o$ and $O'_o t'_o$ (Figs. 523*b* and *c*). The ray $I_o t_o$ is drawn and extended indefinitely; also the ray $II_o t'_o$ is drawn to intersect the ray $I_o t_o$ and thus to locate the point T_o on the map. The positions of other points in the photographs are plotted in a similar manner.

525. Contour Lines.—If contours are to be drawn, the elevations of many points, such as T, may be found in the following manner: In Fig. 523*a* we have the similar triangles Itt_1 and ITH, the point H representing the point in the horizon plane vertically above T. Hence, the distance HT represents the elevation of the point T below the horizon plane of the station I. In Fig. 523*c* the right triangle $I_o t_o t''$ is drawn as an auxiliary construction, the distance $I_o t_o$ being equal to It_1 in Fig. 523*a* and the distance $t_o t''$ being made equal to $t_1 t$ in Fig. 523*a*; this is as if the triangle $It_1 t$ were rotated about the axis It_1 (Fig. 523*a*) into the horizontal plane and shown thus in Fig. 523*c*. The line $I_o t''$ is extended to T'' (Fig. 523*c*); the distance $T_o T''$ then represents to scale the elevation of the object T below the horizon line. Likewise, the elevation of any other

[1] It is the focal length of the photograph, rather than that of the lens, since there is nearly always a shrinkage of photographic paper upon drying after the developing and washing processes. This, of course, causes the focal length of the photograph to be less than that of the lens which formed the photographic image.

object pictured in the photograph can be determined by a similar construction. There are graphical devices for use in the drafting room by means of which this procedure may be facilitated.

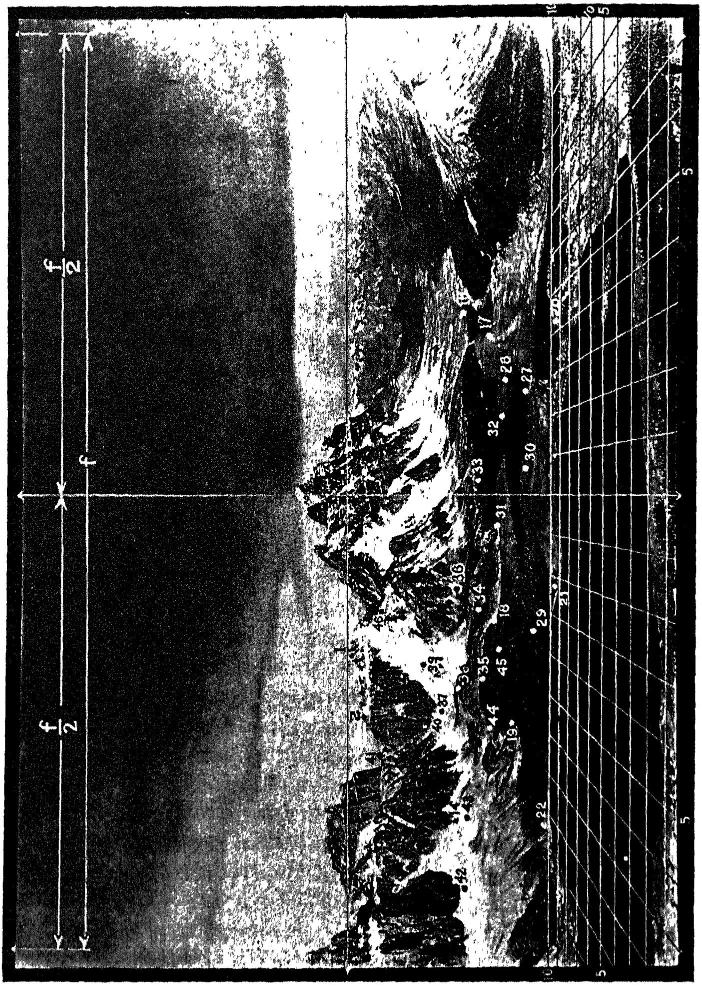


FIG. 526a.—Photographic view No. 3.

526. The Perspectometer.—It is evident from Fig. 523c that had the distance T_oT'' been known, it being (to scale) the elevation of the object T below the horizon H, the distance I_oT_o could have been determined without the use of the intersecting ray II_oT_o. In other words, if the elevation of an object which appears in any photo-

graph is known, its position can be plotted without reference to any other picture. This condition has a particularly valuable application in plotting shore lines of water surfaces, for if the elevation of the surface of a lake is known, its shore line appearing in a photograph can be completely mapped, independently of other data. In practice, a special device called the *perspectometer* is used for such features, but space here does not permit a complete description of its

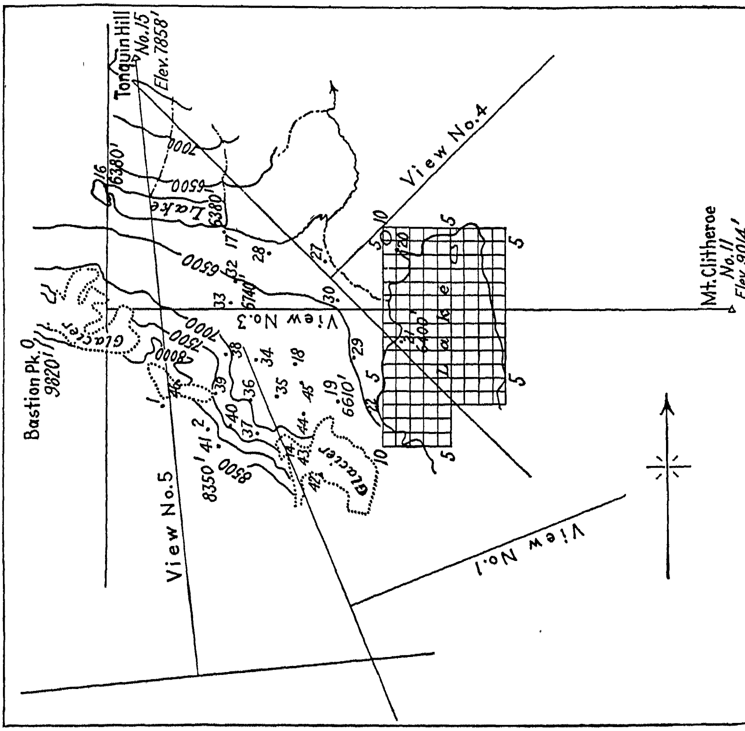


FIG. 526*b*.—Portions of map constructed from photographic views.

construction. Briefly stated, it consists of a grid system drawn according to the principles of perspective and in such manner that, for a photograph taken with a camera of known focal length, the size of each square in the picture represents a square area of known size on the map. Hence, if a system of corresponding rectangular squares is drawn upon the map, the features in the photograph may be readily transferred to the map. Figures 526*a* and 526*b* illustrate the principles described above and show a portion of a map constructed from photographic views, and a reproduction of view 3.

These figures are adapted from a bulletin of the Topographical Survey of Canada (Ref. 6, p. 820).

Figure 526*a* illustrates the use of the perspectometer by means of which the lake is mapped in Fig. 526*b*. The positions of the notches which fix the principal and horizon lines of the picture are shown, as are also the f and $\frac{f}{2}$ distances. Only a few of the points are shown which were used in plotting the map and the contour lines, and it is evident that many more points could be chosen in the view for use in plotting. To simplify the map in this illustration, only the 500-ft. contour lines are drawn. In the original publication the contour interval is 100 ft., and an excellent map to the scale of $\frac{1}{40,000}$ is there produced.

527. Surveying Cameras.—The instruments used in terrestrial photographic surveying are either *phototheodolites* or *surveying cameras.*

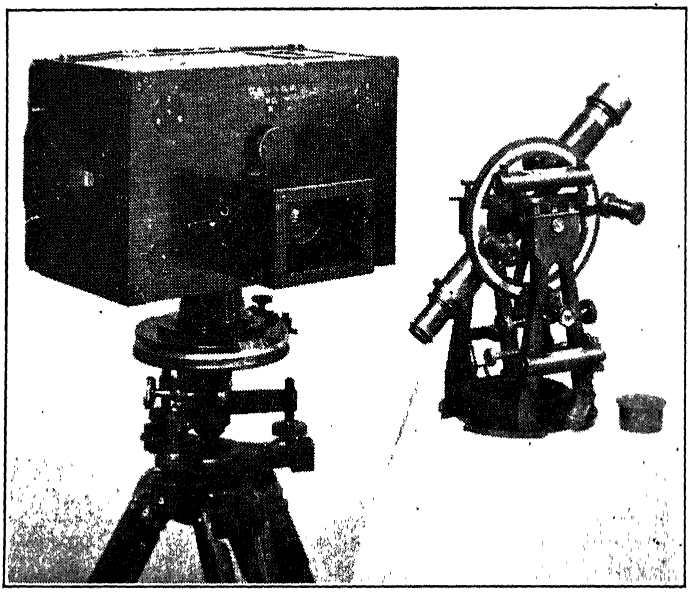


Fig. 527.—Phototopographic camera with theodolite.

The phototheodolite combines the horizontal plates and vertical arc for angle measurements with the camera. Numerous designs have been used in practice, but all are complex and the adjustments are subject to disturbance unless the instrument is handled with care.

The surveying camera is designed in such manner that the camera and the transit are in separate parts, but both are arranged to operate on the same tripod. Because all the important photographic surveys

on the American continent have been conducted in isolated regions where the equipment is subject to rough handling in transport, it has been found desirable to use the surveying camera and transit in combination, rather than the phototheodolite which has had extensive use in Europe.

In Fig. 527 there is shown the camera as designed and used by the U. S. Coast and Geodetic Survey. For a complete description see Ref. 11, p. 820. Briefly described, the camera consists of a metal box housed within a mahogany frame, the whole being supported on a truncated aluminum cone which is secured by means of capstan-screws to the vernier plate of the horizontal motion. The transit alidade, consisting of the standards, vertical arc, and telescope, rests on a base plate which may also be secured to the vernier plate of the lower motion by the same capstan-screws that are used to hold the camera in position. Hence, the camera and transit alidade are interchangeable. To the outside of the metal frame of the camera there are attached two pairs of level tubes, one pair for use when the camera is placed with its longer edge horizontal, and the other pair for use when it is placed with the longer edge vertical. These tubes are viewed through openings cut in the wood box frame. The lens is designed to operate with a fixed focus and is shaded by a projecting hood.

Two other surveying cameras should be mentioned: Dr. Deville's camera (Ref. 6, p. 820), used extensively with excellent results by the Topographical Survey of Canada, to which the Coast Survey type is similar; and the panoramic camera, used successfully in Alaska by the United States Geological Survey (Ref. 2).

The introduction of photographic surveying methods and the design of the cameras first used on the American continent are due to the work of Dr. E. Deville of the Topographical Survey of Canada.

528. To Fix Principal Point, Horizon Line, and Principal Line.— It is necessary in terrestrial photographic mapping that the positions of the principal point, the horizon line, and the principal line be indicated on the photographs. This is accomplished by marks, usually notches cut into the frame which holds the plate, with the result that these marks appear in each photograph. To determine the positions of the notches the following procedure is used: The camera is set up and carefully leveled at a station where many definite points are visible on or near the horizon line, and an exposure is taken. The camera is then rotated through a right angle about the optical axis and is carefully leveled, and a second exposure is taken. With these two photographic prints at hand, a transit is set up at the camera station, and the line of sight is directed along the horizon

line to detect points on this line which appear near the edges of each of the two prints.

Let the points A and B (Fig. 528) represent two horizon points thus located on the photograph with the camera in its first position. The principal point will then lie on the straight line connecting these points. Let C and D represent two horizon points on the photograph taken with the camera in its second position. The principal point will lie on the straight line connecting these points. The positions of points on the second exposure may now be transferred to the first by means of tracing paper or by proper measurements, and the points C and D may thus be located on the first exposure. The point of intersection of the lines AB and CD fixes the position of the principal point O. This point is at a distance x from the left edge of the camera plate, and at a distance y from the upper edge. Accordingly, the distance x is laid off precisely along the top and bottom of the camera frame and the points R and S are marked, either by pinholes or by notches cut into the frame. Likewise, the distance y is laid off along the left- and right-hand edges and the points T and U are fixed. All photographs will now bear on the dark edges of the prints these four marks, and the line drawn on the photographs between the notches T and U will indicate the horizon line. Likewise, the line connecting the notches R and S will indicate the principal line, and the intersection of the two lines will indicate the principal point O.

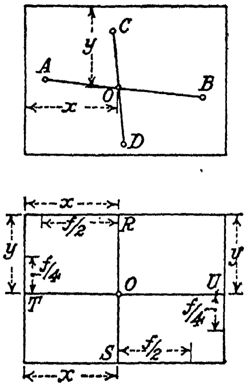


FIG. 528.—Fixing the principal lines on the camera plate holder.

529. Deville's Method of Determining Focal Length.—It is essential in photographic surveying that the focal length of the camera lens be known precisely. Makers of cameras supply this information, but it is often desirable to measure the focal length. Assuming that this determination is made in connection with that of the preceding article, Deville's method is as follows: With the transit in position at the camera station, measure carefully the horizontal

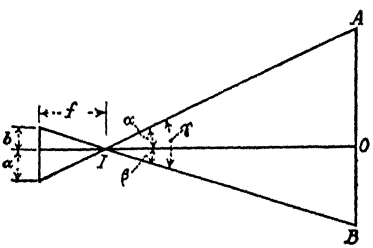


FIG. 529.—Determining the focal length.

angle γ (Fig. 529) between the two points A and B. Also, measure precisely on the print the distances a and b, being the abscissas of the points A and B on the horizon line. From the figure,

$$\tan \alpha = \frac{a}{f};\ \tan \beta = \frac{b}{f};\ \text{and}\ \tan \alpha \tan \beta = \frac{ab}{f^2}.$$

Also

$$\tan (\alpha + \beta) = \tan \gamma = \frac{\frac{a}{f} + \frac{b}{f}}{1 - \frac{ab}{f^2}}$$

from which

$$f^2 - \frac{a + b}{\tan \gamma} \cdot f - ab = 0$$

The solution of this quadratic yields the equation

$$f = \frac{a + b}{2 \tan \gamma} + \sqrt{\frac{(a + b)^2}{4 \tan^2 \gamma} + ab} \qquad (1)$$

Example: On a photographic negative the abscissas, a of point A, and b of point B, are measured and found to be 2.765 in. and 2.140 in., respectively. The horizontal angle γ between the two points A and B, as measured with a transit, is 41°07′. Determine the focal length.

From Eq. (1),

$$f = \frac{2.765 + 2.140}{2 \tan 41°07'} + \sqrt{\frac{(2.765 + 2.140)^2}{4 \tan^2 41°07'} + (2.765 \times 2.140)}$$

$$= 6.526 \text{ in.}$$

530. To Fix the Relationship between Level Tubes, Horizon Line, and Camera Plate.—To render the adjustments of the camera as rigid and permanent as possible, the level tubes are fixed to the metal frame of the camera box. Therefore, having determined the position of the horizon and principal lines by the method explained in Art. 528, it is necessary (1) to determine the position of the bubble in that level tube parallel with the camera plate, when the horizon line of the picture bisects the notches in the camera frame; and (2) to determine the position of the bubble in that tube perpendicular to the camera plate when the plate is truly vertical.

The procedure for the first determination is as follows: The camera is set up at a station for which the horizon line has been found by a transit instrument, as explained in Art. 528. The camera is first carefully leveled, then by means of the foot screws the bubble

in that tube parallel with the plate is moved to one end of its race, its position is carefully noted, and an exposure is made. Another plate is inserted, the bubble is moved to the other end of its race, its position is noted, and a second exposure is made. The first exposure will show the two notches indicated at M and N (Fig. 530), and the horizon line will be pictured as at AB. In the second exposure the notches will appear at M and N, and the horizon line will be indicated at CD. By simple proportion the position of the bubble can be calculated such that the horizon line will cut the notches at M and N.

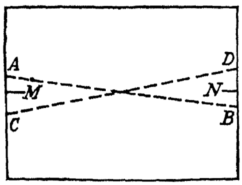


FIG. 530.—Relationship between level tubes and principal lines.

Assuming that the operation just explained was made with the long edge of the camera box horizontal, a similar procedure will be required for that position of the camera when the long edge is vertical.

The second determination may be made as follows: A level is set up near the camera and both instruments are brought to the same elevation. Through the telescope of the level, some well-illuminated and definite object on the horizontal cross-hair is selected. The ground-glass plate of the camera is placed in position, and the camera and the telescope of the level are turned until the image of the selected object is reflected from the back surface of the plate toward the telescope of the level. The camera is leveled by trial until the reflected image of the object falls on the horizontal cross-hair as indicated by sighting through the telescope of the level. The ground-glass plate is now in a vertical plane, and the position of the bubble perpendicular to it is carefully noted. This position of the bubble is maintained when exposures are made.

For other cameras having spherical bubble surfaces or adjustable level tubes, a knowledge of the adjustments of the transit and level instruments will suggest the proper modifications to be made to the procedures here described.

531. Stereophototopography; General.—The office work of constructing the map from the photographic views is great, especially where contour lines are drawn. This condition has led to the development of a method whereby the three dimensions of the terrain may be pictured and measured in the stereoscopic view. This method has the following advantages over the photographic method: (1) It eliminates the use of intersections from two widely separated camera stations (Art. 523); (2) it renders the location of points and the interpretation of features more legible and definite; and (3) by virtue of the stereoscopic magnification, the spatial relations

may be more precisely interpreted and the obstruction to view by leaves on trees and by other vegetation is minimized.

The method requires expensive apparatus of special design and also a high degree of skill on the part of the operator. For these reasons, it is doubtful if the method will have a wide application to terrestrial surveys now that aerial methods are available. Many of the principles of terrestrial stereoscopic methods, however, are applicable to the overlapping photographs of aerial surveys, and future developments will be directed to improvements in these applications. Only a brief statement of the principles involved can be given here.

532. Principles of Stereoscopic Vision.—The phenomenon of binocular or stereoscopic vision is exceedingly complex and investigators do not yet agree in its explanation; but the derivation of a few useful principles are available, and those presented here follow, in part, the excellent treatments of this subject by Higgins (Ref. 14, p. 820) and by Judge (Ref. 16).

1. *Angles of Parallax.*—In all binocular vision the spatial perception of objects in the near foreground is effected in part by the stereoscopic principle of parallax, and in part by the relative size of adjacent familiar objects. This principle is illustrated in Fig. 532*a* in which the points I and I' represent the eyes of an observer. Rays of light coming from an object Q at a great distance will be practically parallel and will impress the retinas of the two eyes at the points q and q' respectively, and the object will appear indefinitely distant. The rays coming from the two points Q_f and Q_b (representing the front and back points, respectively, of a nearby object Q) impress the retina of the left eye at q, and of the right eye at the points q_f and q_b. The object then seems to be at a definite distance, and the point Q_f appears to be nearer the observer than the point Q_b. The angles ϕ_1 and ϕ_2, subtended by the rays of vision at the points Q_f and Q_b, are the *angles of parallax* at these points, and the angle $\beta = \phi_1 - \phi_2$ is termed the *differential parallax* of the object Q.

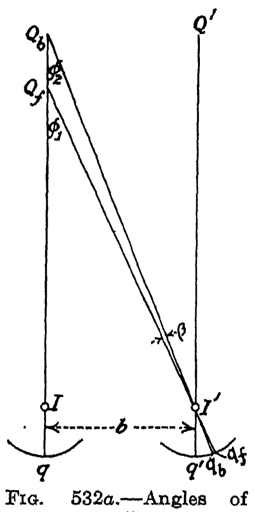


FIG. 532*a*.—Angles of parallax.

Two physiological conditions which fix the limits of stereoscopic vision are (1) that the distance between the human eyes is, on the average, about 2½ in., and (2) that the smallest parallactic angle which the eye can sense is, on the average, about 20″ of arc. Accord-

ingly, if we apply these limiting values to the conditions illustrated in Fig. 532*a*, it may readily be computed that the angle ϕ_1 reaches the limiting value of 20″ when the distance IQ_f becomes approximately 2,100 ft. Beyond that distance the phenomenon of stereoscopic vision becomes inoperative and distances are judged by other optical principles, such as the comparative sizes of objects, the effects of light and shade, etc. Likewise, if the distance between the front and back of an object is so small, in comparison with the

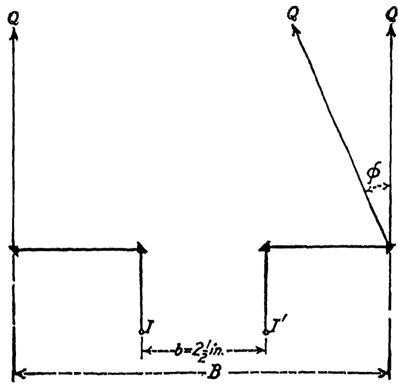


FIG. 532*b*.—Magnification of angles of parallax.

distance to the object, as to render the differential parallactic angle less than 20″, no visual evidence of solidity or depth can be perceived by stereoscopic vision, though it may be evident by other means, as stated above.

2. *Magnification of Angles of Parallax.*—It has been found that the range of stereoscopic vision can be increased by increasing the angles of parallax. This may be accomplished either by increasing (apparently) the base distance *b* or by magnification of the field of view. The usual means of increasing the effective base *b* is by the use of prisms arranged somewhat as shown in Fig. 532*b*. By this means the angle of parallax is increased by an amount equal to the ratio of $\frac{B}{b}$, and similarly for all parallactic angles in the field of vision. Also, if in the rays of vision thus arranged a system of lenses be introduced to magnify the objects sighted, a corresponding increase in the parallactic angles is effected. Thus, if by prisms the natural base of vision (2½ in.) is increased three times, and if a system of lenses is used to magnify the field four times, the effect of stereoscopic vision will be increased $3 \times 4 = 12$ times. Both of these means are used in many binocular field glasses now on the market.

3. *Law of Stereoscopic Perception.*—A third principle of stereoscopic vision is that the degree of stereoscopic perception varies as the square of the distance; *i.e.*, the precision of estimating the depth of objects decreases with the square of the distance to the object sighted.

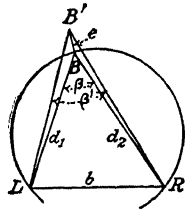


FIG. 532*c*.—Stereoscopic perception.

In Fig. 532*c* let L and R represent the positions of the left and right eyes respectively, separated by the distance b; let an object be observed at B, and a circle constructed through the three points L, B, and R. Suppose the object B to have a depth dimension of $BB' = e$, normal to the circle; also let the angle $LBR = \beta$ and $LB'R = \beta'$; and $-d\beta = (\beta - \beta')$ = the differential parallax. Then

$$e = -\frac{d\beta d_1 d_2}{b}$$

Since d_1 and d_2 are large compared to b, for all practical purposes $d_1 = d_2 = d$. Therefore

$$e = -\frac{d\beta d^2}{b} \tag{2}$$

which shows that the differential parallactic angle $-d\beta$ decreases with the square of the distance d to the object.

533. Applications of Stereophotography to Terrestrial Surveying.—In stereophotographic surveying, two plates accurately fixed in the same vertical plane are exposed, one at each end of a base line. When the resultant prints are placed in the stereoscope, the terrain appears vividly in its spatial relations. The surveyor then has the iconometrical problem of drawing the map from the three-dimensional view, which procedure is explained below. It is necessary to limit the length of the base line so that most of the terrain pictured in each photograph will appear in the stereoscopic view. In constructing the map of the area viewed from a given base line, the field location of but one of the two camera stations is required.

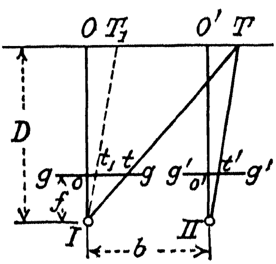


FIG. 533.—Stereophotography.

The principle of the stereoscopic method, as applied to terrestrial surveys, is illustrated in Fig. 533 where are shown two camera stations I and II; two picture traces gg and $g'g'$ at a focal distance f from each of the two stations; the central rays IO, IIO'; and an object

T at a distance D, being pictured on the trace gg at t and on the trace $g'g'$ at t'.

Let the line It_1T_1 be drawn parallel to $IIt'T$. Then in the similar triangles It_1t and IT_1T, we have the relation that $\frac{T_1T}{t_1t} = \frac{D}{f}$. But $t_1t = ot - ot_1$, which equals the displacement of the pictured position of the object T in the two stereoscopic views; this is termed the parallactic displacement p. Hence, from the proportion given above, since $T_1T = b$, we have the equation,

$$D = \frac{fb}{p}$$

Stated as a rule, *the distance to any object is determined by the product of the focal length of the camera and the length of the base between stations, divided by the parallactic displacement.* The values f and b are constant for any stereoscopic pair of photographs, hence the quantity p is a measure of the distance to any object in the stereoscopic view.

534. Stereocomparison.—The method here described was first suggested by Deville and was later applied by Pulfrich of the firm of Carl Zeiss of Jena. The Brock stereocomparator is shown in Fig. 551*a*.

Figure 534 is a diagrammatic representation of the method of measuring the three coordinates x, y, and p, necessary to determine the position, relative to the camera station, of any point in the field of view. On the frame MM, two photographs, composing a stereoscopic pair, are placed in position with the principal points at the points O and O'. From the frame NN, two pointers W and W' are projected into the field of view and in contact with the photographs. The pointer W' can be moved (translated) with respect to W, the amount of the movement being registered on the P-scale. When W' is fixed with respect to W, both pointers can be moved in a right or left direction, the amount of the movement being recorded on the X-scale; also the pointer frame and table NN can be moved up or down, the amount of movement being registered on the Y-scale.

Of course, the two photographs, when viewed through a stereoscope, appear as a landscape in its spatial relations. These relations are usually magnified, as explained in preceding articles. Now let it be supposed that the two pointers W and W' are brought into position at the points O and O', respectively. The appearance in the stereoscopic view will be that of a single pointer W, called the *floating mark*, at an indefinite distance behind the landscape. Let the crest of a hill A be represented in the two photographs by a and a'. By the X and Y motions let the pointers be moved to the points a

and O'', the corresponding values being measured on the X- and Y-scales. The floating mark will now appear to be in the direction of the top of the hill A, but indefinitely far behind it. If the pointer W' is now moved to the left, the floating mark will appear to move forward in the landscape until it apparently touches the ground at A.

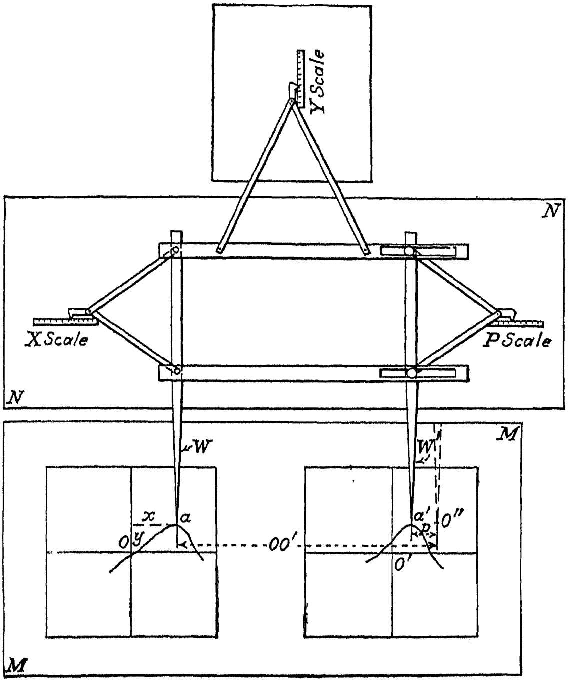


FIG. 534.—Stereocomparison.

The pointer W' has now moved from O'' to a', and the distance p is a measure of the parallactic displacement of point A. The values of the X, Y, and P coordinates are measured, and the position of the point A is thus determined.

535. Iconometric Interpretation of the Stereoscopic View.

1. *Plotting Points.*—The method of constructing the plan or map from the stereoscopic view may be explained by reference to Fig. 535, where is represented the plotted position of the camera station at I_o, the picture trace gg of the left-hand photograph of Fig. 534, and f the actual focal length of the lens. The distance x laid off on the picture trace fixes the direction of the ray I_oa_o which must contain the plan view of the object A. As shown in Art. 533, $D = \frac{fb}{p}$.

Therefore the point where the perpendicular erected at k intersects the ray I_oa_o locates the position of the object on the map at A_o. By reference to Fig. 523a it is seen that the elevation of the point T below the horizon plane was determined by the measurement of the distance tt_1 on the photograph. Likewise in Fig. 535, the elevation of the point A above the horizon plane of the camera station I is determined by means of the value of the distance y, laid off perpendicular to the ray I_oa_o at the point a_o. The perpendicular erected at the point A_o intersects the ray I_oa at the point A, and the distance A_oA (to the scale of the map) represents the elevation of the object A above the horizon plane of the camera.

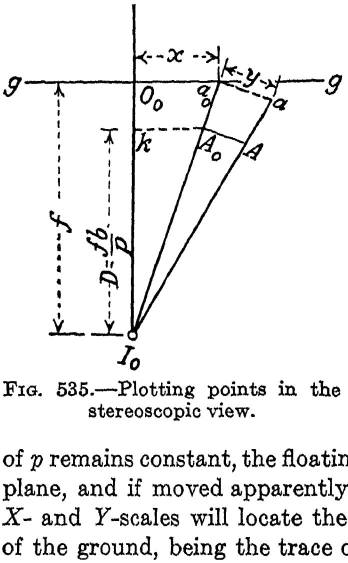


FIG. 535.—Plotting points in the stereoscopic view.

Thus it is seen that the values of the coordinates x, p, and y, respectively, determine the direction (azimuth), horizontal distance, and elevation of the object.

2. *Plotting Profile and Contour Lines.*—Since f and b will be constants for a given pair of views, it is evident, from the equation given in the preceding article, that if the value of p remains constant, the floating mark will indicate points in a frontal plane, and if moved apparently along the surface of the ground, the X- and Y-scales will locate the positions of points to yield a profile of the ground, being the trace of the given frontal plane.

If the Y-ordinate remains constant, then the floating mark will apparently move in a horizontal plane and if moved apparently in contact with the ground, it will trace out a contour line whose position on the map is located by the readings of the X- and P-scales. Also, if the traces of the pointers W and W' (Fig. 534) are marked on the photographs, the successive contour lines will appear in the stereoscopic view, and, of course, these will appear also in perspective on each photographic view.

AERIAL SURVEYS

536. General.—Much misapprehension exists as to the character of aerial photographic maps. The uninformed are apt to think that the only requirements to secure an aerial map for any purpose are to make from an airplane a sufficient number of exposures, to print the photographs, and to match the edges. Such a mosaic

would, of course, possess certain values, as stated below, but it would not serve the purposes of an engineering or a topographic map. It is true that vertical aerial photographs approximate the orthographic projection of the terrain to a far greater extent than do oblique or terrestrial views, but they are perspective views nevertheless, and because they are taken from a rapidly moving airplane, the position and elevation of which are not definitely known, they are not subject to the definite treatment which can be given terrestrial photographs. The distinction between mosaics and maps is important, the latter being of greater concern to the topographer and the engineer, and presenting the greater difficulties.

Mention should be made of the notable progress which has been made in this field in other countries, especially in Germany. There instruments and methods have been developed which permit the use either of terrestrial views or of oblique or vertical aerial views, and which construct the map directly from the stereoscopic view of the landscape. The methods are too complex to receive more than brief mention here, and what is said in the following articles pertains for the most part to the use of vertical aerial views and to methods which require but little special equipment for constructing the map from the photographs.

The process of aerial surveying consists of four distinct parts: (1) photography, (2) flying, (3) ground control, and (4) mapping. The first two parts are not within the qualifications or training of the engineer, and these divisions of the work should be executed by the photographer and the pilot. The ground control and the mapping are executed by the engineer, and accordingly these will be the principal subjects of the discussion in the following pages of this chapter, although enough information about the photography and the flying will be given to enable the engineer intelligently to plan and execute his survey as a whole.

For mapping use, the aerial camera is mounted, as nearly as may be, with its optical axis vertical; accordingly, the principal point of the picture, called the *nadir point*, is always near the point vertically beneath the camera at the instant of exposure. It is seldom possible to determine the nadir point in any picture because of the tilted position of the plane. Hence, the principal point rather than the nadir point is most commonly considered in any discussion of aerial mapping methods.

Aerial surveys are accompanied by ground-control measurements, to locate the positions of primary stations on the map. In the procedure of mapping, to be explained below, secondary control points are located on the map by office methods. This control is termed

map control, and is entirely distinct, in the manner in which it is determined, from the ground control.

Aerial photographs are taken so as to provide a considerable amount of overlap between adjacent views, in order that the pictures may be correctly oriented with respect to each other, and in order that stereoscopic vision may be afforded. The overlap, in the direction of flight, of two adjacent views is termed *end lap;* and the overlap, between two adjacent flight series, of two adjacent views is termed *side lap.*

537. Cameras.—The camera instruments and appurtenances are of the highest grade, the workmanship being comparable in every way with that of the transit instrument. Ordinarily, films are employed, rather than plates.

The principal parts of the film camera are: (1) a high-grade, anastigmat lens; (2) either a between-the-lens or a focal-plane shutter, with its speed adjustable approximately from $1/50$ to $1/250$ sec.; (3) a single circular level, or two level tubes at right angles with each other, mounted on the back of the camera frame; (4) a device actuated pneumatically or by springs to exert pressure on the film against a glass or a metal plate, to insure flatness of the film at the instant of exposure; and (5) a view finder to determine the true direction of flight (*i.e.*, the crab of the plane) and the ground speed, from which to calculate the correct timing of the exposures.

The principal point of the picture is indicated by notches cut in the frame of the camera, or by lines photographed near the edges of the picture. If these marks are not supplied by the maker, they can be added by adapting the methods given in Art. 528 for the terrestrial camera.

537a. Lens.—The uses made of photographs in map construction impose rigorous requirements in the quality of the lens of the camera. These are being successfully met in recent manufacture to the extent that distortion in the resultant pictures due to imperfections in the lens is reduced to about one part in one thousand. This value is much less than the size of errors due to other sources and permits enlargements of two or three diameters without serious resultant errors.

537b. Film.—The film is approximately 9 in. wide by 75 ft. long, and yields about 100 exposures $7\frac{1}{2}$ by 9 in. in size. The camera is supplied with a magazine, spools, and winding apparatus, driven either by wind motor or electric motor. The film is wound and exposed automatically during flight.

Hypersensitized panchromatic film, when used with proper color filters, eliminates the effect of haze to such an extent as to permit

exposures to be made whenever the sun has an altitude of 15° or more. This is a distinct advantage in the high latitudes during the winter months. The shutter speeds may be regulated effectively to prevent blur.

537c. Kinds of Cameras.—Three makes of cameras (Brock, Bagley, and Fairchild) are now in use for mapping purposes and others are available for the more general uses of aerial photography.

1. *Brock Camera.*—This camera (Fig. 537*a*) has been designed to yield photographs suitable for the special process of map construction employed by Brock and Weymouth (see Art. 551*a*). The principal feature of the camera is the provision for the use of glass-plate instead of film negatives. The emulsion of a film negative is not always perfectly fixed to the celluloid base, and in the process of developing and

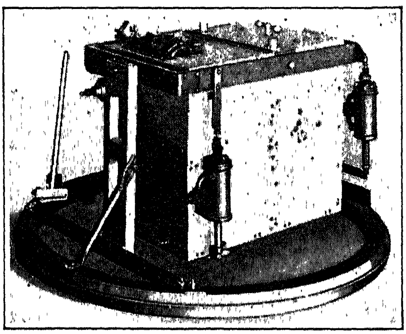


Fig. 537*a*.—Brock aerial camera.

fixing the negative, the emulsion may creep slightly in places. This condition causes distortion in the photographs, which, unless proper care is exercised, may introduce serious errors in the map. To eliminate this source of error and to provide a picture plane (negative) which shall be as nearly as possible a plane surface accurately perpendicular to the optical axis, glass-plate negatives are used. These are more nearly perfect in their perspective qualities than film negatives and make possible certain mechanical restitutional processes such as rephotography, projections, enlargements, etc., of which the film negatives would not be capable.

A suitable arrangement for automatically exposing the plates in flight is provided. A disadvantage of the instrument is the weight of the glass plates, this factor being an important one where long flights and high altitudes are necessary.

2. *Bagley or Air Corps Camera.*—This camera, shown in Fig. 537*b*, is equipped with five lenses so arranged that when mounted in the airplane one axis is vertical and the other four are inclined at an

angle of 35° with the vertical, one to the right, one to the left, one backward and one forward. All lenses operate simultaneously, and by means of a specially designed transforming camera the four wing pictures are projected into the picture plane of the central lens. The result from each 5-lens exposure is a single picture, in every way similar to one taken with a single wide-angle lens.

The great advantage of this camera is the wide extent in all directions of the picture, the ground distance included in the photograph being about three times the height of the plane. This wide

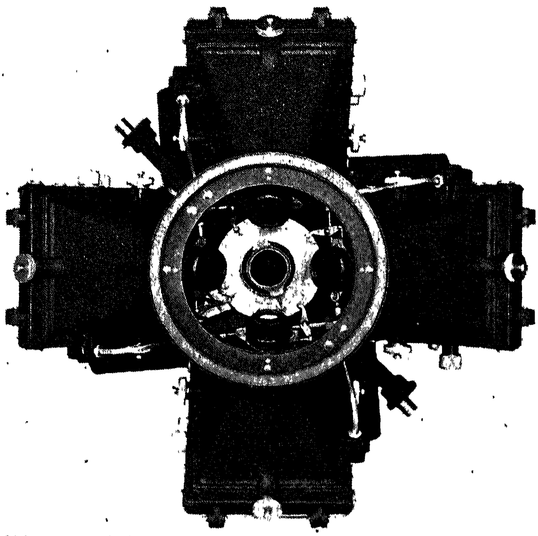


Fig. 537b.—The Bagley or Air Corps camera.

angle of vision increases the angles of parallax (Art. 532) for points near the edge, thus increasing the precision with which elevations of points can be determined from the photograph. The scope of the picture also permits the use of a long base line from which to establish the secondary control for plotting the map (Art. 549). The disadvantages of this type of camera are the need for a large amount of special equipment, and the fact that the short focal length, 6½ in., limits the use of the camera to relatively small-scale maps.

The Bagley camera is best adapted to surveys of large areas to be mapped to a small scale such as 1 in. equals 2000 ft. or more. Principally because of its breadth of view, it is most economical in operation for small-scale maps. By means of the large parallactic angles involved, elevations of points for use in contour construction

can be determined with probable errors ranging between 7 and 20 ft. (Ref. 1, p. 820).

3 *Fairchild Camera.*—A photograph of this camera is shown in Fig. 537*c*. A film is used, each exposure yielding a negative approximately 7 by 9 in. The camera is provided with an inter-

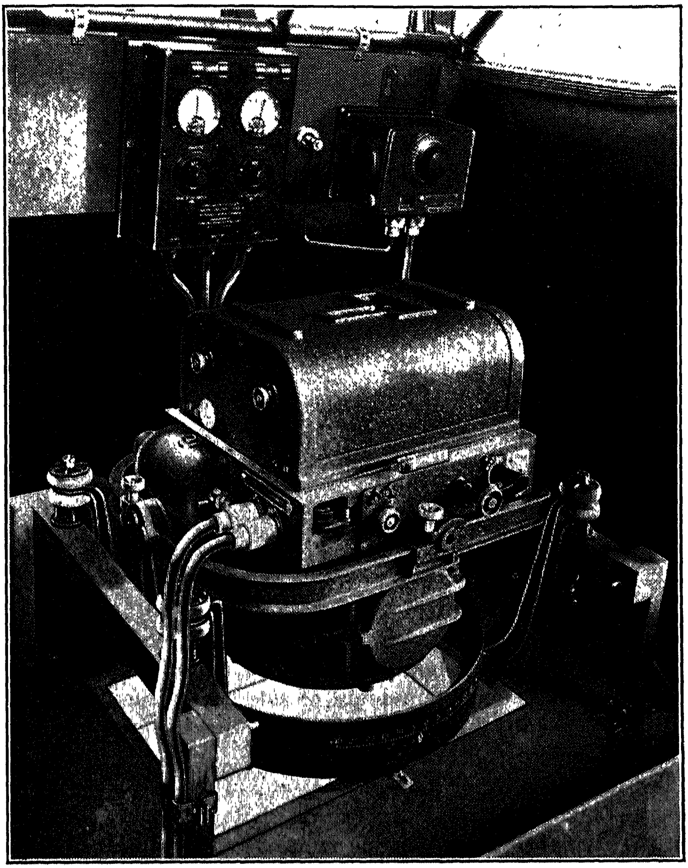


Fig. 537*c*.—Fairchild aerial camera.

changeable set of cones which operate with focal lengths of 8, 10, 12, 15, and 20 in. It is mounted with the optical axis vertical for most mapping purposes, but by a separate interchangeable mechanism it may also be mounted for oblique views (see Fig. 552*b*).

538. Camera Operation.—During flight, the operation of the camera requires careful attention to the following matters, in addi-

tion to a proper knowledge of the details of manipulating the particular camera in use: (1) to determine, before entering upon the area to be mapped, the ground speed of the camera and the area to be included in each picture, and thus to regulate the time interval between exposures so as to secure the proper amount of end lap between successive photographs; (2) to insure, as nearly as possible, a truly vertical position of the optical axis at the instant of exposure; (3) to orient the camera so as to eliminate the effect of a crabbed

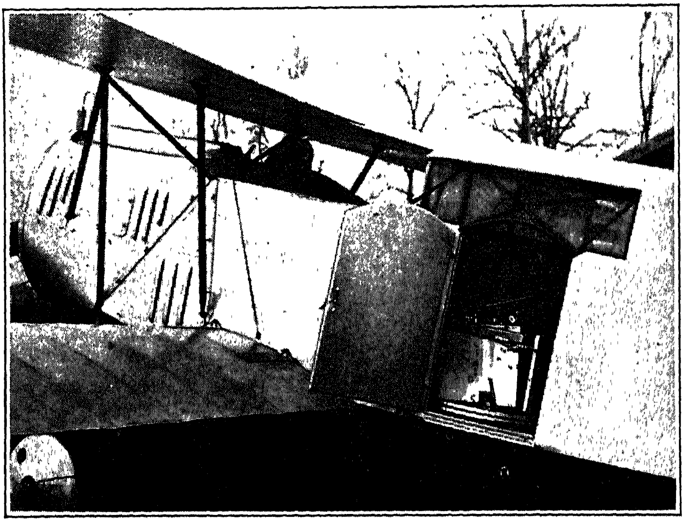


Fig. 538.—Aerial camera in airplane.

position of the airplane when cross winds are encountered; and (4) to give the pilot information to enable him to fly a proper course and to secure an adequate amount of side lap on successive flights across the terrain. Fig. 538 shows the Brock aerial camera in position for flight.

The information listed under (1), (2), and (3) is secured by means of a practice flight over the course to be photographed. During this flight the pilot throttles his engine sufficiently to maintain a constant speed and determines the amount of crab necessary to keep the airplane on the desired course. Having made these adjustments, he signals to the observer, who by use of the view finder, determines the *angle of crab* of the plane. He then rotates the camera to eliminate the effect of crab, and determines the time interval for exposures, after which he levels the camera and signals the pilot that he is ready. The pilot brings the ship about and comes up to the point of beginning, where the photographic work is begun.

It is practically impossible for a pilot to fly a course, the direction of which will be satisfactory for mapping purposes, without a guide map and ground objects to guide him. The usual procedure is to outline on the best available map the area to be covered, and to project the flight courses upon the map before the field work is begun.

If the methods of surveying employed require field work subsequent to the photographic work, it is important, in order that serious delays may be avoided, that the means for developing and printing the pictures be provided at or near the field base.

538a. Number of Exposures.—The number of exposures can be readily calculated from the given conditions as to the scale fraction (Art. 544), the size of the prints, and the ground area to be covered.

Thus let l = length of print in direction of flight

w = width of print transverse to direction of flight

P_e = percentage of end lap (in direction of flight)

P_w = percentage of side lap (transverse to direction of flight)

L = net ground distance corresponding to l

W = net ground distance corresponding to w

$$S = \text{scale fraction} = \frac{\text{height of camera (feet)}}{\text{focal length (inches)}} = \frac{H}{f}$$

A = Net area of each print.

Then $L = Sl(1 - P_e)$ and $W = Sw(1 - P_w)$.

The net area of each print is $A = LW$.

Example 1: The scale fraction = 600 ft. per in.; l = 6 in.; w = 8 in.; P_e = 0.6, and P_w = 0.5. Determine L, W, and A.

$$L = 600 \times 6 \times 0.4 = 1{,}440 \text{ ft.}$$
$$W = 600 \times 8 \times 0.5 = 2{,}400 \text{ ft.}$$

and

$$A = 80 \text{ ac.} = 0.12 \text{ sq. mi.}$$

538b. Timing the Exposures.—From the considerations of the preceding article, it is only necessary to know the ground speed of the plane to calculate the interval of time between exposures. Thus if V is the ground speed in feet per second (very nearly 1.5 times the speed in miles per hour) and T is the interval between exposures,

$$T = \frac{L}{V}$$

Example 2: For L = 1,440 ft. as in example 1, and for a ground speed of 90 miles per hour, determine the time interval between exposures.

$$T = \frac{1{,}440}{1.5 \times 90} = 10.7 \text{ sec., or, say, 10 sec.}$$

539. Sources of Error.—The principal sources of error arising from the photographic instruments and methods are as follows: (1) imperfect adjustments of the camera, (2) lens distortions, (3) creeping of emulsion on the film, and (4) distortions resulting from the printing process.

An understanding of the adjustments of the aerial camera follows from a knowledge of those which apply to the terrestrial surveying camera (Arts. 528–530). They are (1) that the optical axis shall lie in a plane perpendicular to the plane of the negative plate (or plate against which the film is pressed at the instant of exposure); (2) that the plane of the plate shall be parallel to the axes of the level tubes (or axial plane of the universal level surface); and (3) that the principal point, as determined by the intersection of lines connecting the side notches or by the intersecting diagonals of the picture, shall lie on the optical axis.

The construction of the camera is such that most of these relations are permanently fixed at the factory and do not permit field adjustment. Ordinarily, the magnitude of the errors arising from these sources is much less than that from other sources, and the errors may be neglected. The photographer and the engineer, however, should be alert to detect any serious disturbance of these adjustments.

The effect of creeping emulsion on the film is not serious for the methods here described. In the Brock process, this source of error is eliminated by the use of glass plates instead of film.

Distortions in developing and printing are ordinarily not serious, but in the best work, especially if enlargements are to be made, this source of error is minimized if the position of the principal point and the focal-length distances are fixed by means similar to those previously described (Art. 528). If oblique views are to be used in mapping, these data are necessary.

540. Flying.—The character of the results achieved in aerial surveying are dependent largely on the ability of the pilot (1) to fly a straight course in varying wind current, (2) to maintain his plane at a uniform altitude above the ground, and (3) to keep his ship on an even keel. Within the limits required for mapping purposes, these are exacting conditions and can be met only by a highly skilled pilot. The following statements indicate probably the best performance in flying under present conditions (Ref. 22, p. 821), though improved aviational aids will undoubtedly yield better results in future work.

(1) A flight at 10,000 to 15,000 ft. above the ground can be held to [the desired alinement] within about ¼ mile in 20 miles (¾°), provided a good aeronautical map is available . . . A deviation of not more than 2°, or

about $\frac{3}{4}$ mile in 20 miles, will probably be possible in the near future in flights without guide maps.

(2) Changes of elevation of the airplane will not greatly exceed 100 ft. in flights over flat or rolling country.

(3) Fifty to eighty per cent of a series of photographs taken will be tilted less than 1°, only 2 to 5 per cent will be tilted more than 2°, and rarely will any photograph be tilted more than 3°.

It should be added that these tests were made at altitudes of 10,000 ft. and above. The air currents at low altitudes are full of pockets which render the results unsatisfactory when the photographs are taken at altitudes below 7,000 or 8,000 ft., and quite impossible for accurate mapping purposes when taken as low as 3,000 or 4,000 ft.

Weather conditions greatly affect the amount of time required to complete the photographic field work. It has been observed in various regions having an annual rainfall of about 30 in. that, on the average, not more than 1 day in 3 is suitable for photographic mapping, and that on about one fourth of these available days, the period suitable for good work will not exceed 3 hours (Ref. 20, p. 821). It is difficult to judge cloud formations at a distance of a few miles only, and it is evident that unless the aviator is in a position to make use of all 1, 2, and 3-hour periods, the total time of the field work will be increased about 25 per cent.

541. Ground Control. *Horizontal.*—Aerial photography, in general, has been more extensively applied to mapping comparatively large areas at intermediate or small scale than to mapping small areas at large scale. Under these conditions, control schemes of either triangulation or traverses are required, as for mapping by terrestrial methods. In aerial surveys it is necessary that control points be specially marked by white material or else be at or near objects which will show clearly on the photographs, such as crossroads, fence corners, etc. As a rule, triangulation stations on mountains or the highest hills are not as suitable as traverse stations along highways and railways, both because hilltop stations usually require special marking in order that they may be identified on the aerial photographs and because their positions are subject to large displacements on account of their height above the datum plane.

As stated more in detail in Art. 553, the errors in aerial maps amount to perhaps 1 per cent or more, and accordingly it seems that the ground-control stations will be located satisfactorily by a traverse for which the error of closure is not greater than $\frac{1}{500}$. This accuracy can be maintained or exceeded by careful transit-stadia methods (Art. 252, p. 348), or by a rapid transit-tape traverse (Art. 225, p. 306).

The number and distribution of the ground-control stations will vary with the scale of the map, but it has been found that when the radial method of map control is used, a ground-control point should appear at intervals of not more than eight photographs in a flight series. For small-scale mapping, *i.e.*, 1 in. = 1,000 ft. or more, the ground distances between successive photographs would be 3,000 ft. or more, thus making the distance between the required ground-control points $8 \times 3{,}000 = 24{,}000$ ft. or roughly $4\frac{1}{2}$ mi. The distance between adjacent flight series under these conditions would be about 5,000 ft., or roughly 1 mi. In a tract of many square miles it is evident that one ground-control point per 3 sq. mi. of territory will be sufficient.

For intermediate-scale maps it is found that best results are secured by a ground traverse running near the center line of a flight series, thus determining accurately a central point in each photograph, and requiring about four ground-control points per square mile.

The values given above indicate the probable limits within which satisfactory results may be secured, but the numerous factors involved prevent the assignment of more definite values.

Vertical.—By the methods of mapping herein described, the contours are located by field parties, as in terrestrial work. Hence vertical control is established in the same manner as in ordinary topographic surveying, described in Chap. XXV.

542. Applications of Stereoscopic Vision to Aerial Surveying.—When suitably arranged, two aerial views which overlap provide a stereoscopic view of the region appearing in the views. A simple and suitable arrangement of mirrors as shown in Fig. 542 may be used for this purpose. The Pellin stereoscope, of this type, is a convenient instrument which may be purchased in the market.

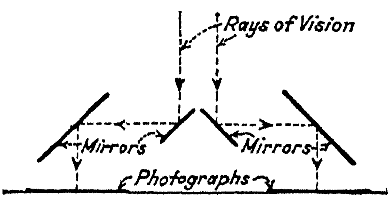


FIG. 542.—Mirror stereoscope.

Use is made of stereoscopic vision (1) when contour lines are drawn either on photographs in the field or on map projections constructed from photographic data, the stereoscopic view being used to guide and aid the topographer in his interpretation of the ground forms; and (2) if the photographs are reprojected and rephotographed so as, in effect, to bring the two picture planes of the stereoscopic pair into the same horizontal plane, when they may be treated in a manner similar to terrestrial views as described in Art. 533.

It has already been shown that the stereoscopic perception of depth in a view decreases with the square of the distance, or $e = -d\beta\frac{d^2}{b}$ (Art. 532) and, accordingly, no spatial relations could be perceived in aerial views taken at a distance $b = 2\frac{1}{2}$ in. apart. But it has also been stated that the stereoscopic effect increases with the ratio of $\frac{B}{b}$, in which B is the distance between photographs and b is the distance between the observer's eyes. Hence if the base b is increased so that the base $B = db$, the perception of spatial relations becomes natural and the effect is that of looking at a model of the landscape to the scale of the photograph.

Since the base B and the distance d are to be interpreted in terms of the scale fraction $\frac{H}{f}$, then to calculate the distance between aerial views which will yield natural proportions of relief in the stereoscopic view, we have the relation that $B = \frac{bH}{f}$. For example, if the height of lens is 10,000 ft., the focal length is 12 in., and b is $2\frac{1}{2}$ in.,

$$B = \frac{2\frac{1}{2} \times 10{,}000}{12} = 2{,}100 \text{ ft. (nearly)}$$

If this base is increased to, say, 4,200 ft., the proportions of relief are doubled in relation to the horizontal dimensions and an exaggerated effect of the relief is secured. Since the distance, to the scale of the photographs, between the nadir points of overlapping views is usually greater than the distance between the observer's eyes, the relief effects are enhanced in nearly all the stereoscopic views of aerial photographs.

543. Mapping.—The iconometry of aerial photographic surveying is much less accurate than that of terrestrial work because (1) the position of the airplane both in plan and in elevation is not known accurately, and (2) the optical axis of the camera can not be held in a vertical position at the instant of exposure. The character and magnitude of the resultant errors will be discussed in the following articles. The first condition mentioned is relatively unimportant, but the second condition is serious and results from the fact that in the rapidly moving plane it is impossible, by means of any control yet devised, to prevent tilt of the plane and therefore of the camera plate at the time of exposure. Level tubes are unsatisfactory because the bubble itself is subject to the accelerations of the plane. Thus, if the plane is banked while flying in the direction of a curved line, the bubbles may remain centered. Gyroscopes have been tried but have proved impracticable. Accordingly, the best practice

at the present time is to fly at relatively high altitudes where the air strata are most uniform, to keep a course as straight as possible, and to photograph the position of the bubble, or bubbles, on each picture. By this means the amount and direction of tilt can be determined, not always with precision, but with sufficient accuracy to be of value.

Because of the conditions stated, the mapping process can not be based upon such definite geometrical principles as those used in terrestrial work. Instead, a series of graphical adjustments of the data are so made as to secure the best possible results under the conditions imposed. These methods are described in the following pages.

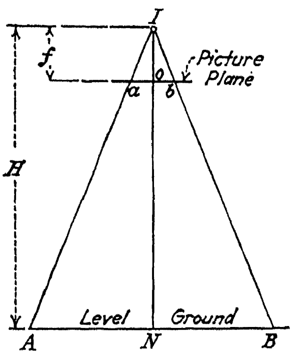


FIG. 544.—Scale fraction.

544. Scale Fraction.—In Fig. 544, by similar triangles we have the relation $\frac{H}{f} = \frac{AB}{ab}$, in which H represents the height of the camera lens and f its focal length; A and B are two ground points of equal elevation lying in a vertical plane which contains the optical axis; and a and b are the pictured positions of A and B.

If the ratio of the ground distance AB to the corresponding picture distance ab is designated as S, then $S = \frac{AB}{ab} = \frac{H}{f}$. This ratio is termed the *scale fraction* of the photograph, and since the focal length of a lens is expressed in inches and the height of lens is expressed in feet, the scale fraction is expressed in the same units. Thus, for a given photograph, if the focal length of the lens is 10 in. and the height of lens is 6,000 ft., then the scale fraction $S = \frac{6{,}000 \text{ ft.}}{10 \text{ in.}}$, or 600 ft. = 1 in. For a given lens, the scale fraction changes with and is directly proportional to the height of lens H.

In order to secure the better flying conditions of high altitudes, it is a common practice where a relatively large-scale map is desired, say 1 in. = 400 ft., to take the photographs at an altitude which yields pictures to a scale of 1 in. = 800 ft., and then to secure the desired scale by rephotographing the original negatives to an enlargement of two diameters. Such enlargements are entirely satisfactory for mapping purposes up to about three diameters, beyond which the definition becomes seriously faulty.

In Table 544 are given the variations in the scale of the photograph which correspond to a 100-ft. change in the height of lens. Thus.

TABLE 544.—SCALE VARIATIONS

Focal length, inches (f)	Change in scale for 100-ft. change in height of lens, feet per inch
8	12.5
10	10.0
12	8.3
15	6.7
20	5.0

for a camera where $f = 12$ in., and for which the altimeter shows a variation of 100 ft. in the height of lens between two successive exposures, the scale of the two photographs would differ by 8.3 ft. per in. Likewise, a difference in ground elevations of 100 ft. will be accompanied by the scale variations indicated. Thus when $f = 8$ in. the dimensions scaled along two adjacent 100-ft. contour lines will differ by 12.5 ft. per in.

545. Effects of Variant Ground Elevations (Parallax).—The errors in dimensions scaled from aerial photographs because of the variance in ground elevations are illustrated in Fig. 545*a*. In this figure, *I* and *II* represent two positions of the camera when exposures were made. The optical axis is assumed to be vertical, and therefore the principal points O_1, O_2 are projected vertically above the nadir points N_1, N_2. *A* and *B* are two ground points located on the line of flight, *A* being below the assumed datum plane *DD* and *B* being above it. The two camera stations *I* and *II* are at the same height above the datum. A' and B' are the orthogonal projections of *A* and *B* on the datum plane.

The perspective views of the points *A* and *B* on the photographs are as shown by the points where the rays *IA*, *IB*, etc., intersect the pictured position of the line of flight *hh* at a_1, a_2, b_1 and b_2. This line of flight may be considered the principal line of the photograph.

In the plan view of the photographs, these perspective views have been superimposed in such manner that the two points O_1 and O_2 are spaced a distance apart equal to the ground distance between the nadir points N_1 and N_2, divided by the scale fraction of the photographs. Thus these two photographs are correctly oriented with respect to each other.

An inspection of the two pictures shows that the points a_1' and a_2' are identical, as are also the points b_1 and b_2'. And it is observed that these are the pictured positions of the points A' and B' lying in the datum plane. It is evident, therefore, that the pictured positions of all points along the line of flight having the same elevation as the datum plane will be without error in the photographs.

But the point A is pictured at a_1 from station I, and at a_2 from II. Thus the distance a_1a_2 in the plan view of the photographs represents the total displacement of the pictured positions of the point due to the elevation $A'A$ of the point A below the datum. Similarly, b_1b_2 represents the displacement of the pictured positions of B due to the elevation $B'B$ of the point above the datum.

It may be noticed that the relative positions of the two pairs of points a_1a_2 and b_1b_2 are interchanged in the plan view because one point A is below and the other point B is above the datum.

If another datum plane were assumed having an elevation either above or below that shown in the figure, the pictured positions of the points referred to that plane would be differently placed along the principal line hh. The relationship between the pictured points, however, would remain unchanged, and the amount of change of all points would be directly proportional to the change in the scale fraction. Accordingly, in so far as the correct relationship between points plotted on the map is concerned, the elevation of the datum is immaterial. It is usually assumed to have an elevation below that of the lowest point in the terrain.

The error here described is termed the *parallax error*, reference being made to the difference in the values of the parallactic angles α and α' (Fig. 545a); but this use of the term parallax should be carefully distinguished from that which is considered in connection with stereoscopic vision.

This source of error, as it affects the plotted positions of objects not in the direction of flight, is illustrated in Fig. 546. For the time being, neglecting camera station III and its corresponding data, the figure represents two camera stations I and II, the two corresponding photographs in perspective and in plan views, and the elevation of the terrain.

The two camera stations I and II are assumed to be at the same elevation above the datum plane, hence the different positions of the pictured points a_1a_2, b_1b_2, and c_1c_2 of the objects A, B, and C, as shown in the oriented superimposed photographs I and II, are due entirely to variant ground elevations. It is evident that, from the camera station I, the point C will be observed along a ray which lies in the plane whose trace is O_1c_1. Hence, the plotted position of the ray O_1c_1 (plan view) gives the true direction but not the true distance (since the object is not in the datum plane) of the point c from the point O_1. Similarly, from the camera station II, the object C will be observed along a ray which lies in the plane whose trace is O_2c_2, and the plotted position of the ray O_2c_2 (plan view) gives the true direction but not the true distance from the point c to O_2. There-

fore, the point where the two lines O_1c_1 and O_2c_2 cross locates the true position of c with respect to both O_1 and O_2. In a similar manner, the true locations of the points a and b are found.

545a. Magnitude of Parallactic Displacement.—In Fig. 545*b* there is represented the effect of the displacement due to relief, for the case

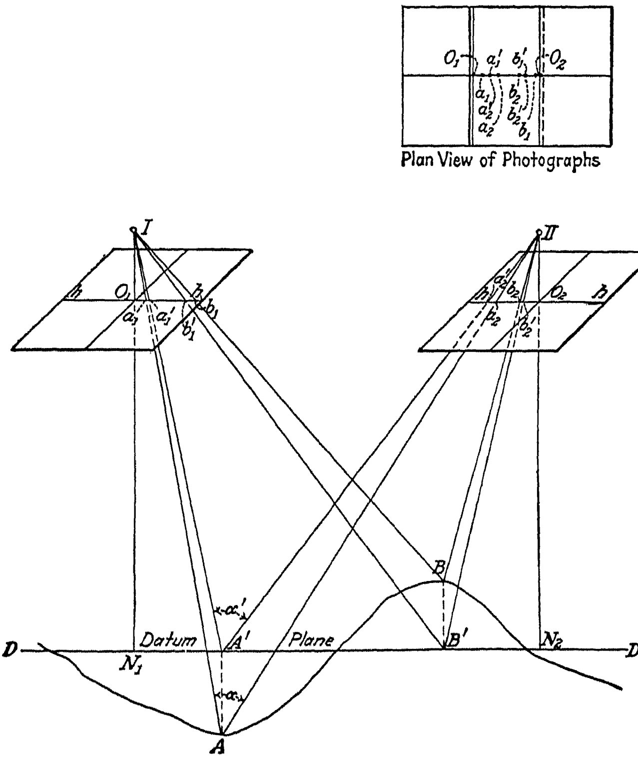


FIG. 545*a*.—Effects of ground-elevation parallax.

of a point A which lies in the vertical plane containing the two camera stations I and II. From the camera station I, the ray toward A pierces the picture plane at the point a_1, and the ray toward the point A_0, in the datum plane vertically below A, pierces the picture plane at a_0. The distance a_0a_1 represents the displacement error r_1, due to relief, for the camera station I. Similarly r_2 represents the relief error at station II. Then from the figure $\frac{O_1a_1}{f} = \tan \alpha$, in which f = focal length of the lens, O_1a_1 is the picture distance from the principal point to the pictured position of A, and α is the angle which the ray IA makes the vertical.

Then

$$\frac{r_1}{A_0A_1} = \frac{f}{H}; \text{ also } A_0A_1 = h \tan \alpha$$

Hence,

$$r_1 = \frac{hf \tan \alpha}{H}$$

or since $\frac{H}{f}$ is the scale fraction S

$$r_1 = \frac{h}{S} \tan \alpha \tag{3}$$

Example 1: Let $h = 200$ ft., $f = 12$ in., $H = 10{,}000$ ft., and $O_1a_1 = 4.30$ in. Find the parallactic displacement.

$$S = \frac{10{,}000}{12} = 833 \text{ ft. per in., and } \tan \alpha = \frac{4.30}{12} = 0.358.$$

Then

$$r_1 = \frac{200 \times 0.358}{833} = 0.086 \text{ in.}$$

The total parallactic displacement r is found by adding the two values r_1 and r_2.
Hence,

$$r = \frac{h(\tan \alpha + \tan \beta)}{S} \tag{4}$$

If the ground distance C between the nadir points N_1 and N_2 is known, then from similar triangles $IAII$, and A_1AA_2, we have $\frac{C}{A_1A_2} = \frac{H - h}{h}$, and since

$$A_1A_2 = A_2A_0 + A_0A_1 = a_2a_0\frac{H}{f} + a_0a_1\frac{H}{f} = r_2S + r_1S = rS$$

then

$$\frac{C}{rS} = \frac{H - h}{h}, \text{ or } r = \frac{hC}{S(H - h)} \tag{5}$$

Example 2: Let $C = 3{,}000$ ft., $\frac{H}{f} = S = 800$ ft. per inch, $h = 300$ ft., and $H = 7{,}500$ ft. What is the total parallactic displacement?
By Eq. (5)

$$r = \frac{300 \times 3{,}000}{800 \times 7{,}200} = 0.156 \text{ in.}$$

It is obvious that points below a given datum will be displaced toward the nadir point, just as points above the datum are displaced away from it.

The principal applications of the preceding formulas are as follows: (*a*) The parallactic displacement of any single point in a photograph along its radial from the nadir point is computed by Eq. (3); (*b*) in

adjusting adjacent photographs by the straight-line or radial methods (Arts. 549*b* and 549*c*) the central point of each photograph is practically without error, hence the correct distance between the principal points of adjacent photographs is calculated by Eq. (4); and (*c*) the displacement of points in the direction of flight, whose positions can not be located by intersections, is computed by Eq. (5).

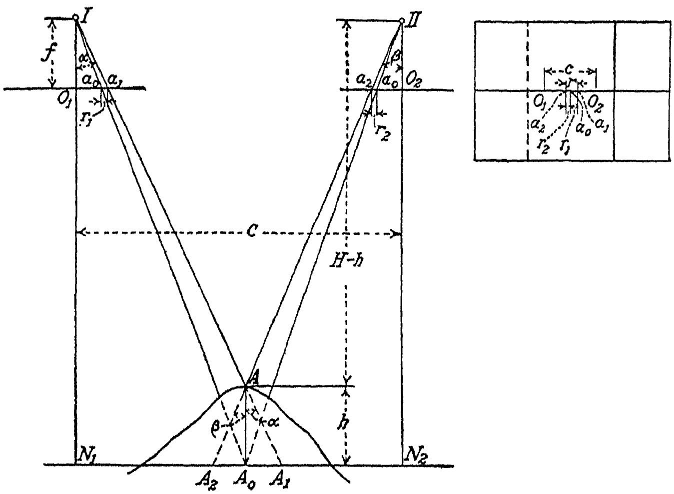


FIG. 545*b*.—Magnitude of parallactic displacement.

The statement may be repeated that the true location of the points *a* and *b* (Fig. 545*a*) is now independent of the elevations of these points above or below the datum, but is dependent only upon the scale fraction to which the distance O_1O_2, representing the ground distance N_1N_2, has been plotted.

The conditions here described are further illustrated in Fig. 545*c*, in which certain perspective effects in aerial photographs are shown. The principal line *hh* in the picture plane may be assumed to represent the direction of flight. The effect of ground-elevation parallax is shown in the pictured position of the road which leads over the hill. The plan shows the road to be straight between the points *B* and *K*, and the profile shows a vertical curve between the points *D* and *H*. The photographic perspective view shows the road straight where the profile shows a uniform grade, but deflected away from the principal point of the picture *O* for the ascending grade and deflected toward the principal point for the descending grade. For the vertical curve, also, the photographic perspective view shows a horizontal curve, concave toward the principal point *O*. Had the profile shown a sag

instead of a summit, the horizontal curve in the perspective view would have been reversed, *i.e.*, concave away from the point O.

Other perspective effects are shown also (see Art. 521*c*). Thus, the straight roadway, on level grade parallel with the direction of

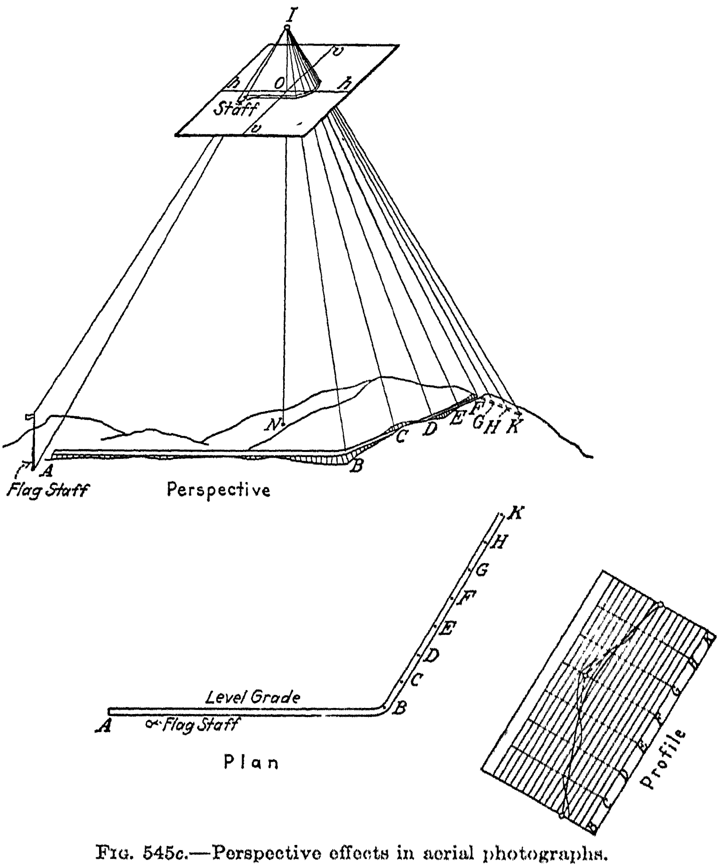


FIG. 545*c*.—Perspective effects in aerial photographs.

flight, is shown in the perspective view by straight lines parallel with the principal line *hh*. The flagstaff in the perspective view shows its top displaced outward from the principal point; also, since the staff is a vertical object normal to the picture plane, its pictured position is a line which, if prolonged, passes through the point O.

546. Effects of Variant Height of Lens.—In Fig. 546 the point *III* represents a camera station at which the optical axis is coincident with that of station *II*, but the elevation of which is higher above the

datum plane. The plan views of photographs *I* and *III* have been superimposed to the same scale fraction as that used for photographs *I* and *II*. Also, the plotted positions of the points of intersection appearing in the plan view of photographs *I* and *II* have been added for the purpose of comparison. The important result that should

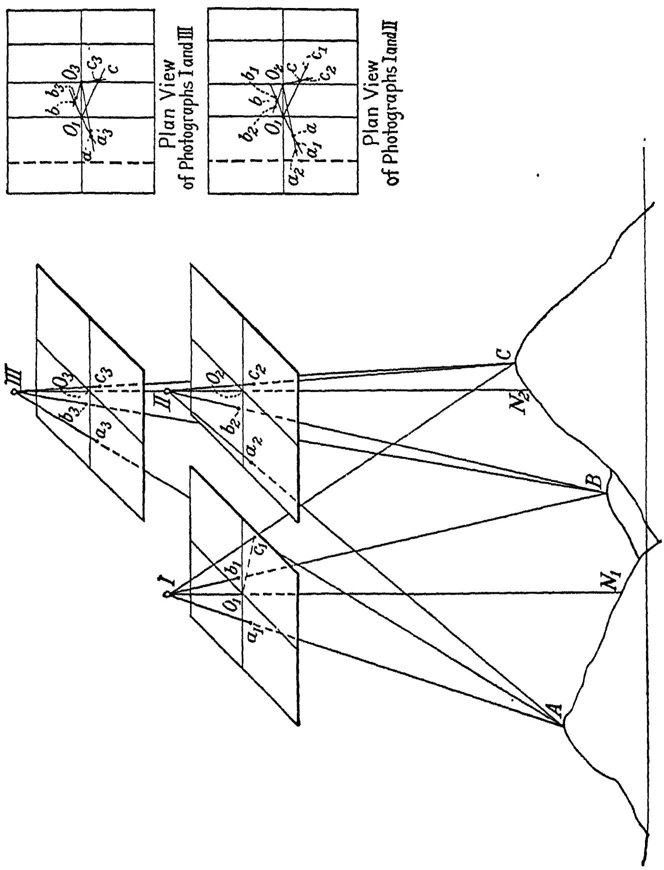


FIG. 546.—Effects of height of lens and of ground-elevation parallax.

be noted is that, for the position of the camera at station *III*, the plotted positions of the points a_3, b_3, and c_3 in each case, fall along the respective rays O_3a, O_3b, and O_3c, but nearer the point O_3 than the point of intersection of each pair of rays. This shows that the effect of increasing the elevation of the camera is exactly similar to that which would result by depressing all ground points a like amount, the datum remaining unchanged. Had camera station

III been taken as lower than station *II*, then conditions would have been reversed, and the points a_3, b_3, and c_3 would have fallen along the rays as shown, but at a distance greater in each case than the true distance O_1a, etc.

The effect of variant ground elevations is variable and becomes zero in the case of points lying in the datum plane, but the effect of variant height of lens affects every pictured point in the photograph.

547. Effects of Tilt.—The nature of the errors in the aerial photograph due to the tilted condition of the camera may be shown by the

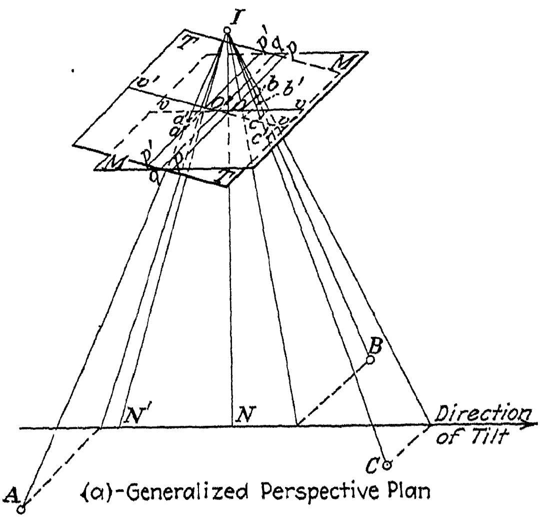


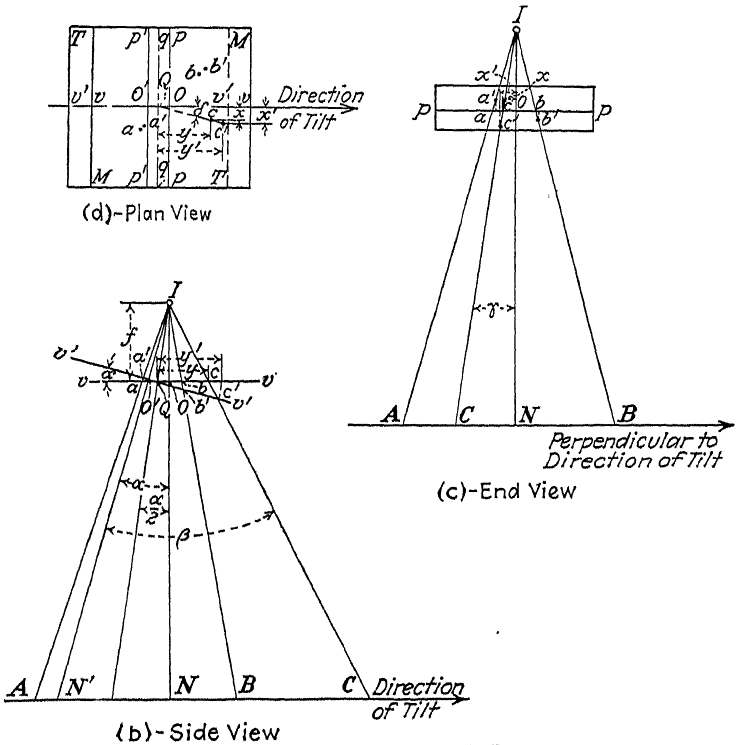
FIG. 547*a*.—Effects of tilt.

aid of Figs. 547*a–d*. Here are given (1) a generalized view (Fig. 547*a*) which includes a representation (*a*) of the ground stations *A*, *B*, *C*, and the nadir point *N*, (*b*) the photographic picture planes *MM* (assumed to be horizontal) and *TT* (assumed to be tilted through the angle α, the direction of tilt being in the vertical plane passing through the principal line *vv*), (*c*) the point of view *I*, and (*d*) the rays of vision from the point of view *I* to the ground stations; (2) a side view of the photographic picture planes (Fig. 547*b*), being an orthographic projection on a vertical plane which is parallel with the direction of flight; (3) an end view of the picture planes (Fig. 547*c*); and (4) the plan view of the picture planes (Fig. 547*d*), in which it is assumed that the plane *TT* is rotated into the plane *MM*, about the intersection trace *qq*.

The generalized view (Fig. 547*a*) shows, further, the central ray which pierces the plane *MM* at the principal point *O* and the nadir

point N; also the principal ray which pierces the plane TT at its principal point O' and the ground at the point N'; also the line of intersection of the two planes qq, being perpendicular to the direction of tilt vv.

Since the positions of all points pictured in the horizontal plane MM are without error, the positions of all points in the line of intersection of the two planes qq, will be without error, and this may,



FIGS. 547*b–d*.—Effects of tilt.

therefore, be called the equal-scale line for both planes. Also, since the vertical plane through I, which lies in the direction of flight vv, cuts the equal-scale line at Q, the direction of the displacement of each point in the tilted plane will be radial from the point Q. Thus in Fig. 547*d* the direction of the line $c'c$ prolonged passes through Q, as do also the lines through aa' and $b'b$.

If the ground points A, B and C should be assumed to lie in a straight line parallel with the direction of tilt vv, the pictured positions of these points would appear at a, b, and c in Fig. 547*b*, also the

positions of these points in the tilted picture plane would appear at a', b', and c'. The total displacement of the two pictured positions of the point C, for example, would be given by the difference between the two distances, the one Qc, measured along the line vv, and the other Qc', measured along the line $v'v'$; these distances have been designated as y and y' for the purposes of the discussion which follows. Likewise, the total displacement of the pictured position of the point B would be given by the difference between the distances Qb and Qb'.

Most often, however, the points do not lie in the vertical plane passing through the line vv, and we have the condition illustrated in Fig. 547a, where the rays passing from the point of view to the ground stations pierce the picture planes in the points a, b, c, and a', b', c'. In this case, the total displacement is a function of the angle of tilt and of the direction of the ray from the point I to the pictured position of the object. The amount of this total displacement may be found by means of the rectangular coordinates of the points c and c' (Fig. 547d) with respect to the origin Q, the axes being the lines vv and qq. Thus, let it be supposed that the angle α represents the angle of tilt of the camera; that β represents the angle which the optical axis IN' of the tilted picture plane makes with the orthographic projection of the ray IC upon a plane normal to the intersection trace qq, shown as Icc' (Fig. 547b); let γ represent the inclination of the ray IC as seen in the end view. Then the ordinates y and y' of the points c and c' respectively (plan view), are seen to be given by the distances Qc and Qc' respectively (side view); likewise, the abscissas x and x' of the points c and c', respectively (plan view), are given by the distances x, x' respectively (end view).

From the coordinates x, x', and y, y', it is evident (plan view) that the total displacement $cc' = \sqrt{(y' - y)^2 + (x' - x)^2}$ or if we let $d_y = y' - y$ and $d_x = x' - x$, $cc' = \sqrt{d_y^2 + d_x^2}$.

Referring to Fig. 547b, it is seen that $O'Q = f \tan \frac{\alpha}{2}$,

and

$$O'c' = f \tan \beta$$

hence,

$$y' = Qc' = O'c' - O'Q = f \tan \beta - f \tan \frac{\alpha}{2} \tag{6}$$

also,

$$OQ = f \tan \frac{\alpha}{2}, \text{ and } Oc = f \tan (\beta - \alpha)$$

hence,

$$y = Qc = Oc + OQ = f \tan (\beta - \alpha) + f \tan \frac{\alpha}{2}$$

then,

$$d_y(\text{outward}) = y' - y = f\left[\tan\beta - \tan(\beta - \alpha) - 2\tan\frac{\alpha}{2}\right] \quad (7)$$

For the small angles of tilt ordinarily encountered (see Art. 540) no appreciable error will result if the tangents of the angles are assumed to vary proportionally with the angles themselves. Thus $2\tan\frac{\alpha}{2} = \tan\alpha$ (approx.) and Eq. (7) becomes

$$d_y(\text{outward}) = f[(\tan\beta - \tan\alpha - \tan(\beta - \alpha)] \text{ (approx.)} \quad (8)$$

The expression for the displacement d_y, as given by Eq. (8), has been derived for a point c in that part of the tilted photograph which is below the assumed horizontal plane through the point Q, and is therefore displaced outward. A point on the opposite side of the equal-scale line qq will appear in that part of the tilted photograph which is above the assumed horizontal plane through Q and will be displaced inward. The amount of displacement, by a derivation similar to that given above, is found to be

$$d_y(\text{inward}) = f[\tan(\beta + \alpha) - \tan\beta - \tan\alpha] \text{ (approx.)} \quad (9)$$

Example: Let $f = 12$ in., $\beta = 20°$, and $\alpha = 2°$.

Find the magnitude of both the outward and the inward displacements of the point.

By Eq. (8) the outward displacement, when the point is below the assumed horizon plane, is

$$d_y \text{ (outward)} = 12[0.364 - 0.035 - 0.325] = 0.048 \text{ in.}$$

By Eq. (9) the inward displacement when the point is above the assumed horizon plane is

$$d_y(\text{inward}) = 12[0.404 - 0.364 - 0.035] = 0.060 \text{ in.}$$

The direction of the radial line Qcc' (Fig. 547*d*) is represented by the angle δ, which ordinarily may be determined directly on the photograph. Thus, the distance $d_x = x' - x = d_y \tan\delta$, or this value may be determined graphically on the photograph by constructing a right triangle cgc', in which $cg = d_y$, and $c'g = d_x$. By assuming values for α, β, and δ, the probable maximum displacement of points in photographs to be taken can be calculated. From such calculations, specifications can be drawn as to scale fraction, permissible tilt, etc.

548. Index Map.—The first step in the mapping process after the negatives are developed is to prepare an index map and to number each negative accordingly. For this purpose the ground-control points are plotted on a sheet of paper to a convenient small

scale. The negatives are then assembled in approximately their relative positions, and these positions are drawn to the reduced scale on the control sheet and are numbered consecutively. By this means each negative is assigned its serial number which is inscribed on the negative and which serves to identify each photograph. Figure 548 illustrates the resultant map.

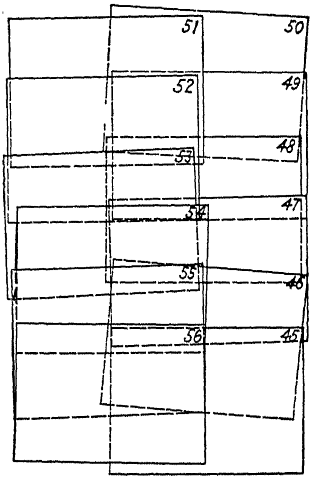


FIG. 548.—Index map.

549. Secondary Map Control; General.—The photographs from which the map is to be constructed are perspective views and are subject to the various effects explained in the preceding articles. Moreover, these aerial views can not be handled by the definite methods applicable to terrestrial views. The means most commonly employed for constructing the map, aside from those processes making use of highly specialized apparatus, is that whereby the data of the perspective view are transferred to the map by a series of adjustments and restitutions which yield a fairly accurate orthographic projection.

Four methods of securing the map control will be described: (1) the *section-line* method, (2) the *straight-line* method, (3) the *radial* method, and (4) the *three-point* method. The general procedure is to determine the correct orthographic projection, to the scale of the map, of a few well-distributed points in each photograph. This set of points may be called the *secondary map control*, to be carefully distinguished from the ground control. The former is an office procedure and the latter is a field procedure.

549a. Section-line Method.[1]—In regions covered by the United States system of land subdivision, it is sometimes possible to assemble and to orient the photographs of an aerial survey with sufficient accuracy to plot the map by means of a traverse around the perimeter of the area and by means of the included section lines. For this method, a considerable number of these lines should be apparent in the photographs and a considerable amount of both end lap and side lap should be secured, so as to enable the topographer accurately to fix the position of a given line joining two photographs by its direction as shown on the two adjacent prints.

[1] The methods described in Arts. 549*a*, 549*b*, and 550 (3) were developed by the United States Geological Survey (see Ref. 25, p. 820).

This method is subject to the irregularities and inaccuracies of the land-survey operations, and for one not well acquainted with the nature of these operations it is hardly to be expected that satisfactory results will be secured. It has the great advantage, however, that the field measurements for horizontal control are considerably less than those required for other methods of control.

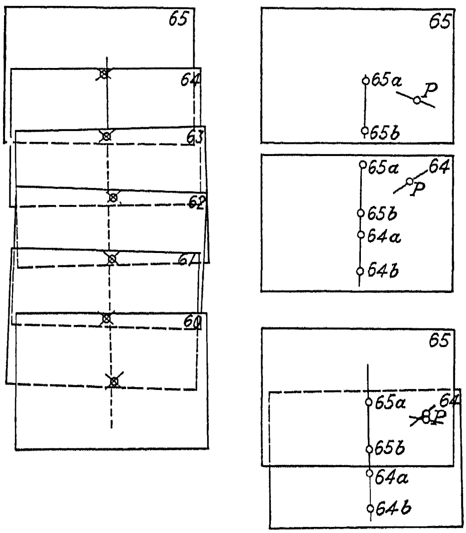


FIG. 549*a*.—Photograph marks for map control, straight-line method.

Briefly stated, the method consists in adjusting the photographs and correcting their scale errors by matching the section lines, the positions of which appear on adjacent prints. The survey, as a whole, is controlled by a traverse near the perimeter of the tract. The method is the least accurate of those described in this chapter.

549b. Straight-line Method.—A convenient method of combining the separate photographs for the aerial map is the straight-line method.

The procedure is illustrated by Fig. 549*a*. The first step is to select a series of photographs extending along a given flight from one control point to the next (as those numbered 65, 64, 63, etc.) and to mark on each one its principal point. By matching images on adjacent views these photographs are arranged, as nearly as may be, in their correct relations. A straightedge is then laid down on the photographs and is adjusted to an axial position with respect to the principal points. This direction is marked by a fine pencil line on the first photograph of the series only, as shown marked by a full line on photograph 65. In the

figure its continuation is indicated by the dotted line on the remaining prints.

The photographs are then disassembled, and, on each photograph on which there is shown a control point, a radial line is drawn from the principal point through the control point, as through *P* on photographs 65 and 64. On photograph 65, also, the pictured positions of two points 65*a* and 65*b* are chosen which lie on the marked straight line and which appear also on photograph 64. These two points are located on 64, and a straight line is drawn through them and continued to the edges of the print. Two more points are now chosen on 64 which also appear on 63, and so the process is repeated on successive prints until another control point is reached, where a new start is made.

The pictures are now ready to be assembled as shown in the lower part of the figure. A sheet of celluloid or tracing cloth is placed over photograph 65 and the straight line containing the two points 65*a* and 65*b* and the radial line including the point *P* are drawn on the tracing. Photograph 64 is slid under the tracing and adjusted so that the points 65*a* and 65*b* coincide with corresponding points on the tracing and the radial line on the picture passes through *P* on the tracing. If the ground is flat and if all sources of error are negligible, these points will coincide precisely; but when parallax, tilt, etc. affect the photographs the points will not coincide and the best adjustment possible, based upon a knowledge of the conditions, must be made. These two photographs are now joined as accurately as this method permits, and the process is repeated with succeeding photographs of the series. Upon the completion of one series, another is begun in like manner.

If the direction in any flight series, between control points, is so broken that a single straight line passes far from the center of some pictures, it is best to use two straight lines which will intersect near the center of some picture intermediate in the series. In this case, it is best to begin plotting control points from this picture and to work both ways from it.

This method may be improved by combining it with the method of radial control to be described in the following article; but in no case should the straight line be abandoned in the attempt to adjust the photographs to such radial control points. The method is best adapted to flat country, a small scale, and pictures with from 40 to 60 per cent overlap.

549c. Radial Method.[1]—This method of providing map control from vertical aerial photographs is based upon the following perspective properties of such photographs: (1) that points near the center of a photograph are nearly free from errors of tilt and of ground-

[1] The radial method described in Art. 549*c* and the three-point method described in Art. 549*d* were devised by Major James W. Bagley, Corps of Engineers, U. S. Army, formerly of the U. S. Geological Survey.

elevation parallax; (2) that all errors due to small amounts of tilt and to differences in ground elevation are nearly radial from the principal point of each photograph; and (3) that, for this reason, objects included in properly overlapping photographs may be located by direction lines drawn to them from the principal points of photographs, the position of the objects being at the intersection of such lines. Just as in plane-table operations, the locations of points are found at intersections of direction lines drawn from two or more stations. Since the view is taken directly above the central point and at a considerable distance from it, the apparent horizontal angles are very nearly the true angles.

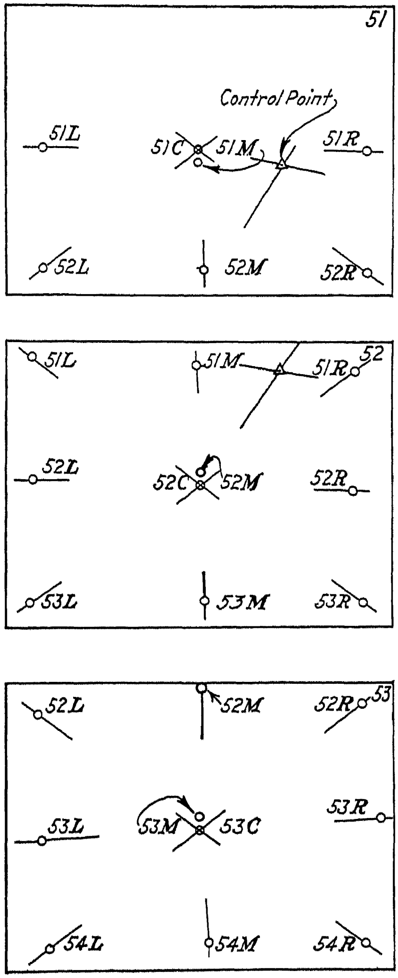


FIG. 549*b*.—Photographic marks for map control, radial and three-point methods.

In applying the method, a group of several consecutive photographs, which includes at least two ground-control points, one near each end of the group, is selected. Upon each photograph certain points (objects) are selected and marked and lines are drawn as follows: (1) the principal point of each photograph is plotted; (2) beginning with the first photograph of the group, as for example, 51, Fig. 549*b*, a definite object 51*M*, which also appears on the adjoining photograph 52, is chosen near the principal point 51*C* and is marked on both photographs; (3) points 51*R* and 51*L* are chosen at objects near the right and left edges of the photograph, respectively, which objects also appear on photograph 52; (4) similarly, points 52*R* and 52*L* are chosen at objects near the lower corners of photograph 51 and these two points must appear on the two succeeding photographs, 52 and 53, opposite the center of 52 and near the upper corners of 53; and (5) 52*M* is

chosen as a point which appears in photograph 52, near its center and also in photographs 51 and 53, near the lower edge of 51 and near the upper edge of 53.

This procedure of selecting and marking points having been carried through the group of photographs, it will be seen that each photograph of the group, except the first and last photographs, will have nine marked points besides the principal point. The first and last photographs will each have six marked points and the principal point. In addition, the first two photographs must include a control point, as shown on Fig. 549*b*, and the last two photographs of the group must also include a control point.

Radial lines are drawn on each photograph from the principal point to all marked points including the two control points.

The method of combining the data of the separate photographs into a map showing the correct relative positions of the selected points and the two control points, is as follows:

A sheet of tracing cloth, large enough to span the entire group of photographs when matched together, is laid over the first photograph of the series, and it is so placed that each photograph in turn can be laid under the tracing cloth in correct relation. Three points, namely, the principal point 51*C*, 51*M*, and 52*M*, are traced. The radial line to each of the other points, including the control point, is traced. Photograph 51 is removed and the tracing cloth is placed over photograph 52 so that the traced position of 52*M* falls on its position as it appears on photograph 52, and the tracing cloth is swung about this point until the traced position of 51*M* falls on its radial line on the photograph. The tracing cloth is now held with weights in this correctly oriented position. Radial lines to the other points are traced. These will give the location of all marked points that appear on photograph 51 and radial lines to three other points, 53*L*, 53*M*, and 53*R*, which appear on photograph 52.

The position of a point thus determined may or may not coincide with its pictured position on either print, depending upon the effects of the various sources of error in the views. If the pictures are without errors due to tilt, etc., the pictured position of each point should lie at the intersection of the two lines drawn toward it; but if displacement errors are present, the pictured position of any point will not coincide with the intersection of the radial lines. This condition is shown in Fig. 549*c*, where the pictured positions of the various points are shown by small circles and the correct locations are shown by the intersections of the radial lines.

When beginning the location of points on the tracing cloth, no attempt is made to use any particular scale, but a definite, though as yet unknown, scale is established by the distance between the two central points 51*M* and 52*M*, which have been traced as they appear on photograph 51. It is essential to accuracy that each central point, such as 51*M* or 52*M*, be selected as a point which lies close to the line connecting adjacent

principal points, and also close to its respective principal point; and that the control point fall within the zone of overlap so that it may be intersected.

As the procedure is continued it is evident that as each succeeding photograph is treated, there have been located previously on the tracing cloth three points which appear along the upper border of the photograph and a direction line to the next central point. The tracing cloth

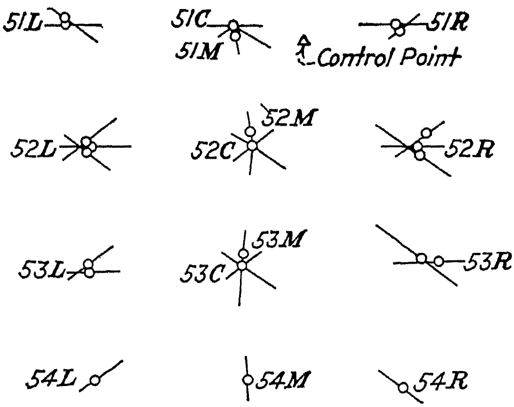


Fig. 549c.—Plotted map-control points.

is shifted until (1) the located positions of the three points appearing along the upper border of the photograph fall on their respective radial lines of the photograph and, the (2) radial line on the tracing cloth to the central point of the photograph now being treated falls on the point as marked on that photograph. When these four conditions are satisfied, the cloth is correctly placed with respect to the photograph, and the principal point is traced and radial lines are drawn. This procedure is similar to the graphical solution of the three-point problem (Art. 417*c*, p. 621).

This process is carried through the group of photographs until the second control point is located. The scale of the data assembled on the tracing cloth can then be determined, for, knowing the actual distance between the two control points from the field measurements, the ratio between this actual distance and the map distance between the two points, as measured on the tracing cloth, establishes the scale of the tracing. Proper reduction or enlargement can now be conveniently accomplished with the pantograph.

The radial method is dependent upon having the optical axis of the camera nearly vertical, because it is assumed that errors due both to tilt and to displacement due to ground elevation are radial from the principal point. If the photographs have been carefully taken under favorable flying conditions and if the terrain is not too rough, the method gives results that are as accurate as plotting practice requires.

To guard against the undesirable combination of rough terrain and large tilt, (1) images of level bubbles in the photographs should be examined, and whenever these indicate excessive tilt, the photograph affected should be treated as an oblique view, its nadir point being plotted as the level bubbles indicate and allowances being made for distortion of direction lines (Art. 547); (2) points selected to be located should represent objects near the average level of the terrain, neither greatly above nor greatly below the datum plane of the map. Use of the stereoscope makes this procedure practicable. Whenever it becomes necessary to use a point which represents a very high or a very low object, correction to the radial line thereto should be made in accordance with the method explained in Art. 545.

Other sources of error which accompany the method have been mentioned in Art. 539. In actual practice, the total effects of all sources of error are as follows: (1) tendency to swing off true alinement; (2) failure to obtain complete coincidence of all four radial lines with their corresponding points. To avoid the effect first mentioned it is necessary to give all possible care to orientation[1] of the tracing cloth over the photograph.

The central direction line should be rigidly held to its corresponding point (due regard, of course, being given to conditions of excessive tilt or great difference in ground elevation) and close examination should be made of the check which is afforded by the back central direction line on the photograph under treatment. Failure of the two lateral direction lines to fall on their respective located points should be adjusted by shifting the tracing cloth along the central radial line until a balance is struck between the triangle of error at one lateral point and that at the other lateral point. The final location of the lateral points should be taken at the centers of the triangles of error, as being the most probably correct positions.

Whenever it is possible to find a definite object at or quite close to the principal point of a photograph, use of the principal point itself may be omitted and that of the object alone may be marked, labeled, and used instead. This simplifies the operation and saves time.

Change in elevation of the camera during flight has no effect upon the accuracy of the method, which is independent of the scale of the photographs. Under favorable flying conditions and with skillful work by the pilot there should be little variation from the chosen level for flight. Therefore, if after having treated several photo-

[1] This difficulty is almost entirely eliminated by having photographs of great scope to the rear of the principal point, as is provided by the Bagley five-lens cameras.

graphs of a group by the radial method it is found that the ratio between distances as measured on the tracing and as measured on the photographs changes considerably, and if there is no evidence of a considerable change in elevation of the terrain, it is indicated that there was an error in the plotting rather than that a large change in flight level of the plane was made. Hence, a search should be made for a possible gross error in plotting.

549d. Three-point Method.—This method is similar in principle to the graphical solution of the three-point problem as used in plane-table work (see Art. 417*c*).

The procedure may be explained by reference to Fig. 549*b*. By this method, three ground-control points, the positions of which appear in two adjacent photographs, must be located prior to the plotting. For best results the three points should be equally spaced across the field of the photograph, transverse to the direction of flight, one point appearing near the principal point, one appearing near the left edge, and one appearing near the right edge. Such points might be those numbered 51*M*, 51*L*, and 51*R* showing on photographs 51 and 52 (Fig. 549*b*) and for the purposes of this article these will be assumed to be ground-control points.

A similar group of ground-control points should be located along the direction of flight at a distance not greater than 8 to 10 mi. in the case of small-scale multi-lens photographs, 4 to 6 mi. for single-lens intermediate-scale photographs, and 1 to 2 mi. for enlarged views and for most accurate results. If, on reaching the second group of control points, discrepancies appear in the data of the tracing, the plotted positions of these control points are adjusted as were those of the first group, by working backward until coincidence is obtained in the location of the secondary points in the two directions. Gross errors may in this way often be readily detected and eliminated.

The positions of the ground-control points are plotted on tracing cloth carefully to a desired predetermined scale which approximates the average scale of the photographs to be mapped. The first photograph (as 51) in the series is selected and each control point is marked by a dot (as 51*L*, 51*M*, and 51*R*), as are also the principal point of the picture (as 51*C*) and a point which appears near the principal point of the next adjacent picture (as 52*M*). A number of secondary map-control points (as 52*L*, 52*R*, and others) are likewise indicated. From the principal point (as 51*C*), radial lines are drawn through the selected points. The second photograph (as 52) is then marked in a similar manner, indicating first all those points which appear in the first photograph (as 51*L*, 51*M*, 51*R*, 52*L*, 52*M*, 52*R*, and whatever other secondary map-control points may have been marked on photograph 51). Then other points (as 53*L*, 53*M*, 53*R*, etc.) which appear also on the succeeding photograph (as 53) are marked. The principal point is found and radial lines are drawn from it through all points.

The first photograph (as 51) is then placed under the tracing and is moved about until the three plotted points on the tracing fall on the respective radial lines on the prints. It is usually possible to make the three points fall on their respective radial lines whether there are errors or not, but if there are errors, the location of the principal point will be in error. If a fourth control point is available, a check is afforded, and in lieu of this, there is the direction on the tracing to the central point of the next photograph. The photograph will be correctly oriented with respect to the plotted control of the tracing, however, when the tracing-cloth points fall on the radial lines, as stated above. Since points near the principal point of each picture (as 51*M* of photograph 51) are without displacement errors, the principal point may be marked correctly on the tracing cloth. From the principal point (as 51*C*) thus located on the tracing, radial lines are drawn on the tracing cloth over each point of the picture that it is desired to locate.

The next photograph (as 52) is now placed under the tracing and oriented in a manner similar to that used for the first, *i.e.*, the picture is moved about until the three plotted control points (as 51*L*, 51*M*, and 51*R*) on the tracing, fall on the corresponding radial lines of the picture. The photograph is now properly oriented so that the principal point and radial lines of the picture may be correctly traced. There now appear on the tracing, intersections of the lines drawn toward the secondary control points (as 52*L*, 52*M*, and 52*R*). These points of intersection are correct locations of the corresponding ground points.

The third photograph (as 53), having been properly marked, is next oriented under the tracing, and the process described in the preceding paragraph is repeated. The map-control points of each successive photograph of the series are thus mapped until the plotted positions of the second group of primary ground-control points is reached. The closeness of agreement between these previously plotted positions and the positions found by the map-control procedure is a measure of the accuracy of the work. The second group of control points also serves to carry the plotting through a second group of photographs.

In carrying the secondary control forward, it is found that additional strength to the control scheme can be secured by orienting, say, three or four pictures in one flight series (as photographs 50, 49, and 48 in Fig. 548), then passing to the next adjacent flight series and orienting a similar number (as photographs 51, 52, 53, and 54), then passing back to the first series, and thus progressing forward by groups of three or four pictures in a series by working back and forth transversely to the direction of flight across the area.

The three-point method is best adapted to conditions where good angles of intersection on secondary control points are afforded, and where the radial lines are of considerable length. These conditions are met best by multi-lens photographs, but the method may be used to good advantage under proper field conditions on single-lens pictures.

Comparison of the three-point method and the radial method shows that both are based on making locations by intersection and resection of radial lines drawn from the principal point or from the plotted position of some object near the principal point of the photograph. The three-point method permits plotting at any desired predetermined scale, while the radial method is limited to plotting at some scale which, while it may be approximately a desired scale, is not definitely known until the plotting between two control points has been completed. The three-point method requires less office work of compilation but involves special arrangement of control points with respect to the aerial photographs. The radial method permits more latitude in the arrangement of control points and is therefore more general in application than the three-point method.

550. Transferring Detail Data from Photographs to Map.—The positions of control points determined by either the radial or three-point methods are ordinarily transferred to the map by means of the pantograph. The instrument is adjusted to the ratio of the distance between primary control points on the tracing to the distance between the corresponding points on the map. All map-control points in that series shown on the tracing are then transferred to the map, and the pantograph is adjusted to the next series.

The positions of details can be transferred from the photographs to the map by either of four methods: (1) photography, (2) use of the pantograph, (3) direct tracing when it is practicable to compile the map at the average scale of the photographs, or (4) the stereographic method.

1. *Photographic Method.*—By this method the significant details on each photograph are inked in, due care being taken to correct, in so far as possible, the displacement errors made apparent by the procedure of plotting the map control as explained in the preceding article. The photographs are then bleached, leaving the inked lines only, and this bleached photograph is reduced or enlarged, as the case may require, by photography, to the scale of the map. This method is rapid, but requires the proper photographic facilities.

2. *Pantographic Method.*—By this method each photograph is properly oriented with respect to the map, and the pantograph is adjusted by the ratio of the distances between map-control points appearing on the photograph to the distances between the corresponding points (previously plotted) on the map. In orienting each photograph, two pairs of points are chosen such that the line connecting each pair passes near the principal point. Thus in photograph 53 (Fig. 549*b*), for example, points 52*R* and 54*L*, also 52*L* and 54*R*, might be chosen. The choice of such pairs of points combines com-

pensative errors and yields the best adjustment which the data afford. Having thus oriented the photograph and adjusted the pantograph, the details are transferred to the map.

3. *Direct Tracing.*—This is in some respects more satisfactory than either the photographic or the pantographic methods. The chief advantage lies in the facility it gives for adjusting details which are displaced as they appear on the photographs because of ground elevation, as, for example, details along a ridge. The method requires that the average scale of the photographs be determined and that the map be compiled at that scale. As a consequence of this requirement, the method is awkward to use unless appropriately graduated boxwood or steel scales are available. The average scale of the photographs may be determined by tracing enough central points from a group of photographs to span the distance between two control points, several miles apart, and then comparing the traced distances with the actual distance between the two control points. The map projection is then prepared at the average scale so determined, or at that scale nearest to the average scale for which graduated boxwood or steel scales are available.

This method is more commonly used than the photographic method and is capable of more flexible adjustment to the variety of conditions which the different photographs present.

4. *Stereographic Method.*—This method makes use of special and complex instruments not only to draw the orthographic projections of the objects which appear in the photographs, *i.e.*, the line map of the objects, but also to draw the contour lines. The method is indicated in Art. 551*a*.

551. Construction of Contour Lines.—Three methods are used to draw the contour lines on aerial maps: (*a*) stereographic methods, (*b*) the use of completely controlled plane-table sheets, and (*c*) the use of photographic plane-table sheets.

551a. Stereographic Methods.—Several methods and instruments make use of stereographic measurements. Most of these are of European design. Two instruments available in the United States will be mentioned here, namely, the *stereocomparator* and the *aerocartograph*.

1. *Stereocomparator.*—As indicated in Art. 534, the stereocomparator is an instrument of precision by means of which the stereoscopic view and the floating mark are utilized to draw the contour lines directly on the photograph or, by the elaborate devices now in use in Germany, the contour lines are drawn autographically to any desired scale on the map. The Brock process (Ref. 24, p. 820) yields photographic plates accurately to scale on which the contour

lines are drawn by means of the apparatus shown in Fig. 551*a*. Prints from these plates may be assembled to form an accurate mosaic contoured map, or by pantograph the data on the photographic plates may be transferred to tracing cloth or to the map.

The principal characteristic of the Brock method is that contour lines are drawn by stereocomparison directly on the photographic negative, without the use of field methods other than the measurements for the ground control. The resulting photographs also

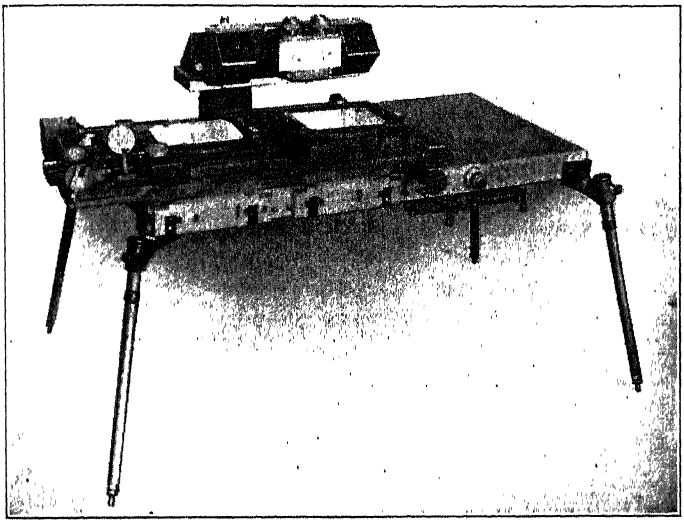


Fig. 551*a*.—Brock stereocomparator.

provide a mosaic free from errors due to tilt and to variant height of lens. The mosaic, therefore, approaches the accuracy of a map. The map is constructed, however, where accuracy or other considerations require it. According to field tests (Ref. 4, p. 820) contours drawn by this method have proved to be correct within 2½ ft. Recent improvements in the instrumental equipment promise elevation errors in contours less than 1 ft.

The results of thorough tests show that the method will yield excellent results, but it is complicated and requires the use of patented apparatus and methods. For these reasons it will not be described in detail here.

2. *Aerocartograph.*—Among the various instruments of foreign make, the aerocartograph is noteworthy. Dr. Hugershoff of the Zeiss firm has developed this instrument and the accompanying methods which permit the use either of terrestrial, oblique, or vertical

views, and which construct the map directly from the stereoscopic view of the landscape. This method has been introduced in the United States and promises excellent results. The instrument and its use are too complex to receive more than brief mention here.

The aerocartograph (Fig. 551*b*) makes use of the principles of stereocomparison which have already been explained. Its principal features are (1) a system of telescopic lenses for viewing the stereoscopic pair of photographs, (2) a mechanism for moving the floating mark in the field of view, and for translating these movements to the drawing table, and (3) a drawing instrument for constructing the map.

The floating mark, which may be called the *index*, in this instrument consists of the fused images of two sets of cross-hairs etched on the lenses of the optical system, one in each of the two telescopes. Hence, in reality this image remains fixed, and its apparent movement in the field of view is effected by moving the photographs, one with respect to the other. These movements of the index simulate in every way what would be observed through a transit telescope in the field, the position and orientation of the transit telescope being the same as for the camera lens which made the photograph.

When the index is fixed on any given object in the landscape, since it is the fused image formed by two telescopes, the effect is the same as though the object were viewed by two transit telescopes, one at the first position and the other at the second position of the camera. Also, the movements necessary to fix the index on any object in the field of view, with respect to any control point, have their counterpart in the azimuth and vertical angles which would be observed on the circles of a transit. Thus, by well understood principles of intersections from two transit stations, the apparent position of the object sighted is determined with respect to other points in the field of view.

The apparent movement of the index is effected by three motions of the photographs, actuated by the observer using his right hand, left hand, and one foot. These movements must be made simultaneously, since one affects the others. These controls are linked to two rods or arms which in their three-dimensional relations to each other are the same as the two lines of sight of the two imagined transits. This condition makes it possible to fix on the drawing the vertical projection, *i.e.*, the x and y coordinates, of the intersection of these lines, or the position of the object sighted by the index. Also, a measure of the elevation of the point sighted is possible. Thus there results the plotted position and the observed elevation of any given point in the field of view.

The great advantages of this instrument are that it (1) permits the use either of terrestrial or of oblique or vertical aerial views, (2) automatically provides for the effects of tilt, ground-elevation parallax, and height of lens, (3) requires a minimum of ground control and by an ingenious arrangement carries the necessary control from one photograph to the next through a flight series, and (4) constructs the map, including contours, to any desired scale.

551b. Controlled Plane-table Sheets.—By this method of constructing contour lines, the horizontal positions of all important objects appearing in the photographs are transferred to the plane-

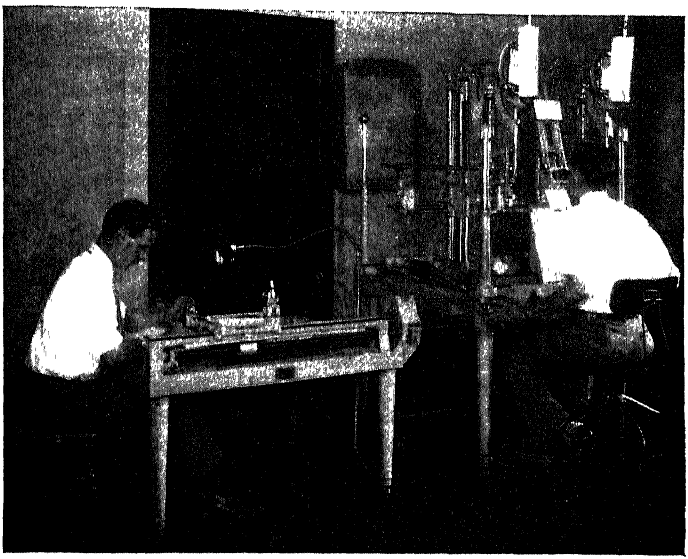


Fig. 551*b*.—The aerocartograph.

table sheet before going to the field, as described in Art. 550. The sheet thus prepared is correct as to scale and shows a wealth of detail far in excess of that available by the usual terrestrial methods. Accordingly, in the field, the topographer is freed from the necessity of locating the horizontal positions of objects, except such as may have been overlooked in the office work, and can devote his entire attention to determining the elevations of points and to sketching contours. He does not have the benefit, however, of a stereoscopic view to check the position of his contour lines because these lines are not drawn directly on the photographs as is the case with the method described in the following paragraph, although the stereoscopic view of the overlapping photographs is helpful.

551c. Photographic Plane-table Sheets.—By this method, each photograph is placed on the plane table where it is utilized in the same manner as the usual plane-table sheet. The topographer then proceeds, by the ordinary plane-table field methods, to locate and to draw the contour lines on the photograph. Stadia measurements are sufficiently accurate for this work. The field work is considerably modified, however, by the following conditions: (1) The wealth of detailed data shown on the photograph enables the topographer, if the country is flat and the photograph is not greatly tilted,[1] to set up and orient the table readily at any point, so that he is not required to carry his horizontal control from station to station; (2) the elevations of many points can be secured by scaling the horizontal distances from the photograph and measuring the vertical angles to the objects; and (3), since the topographer will have the aid of the stereoscope in drawing the contour lines in the office, it will be sufficient in the field, after determining the elevations of controlling points, to sketch only those contour lines which show complex features and which fix the controlling lines of the terrain. The interpolations on uniform slopes can then be done in the stereoscopic view.

In regions of high relief, it is necessary that horizontal distances between pairs of points appearing in each photograph be determined, one pair at a high elevation and another pair at a low elevation, thus to correct the distances on the photographs for displacements, or change in scale due to parallax. Also, in all cases, horizontal distances between pairs of points are measured at fairly regular intervals in the field to provide corrections for the errors in the scale of the photographs due to tilt and to change in the height of lens. These measurements will be most valuable when taken in regions which appear near the edges of the picture.

After the field work is completed, the photographs are brought into the office, all corrections to the positions of important objects are made, and the contour lines are filled in and revised under the stereoscopic view. The map data are now ready to be transferred from the photographs to the tracing or map, and this is done by the methods explained in the preceding article.

The speed of the topographic party in sketching contour lines in the field, by the methods here described, has been found to be from two to four times that attained by the common terrestrial methods, this being due to the fact that so much horizontal control is available and that stereoscopic vision is provided.

[1] An aerial photograph of hilly country can be correctly oriented when the topographer is occupying a point on the ground near the center of the picture.

551d. As to the relative merits of the two methods described in Arts. 551*b* and 551*c*, it may be said that the latter is the more accurate, especially in hilly regions, and is therefore better adapted to intermediate scales and to terrain of high relief; but that if good horizontal ground control is provided, the former is better for small scales and flat country.

552. Mosaics.—Since a mosaic is an assemblage of contact prints, it is (unless constructed by the Brock process or other special method) subject to the displacement errors of the photographs. For this reason, a mosaic cannot be used as a map from which to scale dimensions with such accuracy as is desired in the design of most engineering projects. But the character and amount of detail which appears in the photograph are of such value for many purposes (as tax assessments or the general features of municipal improvements) that the use of a mosaic is more than justified for such purposes. The best results will be secured in flat regions, and with good flying the displacement errors will be kept at a minimum. In regions of high relief a large amount of overlap should be secured so as to permit all but the central portion of the photograph to be trimmed away.

Mosaics are said to be *non-controlled* or *controlled*, depending upon whether or not ground or auxiliary control in the form of existing maps is available to guide the topographer in assembling the pictures.

552a. Non-controlled Mosaic.—In constructing a non-controlled mosaic, the contact prints from the original negatives are assembled, trimmed, matched, and pasted together on heavy cardboard or compoboard. The assemblage may be built up about a single picture which is first pasted in position. The others are then matched as well as may be. Or, still better, a flight series may be assembled by the straight-line method of map control, as described in a preceding article. Adjacent series on either side may now be matched to the central strip, and successive strips may be added until the map is completed. Of course, without ground control, there will be no check on the work and noticeable displacement errors will remain, but this method provides the most satisfactory ready adjustment of the mosaic as a whole.

If the prints are trimmed to a bevelled edge, in pasting them together, the surface will be given a pleasing continuity. This is especially desirable if photographic reproductions are to be made.

552b. Controlled Mosaic.—If there is ground control and a sufficient amount of overlap (60 per cent end lap and 50 per cent side lap) the mosaic may be constructed by means of any one of the methods of map control described in preceding articles. The prints

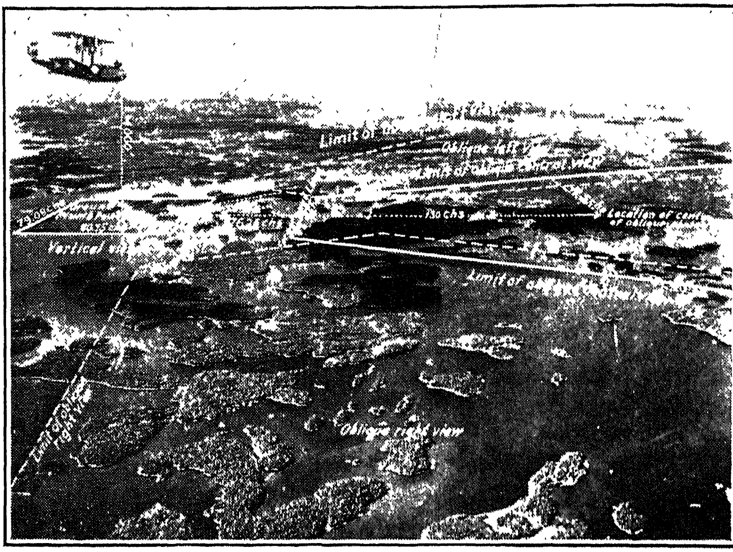


FIG. 552a.—Oblique aerial mapping view.

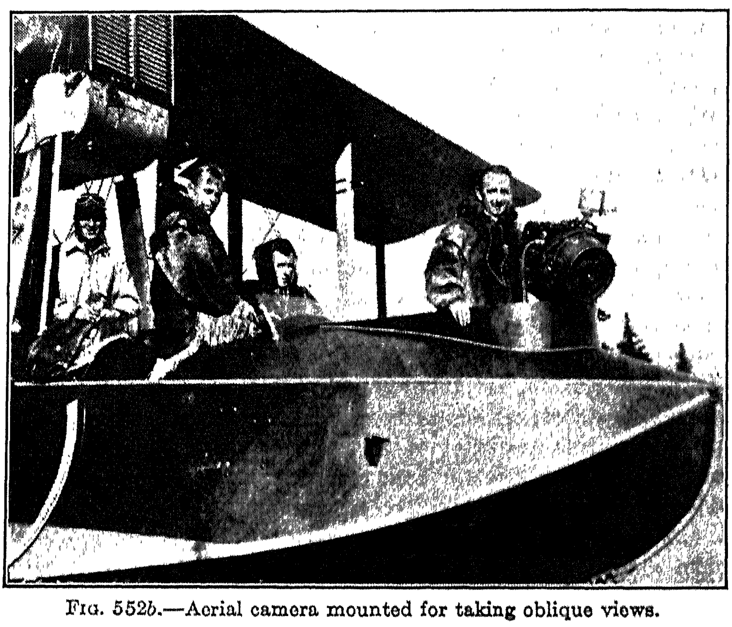


FIG. 552b.—Aerial camera mounted for taking oblique views.

are trimmed to include only the areas near the center of each view and are assembled and oriented by means of the tracing-cloth control sheet. Of course, the prints will not match exactly and the scale will not be uniform throughout the map, but accumulative errors will be effectively reduced and, in relatively flat regions and with good flying conditions, reasonably accurate mosaics can be constructed by this method.

552c. Equal-scale Mosaic.—If the photographs of a given survey are rephotographed and thus brought to the desired scale of the mosaic, the effect is secured as though each negative were exposed at an equal altitude above the datum plane of the survey. All errors due to unequal height of lens are thus eliminated, and although the displacement errors due to tilt and parallax are still present, a mosaic much superior to that previously described will be secured if some method of map control is used.

552d. Oblique Mosaic.—As the oblique mosaic is constructed in the United States, it has little value as a map. The photographs are taken at relatively low altitudes and yield an excellent view of a character that for some purposes is more valuable than the nadir mosaic.

The Topographical Survey of Canada has applied the principle of perspective to oblique views in such manner as to yield maps which, although they are reconnaissance in kind, are most valuable. The scope of the oblique view is much greater than that of a vertical view (see Fig. 552*a*), and the extent of territory which may be covered for a given expenditure of time and money is correspondingly increased. The equipment consists of a Fairchild camera of short focus, mounted as shown in Fig. 552*b*. For the application of this method the terrain must be flat so that the horizon line appears in each photograph as a straight line; and the results are improved if water surfaces of considerable extent appear in the pictures. Because of these requirements, and because the occasions are rare when the atmosphere is sufficiently clear to produce well-defined images of the horizon, the method has had little or no application in the United States.

Oblique views taken in connection with other surveys are often most valuable in exhibiting the general features of the terrain. Such views are taken as an aid in the studies made in connection with various engineering enterprises such as city planning, drainage, flood control, etc.

553. Limitations and Applications of Aerial Mapping.—The character of the various sources of error in aerial surveying have been discussed in connection with each of the four principal parts of this

work, namely: photography, flying, ground control, and mapping. The effect of these errors in combination is such that, on controlled mosaics of good quality, distances of 10 in. or more may be scaled with errors not exceeding 2 per cent (Ref. 19, p. 820), and on maps of similar quality, the errors will not exceed 1.5 per cent.

Maps constructed from enlarged or reduced photographs will not retain the same accuracy as those made at or near the scale of the original negatives. Since the best flying altitudes are from 8,000 to 15,000 ft., by Art. 544 it will be found that the best results are secured at scales from 1 in. = 400 ft. to 1 in. = 2,000 ft. Aerial mapping has little application where accurate large-scale (1 in. = 100 ft. or less) maps are required; nor is aerial mapping employed where the survey is of small extent.

In some cases, from considerations other than that of accuracy, it is desirable to construct maps and mosaics to a large scale. This has been done on mosaics by enlargements of the photographs up to eight diameters, but the definition in such pictures is not sharp. For this reason, the limit of satisfactory enlargement is about three diameters. Such mosaics and maps constructed by enlargement have the important advantage that the expense is greatly reduced, because of the reduced original photographic scale and of the correspondingly increased ground area included in each photograph.

The many variable conditions which affect the costs of both terrestrial and aerial surveys make it impossible definitely to compare the costs of the two methods, although there are reported savings of 10 to 20 per cent in favor of aerial mapping where conditions are favorable.

The principal advantages of aerial surveying are (1) the increased speed with which the map can be secured and plans completed, which factor, in the case of large projects, is often an important consideration because of the saving of interest on the capital invested; (2) the wealth of detailed information supplied by the photographs in addition to the map supplied by ground methods; (3) the absence of personal errors and mistakes accompanying field work; (4) the uniform accuracy maintained in all parts of the map; and (5) the possibility of surveying regions forbidden, or inaccessible, to terrestrial survey parties.

Aerial surveys are used for any engineering project for which the area is sufficiently extensive to render the method economically feasible, and for which an intermediate- or small-scale map is suitable. In addition, the method is especially valuable for such purposes as geological surveying, timber cruising, locating transmission or pipe

lines, tax appraisals, river and harbor improvements, and the various phases of municipal improvements and of city planning.

554. Problems.

1. Given an ordinary square-box, glass-plate camera provided with a tripod; also, attached to the camera for leveling it, either a rod level having two level tubes, or a plane-table declinatoire with attached level tubes. Construct a surveying camera as follows:

a. Fix the principal line and horizon line by marks made on the plate holder.

b. Determine the focal length of the lens.

c. Establish the relations between the level tubes and (1) the horizon line and (2) the negative plate.

2. Make a photographic survey of a portion of the campus, using two methods by which to orient the picture trace: (1) the known azimuth of some object showing in the picture; and (2) the pictured positions of an object from two camera stations, as illustrated in Fig. 523.

3. Given the following conditions which apply to an aerial survey: size of photographs, 7 in. (direction of flight) by 9 in.; end lap (direction of flight) 60 per cent; side lap, 40 per cent; height of lens, 12,000 ft.; focal length, 12 in.

How many exposures will be required for a survey 5 by 7 mi. in extent?

4. Given the conditions of problem 3, if the speed of the airplane is 95 mi. per hour, what time interval should be used between exposures?

5. A mosaic map for city planning purposes is desired at a scale of 100 ft. per inch. The photographs are to be enlarged three times. At what height must the airplane fly if the camera lens has a focal length (*a*) of 20 in.? (*b*) of 8 in.?

6. For a certain photograph the following conditions are given: height of lens, 10,000 ft.; focal length, 12 in.; the distance, on the print, from the principal point to the pictured position of an object, is 3.5 in.; the object is 150 ft. above the datum plane. What is the linear displacement of this point in the photograph?

7. For another point in the photograph of problem 6, the distance from the principal point is 4.2 in. and its linear displacement is found to be 0.063 in. away from the principal point. What is the elevation of the point with respect to the datum?

8. The horizontal distance between the nadir points of two adjacent photographs is known to be 3,200 ft. The scale fraction is 600 ft. per in. and the elevation of an object near the line of flight is 90 ft. below the datum plane. What is the total linear displacement of this point as shown by the two pictures?

9. In a photograph affected by tilt, the following conditions are given: The angle of tilt is 2°30′; focal length, 12 in.; the distance from the principal point to a point in the direction of tilt is 2.2 in. What is the linear displacement of this point and the direction, *i.e.*, toward or away from the principal point, (*a*) if the point is in that part of the picture

which is tilted below the horizontal plane? (*b*) if the point is in that part of the picture which is tilted above the horizontal plane?

References

1. Bagley, J. W., "Stereophotography in Aerial Mapping," *Mil. Engr.*, Vol. 16, No. 88, pp. 303–306, July and August 1924.
2. Bagley, J. W., "The Use of the Panoramic Camera in Topographic Surveying," Bull. 657, Government Printing Office, Washington, D. C., 1917.
3. Bagley, J. W., Roberts, L. B., and Haquinius, E., discussion by others, "Topographic Surveying from the Air," *Mil. Engr.*, November and December 1923, January and February 1924.
4. Bergen, G. T., "Aeroplane Topographic Surveys," *Trans. Am. Soc. Civil Engr.*, Vol. 90, pp. 627–655; discussion by others, pp. 655–679; 1927.
5. Birdseye, C. H., "Stereo-topographic Mapping," *Proc. Am. Soc. Civil Engr.*, Vol. 58, No. 1, pp. 43–70, January 1932.
6. Bridgland, M. P., D. L. S., "Photographic Surveying," Publication 56, pp. 47, Topographical Survey of Canada, Ottawa, 1924.
7. Brock, N. H., "Brock Development of Aerial Mapping," *Mil. Engr.*, Vol. 17, No. 96, pp. 504–509, November and December 1925.
8. Church, Earl, "Topographic Mapping from Aerial Photographs," series of six bulletins, Syracuse Univ., Syracuse, N. Y., 1930–1932.
9. Deville, E., "Photographic Surveying," Dominion of Canada, Ottawa, Canada, 1895.
10. Fiske, H. C., "Topography from Aerial Photographs," *Mil. Engr.*, Vol. 16, No. 89, pp. 399–408, September and October 1924.
11. Flemer, J. A., "Photographic Instruments and Methods," John Wiley and Sons, New York, pp. 435, 1906.
12. Gardner, I. C., "A Magnifying Stereoscope and Camera Lucida," Jour. Amer. Optical Soc., Vol. 11, p. 195, August 1925, (1,500 w.).
13. Gordon, J. W., "Generalized Linear Perspective," Constable and Co., Ltd., London, 1922.
14. Higgins, A. L., "Phototopography," The University Press, Cambridge, xv, pp. 130, 1926.
15. Hotine, M., "Surveying from Air Photographs," Constable and Co., Ltd., London, 1931.
16. Judge, Arthur W., "Stereoscopic Photography," Chapman and Hall, Ltd., London, pp. 240, 1926.
17. Matthes, G. H., "Aerial Photography an Aid to Better Map Making," *Eng. News-Record*, Vol. 92, p. 25, Jan. 3, 1924.
18. Matthes, G. H., "Aerial Photography as an Aid to Map Making, with Special Reference to Water-power Surveys," *Trans. Am. Soc. Civil Engr.*, Vol. 86, pp. 779–802, 1923.
19. Matthes, G. H., discussion by others, "Aerial Surveys for City Planning," *Trans. Am. Soc. Civil Engr.*, Vol. 91, pp. 314–333, December 1927.

20. MATTISON, G. C., "Aerial Survey of the Mississippi River Delta," Sp. Pub. 105, U. S. Coast and Geodetic Survey, Government Printing Office, Washington, D. C., pp. 16, 1924.
21. WEYMOUTH, F. E., "The Brock Process of Making Topographic Surveys from the Air," *Engineering*, London, Vol. 121, No. 3138, pp. 245–248, Feb. 19, 1926.
22. "Aerial Photographic Mapping," Training Regulations 190–27, War Department, Washington, D. C., Jan. 23, 1925.
23. "Plotting Topography from Oblique Airplane Photographs," *Report of Air Board*, Appendix 1, Dominion of Canada, Ottawa, 1922.
24. "Stereophoto Surveying from the Air," *Eng. News-Record*, Vol. 94, No. 15, pp. 604–608, Apr. 9, 1925.
25. "Map Compilation from Aerial Photographs," Bull. 788, U. S. Geological Survey, Government Printing Office, Washington, D. C.

CHAPTER XXVIII

TRIANGULATION

555. General.—The traverse, as a means of determining the relative positions of the points of horizontal control in a survey, has already been discussed. It is composed of a series of straight lines connecting stations occupied by the transit or other instrument. The angle between lines of the traverse intersecting at each instrument station is determined, and the length of each line is measured.

A *triangulation system* is composed of a series of triangles in which one, two, or three sides of each triangle are also sides of adjacent triangles, as illustrated in Figs. 558*a* to 558*d*. The lines of a triangulation system form a network tying together the points or stations at which the angles are measured. The vertices of the triangles are the *triangulation stations.*

By the use of the triangulation method, the necessity of measuring the length of every line is avoided. If it were possible to measure all the angles in a triangulation system and also one side, all with absolute precision, no further linear measurements would be necessary, for when the values of all three angles and the length of one side of a triangle are known, the lengths of the other two sides can be calculated; and since such a procedure would give the length of the side of an adjacent triangle, the process of calculating lengths could be carried through the series of connected triangles. Unavoidable errors in the field measurements, however, make it desirable that the lengths of two or more lines in each system be measured as a means of checking the calculated distances. The lines whose lengths are measured are called *base lines.*

The arrangement of the triangles in most systems affords many different geometrical figures for each of which the theoretical value of the sum of the included angles is known. Also, the sum of the angles about any station should equal 360°, and in any triangle the lengths of the sides should be proportional to the sines of the angles opposite. These known conditions serve as a measure of the precision of the angle measurements and as a means of adjusting the errors in the measurements so as to secure the most probable values of the measured quantities.

It is not necessary that every angle in a triangulation system be measured, since if two angles in any triangle are measured the value of the third can be readily computed. This procedure, however, does not permit the application of the known conditions as a measure of the precision of the measurements, or as a means of adjusting the errors. Accordingly, much thought is given to the selection of stations and to the program of angle observations to the end that enough geometrical conditions can be applied to the observations to secure the desired precision in the calculated positions of all points within the system.

555a. The work of establishing a triangulation system may be divided as follows:

1. Reconnaissance, by means of which a proper selection of station locations can be made.
2. Erection of signals and, in some cases, tripods or towers for elevating the instruments and the signals.
3. Measurement of angles between the sides of triangles.
4. Astronomical observations from which to determine the true azimuths of the triangle sides, also in extensive systems to determine the geographical coordinates, *i.e.*, the latitude and longitude of all points in the system.
5. Measurement of the base lines.
6. Computations, including the adjustment of the observations, the calculation of the length of each triangle side, and the calculation of the coordinates of the stations.

There is a quality of triangulation corresponding to every degree of precision used in traversing. Thus, triangulation may be used for a simple topographic survey covering but a few acres or it may be used to extend control of the highest order across the continent. The relative merits of the triangulation method and the traverse method are based upon the character of the terrain only, and not upon the degree of precision to be attained. If favorable routes are available, the method of traversing is superior to the method of triangulation; but if the terrain offers many obstacles to traverse work (such as hills, vegetation, or marsh), triangulation is the superior method.

The most notable example of triangulation is the transcontinental system established by the U. S. Coast and Geodetic Survey, extending along the thirty-ninth parallel of latitude from the Atlantic to the Pacific Ocean. The system is now being developed to form a network to establish a control for the entire domain of the United States. A permanent reference point for the datum, called the "North American Datum," has been established in Osborne County, Kansas,

and to this point the precise surveys of the United States, Canada, and Mexico are referred.

Because of the character of the terrain near shore lines, the method of triangulation is extensively used in surveys for hydrographic charts, and for maps of the coast line and of navigable rivers.

556. Classification of Triangulation Systems.—Triangulation systems are classified according to (*a*) the average closing angular error in the triangles of the system, and (*b*) the discrepancy between the measured length of a base line and its length as computed through the system from the base line next adjacent.

The Board of Surveys and Maps (composed of representatives of the federal bodies engaged in surveying and mapping) has classified triangulation for the extensive surveys of the United States Government as follows:

	First order	Second order	Third order	Fourth order
Average triangle closure, seconds.	1	3	5	>5
Check on base................	$\frac{1}{25{,}000}$	$\frac{1}{10{,}000}$	$\frac{1}{5{,}000}$	$>\frac{1}{5{,}000}$

First-order triangulation furnishes the primary horizontal control for small-scale mapping operations, the triangle sides often being many miles in length. The system which extends across the continent and from Canada to Mexico is of this order. First- and second-order triangulation call for methods of high precision not often necessary except on very extensive surveys.

Third- and fourth-order triangulation establish points of horizontal control at short intervals in advantageous locations for detail mapping. These orders are often employed in intermediate- and large-scale surveys of limited extent. They call for methods of ordinary precision, and in some cases fourth-order triangulation calls for methods of low precision.

The classification given above relates more particularly to surveys for small-scale maps which cover relatively large areas. For the surveys with which most surveyors and engineers deal, it seems appropriate to retain the designations of *primary*, *secondary*, etc. to indicate the relative degrees of precision in the work. As in the case of traverse work (Chap. XXV), both primary and secondary (sometimes tertiary and quaternary) triangulation may be used on the same survey; also, triangulation and traverse work may be combined to meet best the field conditions.

Herein the principal description of triangulation methods will be concerned with triangulation of ordinary precision. Brief mention

only will be made of methods pertaining to triangulation of low or of high precision. In general, the points of difference are as follows:

1. *Triangulation of Low Precision.*—There is practically no reconnaissance, and the stations are marked with simple signals, often with a single stake or pole. The base lines are measured by the ordinary methods of chaining, or sometimes even by stadia. The angles of the triangles are not necessarily adjusted to meet the known geometric and trigonometric conditions. No correction is made when the instrument is not set up exactly over the station. The true azimuth of one or more lines in the system is not determined by astronomical observation, as in more precise work. The method of graphical triangulation (Art. 415) is often employed.

2. *Triangulation of High Precision.*—The reconnaissance may amount to a preliminary survey. Extensive use is made of tall towers and signals, and of signaling devices for reflecting sunlight or for night work. The *direction instrument* is used for measuring angles (Art. 561*a*). The angles of a system are adjusted by the method of Least Squares, and account is taken of spherical excess (Art. 564*f*). The computations for latitude and longitude of the various points take into account the curvature of the earth.

557. Reconnaissance.—Because of its influence upon the accuracy and economy of the work, the reconnaissance is of the greatest importance. The reconnaissance consists in the selection of stations, and it determines the shape of the resulting triangles, the number of stations to be occupied, and the number of angles to be measured. In this connection are considered the visibility of stations from each other, the accessibility of stations, the cost of necessary signals, the usefulness of stations in later work, and the possibility and convenience of base-line measurements.

The chief of the party examines the terrain, choosing the most favorable sites for stations. At each station the angles to other stations are read, usually with a prismatic compass, and the distances are estimated or measured roughly en route, so that the suitability of the system as a whole can be examined before the detailed work is begun. In open, hilly regions, stations can often be located on summits such that the instrument can be set up on the ground. Under adverse conditions, however, the instrument must be elevated to a height sufficient to enable all adjacent stations to be observed. Above each station is placed a signal, such as one or more square flags attached to a center pole. Stations and signals are described in Art. 560.

Existing maps are of great aid in the reconnaissance for triangulation of high precision, where the distances between stations are large.

Where forest growth is present, the observer must make use of standing trees or guyed ladders or poles to establish visibility with adjacent stations (Fig. 557).

Reconnaissance for triangulation of low precision is either very limited in extent or is omitted entirely, the stations being selected as the work progresses.

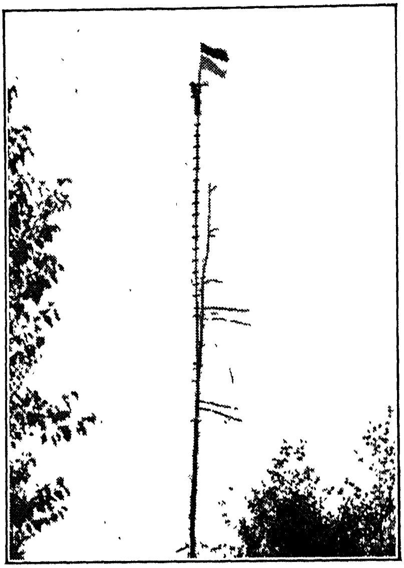


FIG. 557.—Reconnaissance.

558. Triangulation Figures.—In a triangulation system which is long and narrow, a chain of figures is employed, consisting either of *single triangles, polygons*, or *quadrilaterals*, or of combinations of these figures.

A triangulation system extending over a wide area is likewise divided into figures in the form of single triangles, polygons, and quadrilaterals in a more or less irregular scheme, as is illustrated by the system of Fig. 558*a*. The computations for such a system can be arranged to afford checks on the computed values of most of the sides. The base lines should be so placed that as many sides as possible can be included in the routes through which the computations are carried from one base line to the next.

1. *Chain of Triangles.*—The distinguishing feature of the chain of single triangles (Fig. 558*b*) is that there is but one possible route by which the calculation of distances can be carried through the chain. If *AB* be the base line whose length is measured, and if all

the angles of the triangles are observed, the length of the triangle sides in the chain (as AC, BC, BD, etc.) may be calculated progressively along the chain from the measured base line to the triangle side farthest removed from the base line. If two lines are measured as base lines, one at each end of the system, the calculations may be carried from each toward the other, to a triangle side somewhere between them.

The sum of the angles of each triangle should, of course, be 180°. As the sum of the observed angles usually will not equal this

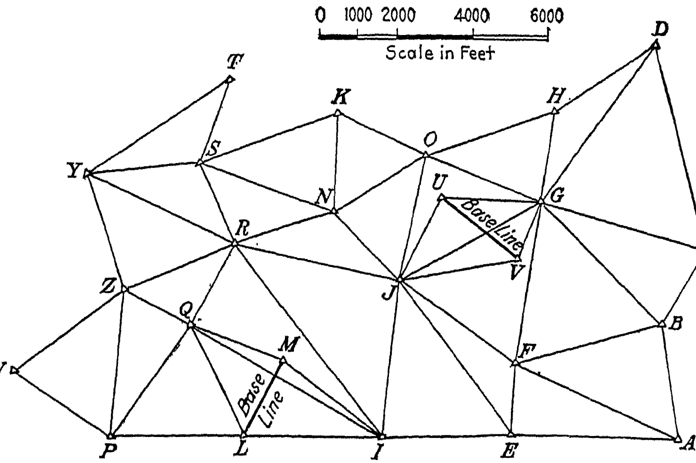


FIG. 558*a*.—Triangulation system.

amount, the angles are adjusted to satisfy this requirement before the distances are calculated. The method of making the adjustment is described later.

2. *Chain of Polygons.*—A polygon is composed of a group of triangles bounded by three, four, five, or more sides and having, in addition to the stations on the boundary of the figure, a station somewhere inside the figure which is at a vertex common to all the triangles of the figure. A chain of such composite figures is illustrated in Fig. 558*c*, in which $BACEF$ is a five-sided polygon with D as the central point, and $FEGJKI$ is a six-sided polygon with H as the central point.

As in the case of the chain of single triangles, the sum of the three measured angles of each triangle should equal 180°. Also, the sum of the angles about the central point should equal 360°. But there is a further condition that the length of any side may be calculated by

two routes and that these two calculated lengths should agree. Assume, for example, that *AB* is the base line. With the length of that line known, the length of *EF* can be found either by way of the triangles *ABD*, *ACD*, *CDE*, and *DEF*, or by way of the triangles *ABD*, *BDF*, and *DEF*. If all the angles were known exactly, the calculated value of the distance *EF* would be the same by one route as by the other. The observed angles are so adjusted by computation that this condition exists, and further, that the sum of the three angles of each triangle equals 180° and that the sum of the angles about the central point equals 360°. A similar adjustment is made for the other polygons in the triangulation chain.

3. *Chain of Quadrilaterals.*—Figure 558*d* illustrates another type of triangulation figure, in which the individual triangles more or less overlie one another. This type usually occurs in the form of the quadrilateral, of which the figures *ABDC*, *DCEF*, etc., are examples. In the individual quadrilateral there is no station at the intersection of the diagonals. Consider one of the quadrilaterals, as *ABDC*. The measurement of the angles gives the angles of four triangles, *ABD*, *ACD*, *ABC*, and *BCD*, in each of which the sum of the angles must equal 180°. In addition, as in the case of the polygons, the length of any line must be the same when calculated by one route as when calculated by another.

For example, consider *AB* as the side of known length and *CD* as the side whose length is required. There are four ways in which the required distance *CD* may be found: (1) by use of triangle *ABD* for the length of *AD* and triangle *ACD* for the length of *CD*; (2) by the use of triangle *ABD* for the length of *BD*, then of *BCD* for the length of *CD*; (3) using triangles *ABC* and *BCD*; and (4) using triangles *ABC* and *ACD*. The four values of *CD* should agree, and will agree if the angles are precisely known. The adjustment of angles must be so carried out as to make their adjusted values satisfy this requirement as well as to make the sum of the three angles of each triangle equal 180°.

558a. Choice of Figure.—Of the three forms of chains of triangulation figures, the chain of single triangles is the simplest, requiring the measurement of fewer angles than does either of the other two. This type of system, however, has the obvious weakness that, aside from the test of precision afforded by the measurement of more than one base line, the only check is in the sum of the angles of each triangle considered by itself. To reach the same precision in the determination of lengths, base lines would need to be placed closer together. As a consequence, this type of chain is not employed in work of the highest accuracy, but it is satisfactory where less precise results are required.

For more accurate work, quadrilaterals or polygons are used when possible in preference to single triangles, quadrilaterals being best adapted for a relatively narrow chain between widely separated

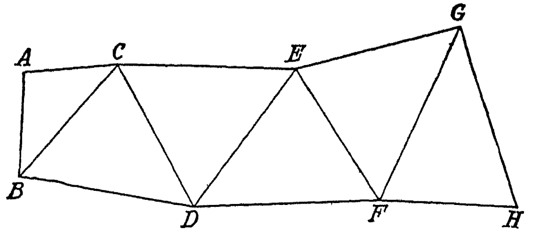


FIG. 558*b*.—Chain of single triangles.

points, and polygons being best adapted to surveys over areas extended in width.

558b. Strength of Figure.—It has been shown in Fig. 21*a*, p. 19, that values computed from the sine function of angles near 0° or 180°

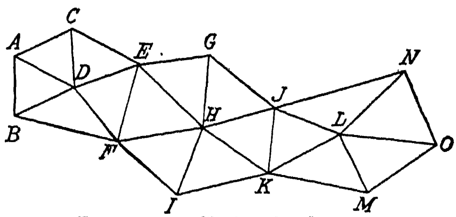


FIG. 558*c*.—Chain of polygons.

are subject to large ratios of error. Since in triangulation computations the sine function is nearly always used, it follows that angles near 0° and 180° are undesirable. It has been found in practice that satisfactory results can be secured for most purposes if the angles used

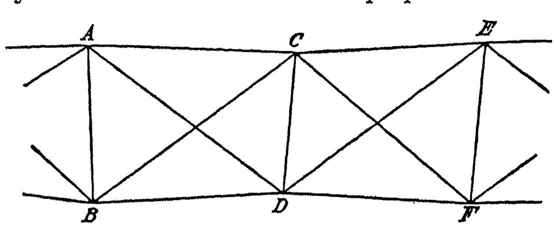


FIG. 558*d*.—Chain of quadrilaterals.

in the computations fall between the limits of 30° and 150°. However, many angles which are measured in the field are not used in computing the lengths of the sides in the system. Such angles may be very near 0° or 180° without impairing the excellence of the system as a whole.

This and other principles may be made clear by reference to Fig. 558*e*. In the figure, let *AB* represent a side whose length is known. This side and all others in the system whose lengths are desired are shown by heavy lines. The sine law, used in computing the lengths, states that in any triangle the sides are proportional to the sines of the angles opposite; accordingly, the angles affecting the computed lengths of sides in each triangle are those opposite the known and computed sides.

Consider the quadrilateral *CEFD*. The length of the side *CD* results from calculations carried through the quadrilateral *ACDB*. Then the length of *CF*, in triangle *CDF*, is computed by use of the known side *CD* and the angles 78° and 88° (13° + 75°); and *EF*, in triangle *CEF*, may be computed by use of the known side *CF* and the angles 93° and 72° (60° + 12°). In these two computations involving the small angles, 12° and 13°, it is noticed that neither one is used separately and so neither, by itself, has any effect upon the length of the side *EF*. Similarly, the side *ED*, in triangle *CED*, is computed by the use of the side *CD*, and again, neither of the small angles, 14° and 15°, is used separately in computing the length of *EF*. Thus it is seen that the side *EF* can be computed by two independent series of computations neither one of which is affected detrimentally by the small angles involved. As a matter of fact, the quadrilateral *CEFD* is a stronger figure than is *ACDB* in which no angle less than 36° occurs.

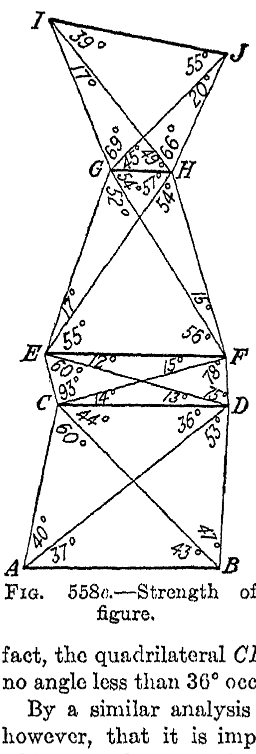


FIG. 558*e*.—Strength of figure.

By a similar analysis of the quadrilateral *EGHF*, it will be found, however, that it is impossible to compute the length of the side *GH* without making use in one series of computations of the angle 15° separately, and in the other, of the angle 17° separately. Therefore, by any possible means, the computed length of the side *GH* must be affected by the large ratios of error resulting from the use of small angles separately. The large degree of uncertainty thus introduced into the computed length of the side *GH* will be effective in all dependent values computed therefrom, as, for example, the length of the side *IJ* in the system shown.

Considerations of economy sometimes render one figure more desirable than another even though it may be the weaker of the two. Hence, the quadrilateral *ACDB* may be more desirable than that of *CEFD* because the progress of the work is advanced more rapidly by the former than by the latter, the ratio of progress being about that of the lengths $\frac{BD}{DF}$.

559. Base Nets.—In a system of triangulation, long sides (within proper limits) are obviously more economical than short ones. It is difficult and expensive to measure long base lines; hence, in practice, the base lines are usually much shorter than the average length of the triangle sides. This condition necessitates the most careful attention to the location of the base lines and the immediately adjacent stations. The figure formed by this group of stations is called the *base net* and is formed so as to permit economical lengths of triangle sides to be used with a minimum loss in the precision of the measured base line.

The arrangements shown in Fig. 558*a* and at the left in Fig. 559 are examples of excellent base nets affording quick and accurate

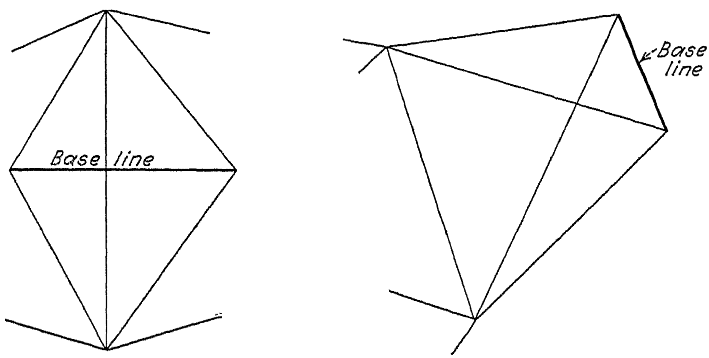


Fig. 559.—Base nets.

expansion of the base line to the longer sides of the system. The form of base net suggested by the quadrilaterals *GHFE* and *GHJI* (Fig. 558*c*) is satisfactory if it can be so laid out as to avoid the small angles there shown. This form is also shown at the right in Fig. 559.

The number of base lines required will depend upon the excellence of the shapes of the triangles in the system. In practice, the accuracy necessary for work of ordinary precision can be carried through a chain of 20 to 60 triangles, depending upon the strength of the figures secured.

560. Signals and Instrument Supports.—Each triangulation station must be marked by some sort of signal visible from nearby stations from which it is to be sighted. The form of signal selected will depend upon the locality and upon the materials available. If the station is one to which observations are taken, but which itself is not occupied for the measurement of angles, no provision need be made for setting the instrument at the station, and, in conse-

quence, a relatively simple signal structure is used. This signal may be one constructed for the purpose or it may be an object already in place such as a flagpole, chimney, or telegraph pole.

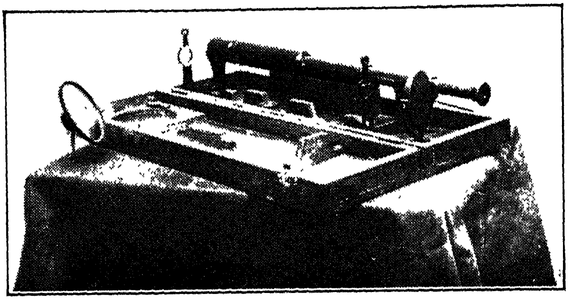


FIG. 560*a*.—Heliotrope, box type.

A pole set vertically in the ground or held firmly in a vertical position by a pile of stones, or by guys or bracing, makes an excellent signal

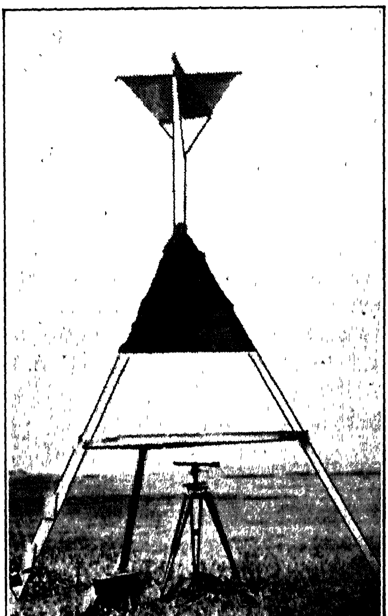


FIG. 560*b*.—Tripod signal.

on a bare summit or in open country. Such a signal is illustrated in the upper left-hand corner of Fig. 560*c*. A white paint mark on a rock cliff is sometimes all that is required. To increase the visi-

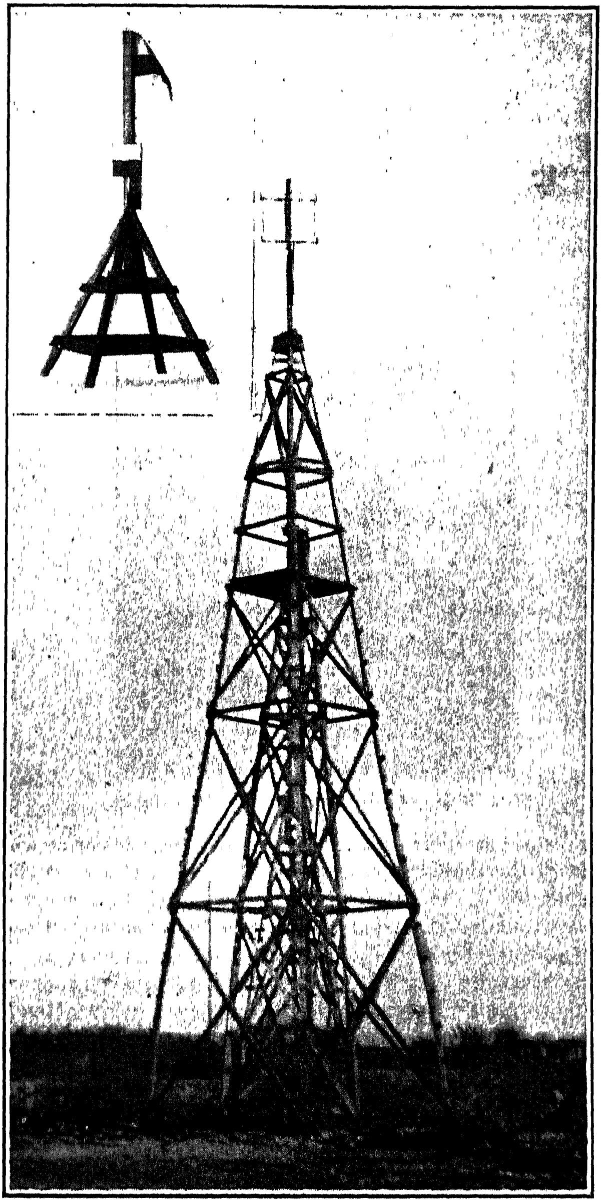


FIG. 560c.—Observing tower and signals.

bility of a pole or tree signal, two square or rectangular targets are sometimes attached, being placed at right angles to each other.

During the middle of the day, unless the sky is overcast, the atmospheric conditions render visibility poor and sighting inaccurate for the distances used in triangulation of high precision (5 to 40 miles and often much greater). Hence, the best time for observing is in the late afternoon or at night.

When the sun is shining, heliotropes of various designs are used as signals for long distances at which flags or poles are invisible. In principle the heliotrope consists of a mirror so arranged as to flash sun-

FIG. 560*d*.—Graphical triangulation signal.

light to the distant station. Figure 560*a* shows the box type in which the line of sight of the telescope is fixed parallel with the axis of the open rings. Therefore, if the heliotrope telescope is sighted at a distant station and by mirror adjustments the beam of light is reflected along the axis of the two rings, the ray is directed to the same station.

For night observations, a dry-cell electric lamp is used as a signal.

When heliotropes or lamps are used there must be an attendant to operate them. Communication between the observer and the attendants is established by the use of code signals flashed back and forth.

At an instrument station it is desirable to have a signal so constructed that the instrument may be placed directly over the station when angular observations are to be taken. In a small triangulation system with triangle sides only a few hundred feet in length, and with but few angles to be measured, a temporary signal which is readily moved may be all that is necessary. A light tripod to which

is attached a plumb line may be centered over the station mark. If the stations are to be used over a longer period, the station may be marked by an iron pipe set vertically in the ground, in which pipe is placed a range pole or similar rod. When the station is occupied, the pole is temporarily removed. Where a tall mast is necessary for visibility, it may be supported in position by three guy wires attached to it near the top. Provision is made for swinging the bottom of the mast to one side when it is desired to place an

FIG. 560*e*.—Built-up tripod for instrument.

instrument at the station. It is necessary that the guyed top be accurately centered over the station.

Where a more permanent signal is required that does not need to be moved to provide for setting up the instrument, a large tripod like that shown in Fig. 560*b* is generally used. Such a signal may be constructed either of round poles or of sawed lumber, with the vertical mast projecting upward from the junction with the legs. The signal should be solidly built and firmly anchored, with the vertical mast centered accurately over the station and made as nearly vertical as possible.

The topographic instructions of the U. S. Geological Survey (Ref. 6, p. 864) give excellent instructions for the construction of signals.

Where the instrument must be elevated to secure visibility, a combined observing tower and signal like that shown in the large photograph of Fig. 560*c* is built. The central structure, built in the tripod form, is designed to support the instrument. Around

this, but entirely separate from it, is the three- or four-sided structure supporting the platform upon which the observer stands. Thus the instrument tower is free from the vibrations caused by movements of the observing party.

For graphical triangulation (Art. 415, p. 616) and for ordinary triangulation of low precision, it is not necessary that the instrument be placed exactly beneath the signal; and some stations are not designed to be occupied by the instrument. For such conditions a single staunch mast may be used, as illustrated in Fig. 560*d*.

Figure 560*e* shows a built-up tripod of a type which may be used if the required elevation of the instrument above the ground is not great.

561. Angle Measurements; General.—Thus far the term *triangulation station* has been used to designate instrument stations, *i.e.*, points where the instrument is set up to measure angles. In most triangulation systems secondary control is established by observations to stations in the vicinity of the primary or *major stations*, but these secondary stations, called *minor stations*, are not used in the extension of the main system of triangles. Obviously, the angle measurements of such stations may be made with a lower degree of accuracy than is required in the main system.

Major Stations.—The degree of accuracy necessary to observations on the major stations in work of ordinary precision may be secured with a 7 or 8-in. repeating instrument of good design (see Art. 561*a*). The average error of closure of the triangles should not exceed 5 sec. (Art. 556). For surveys where less accuracy is required, corresponding modifications should be made in the measurements of the angles.

The instrument is set up at each major station, and angles with vertex at the station are measured by the method of repetition described in Art. 206, p. 275. The method of procedure is stated and the form of notes is shown under field problem 8 with the transit, Art. 216.

The following suggestions are added to those there given: The instrument should be protected from wind and sun; good visibility is necessary, *i.e.*, the air should be free from smoke, mist, or heat waves; after the instrument has been set up, centered, and leveled, the tripod wing nuts should be loosened to free the tripod from any torsion developed while planting it in the ground, and the nuts should then be tightened to a firm bearing; if the stations observed are of some difference in elevation, the horizontal axis of the transit should be leveled with a striding level.

Minor Stations.—These may be definite objects of prominence suitably located for control purposes, such as lone trees, church

spires, flagstaffs, and chimneys, or they may be signals erected at desirable locations. The observations should be made with much the same care as those for major stations, but ordinarily the method of repetition need not be employed. Each angle should be measured, however, once with the telescope direct, and once with it reversed, both verniers being read. Minor stations should be observed from at least three stations, if possible, to provide a check on the computed or plotted positions of these secondary points.

561a. Instruments for Measuring Angles.—For triangulation of ordinary precision, the angles in the system are measured by means of a *repeating instrument* or *repeating theodolite*, which is similar in general design to the ordinary engineer's transit, but which is of larger size and of a higher grade of workmanship. The circle is 7 or 8 in. in diameter, and commonly the verniers read to 10 seconds. An example of this type is shown in Fig. 561*a*.

Because of the refinement necessary in pointing the instrument, a single vertical cross-hair like that in the telescope of the ordinary transit is not suitable. When targets or poles are sighted, cross-hairs placed in the form of an X are used; and when light signals are observed, two parallel vertical hairs are used, the space between them being very small.

For triangulation of low precision, the ordinary transit may be used.

For triangulation of high precision, either the repeating instrument or the *direction instrument* (Fig. 561*b*) is used. In designing these instruments, it was once thought that greater precision could be secured by increasing the size of the circles, but it has been found that because of lost motion, mechanical errors in graduating the circles, etc., nothing is gained by increasing the diameter beyond 10 or 12 inches.

The principal distinguishing features of the direction instrument are (1) that the horizontal plate has but a single tangent motion, and (2) that instead of verniers, micrometer microscopes are used to read the subdivisions of the graduated circle. A photograph of an 8-in. direction instrument is shown in Fig. 561*b*.

The single horizontal motion of the direction instrument is comparable to the upper motion of the ordinary transit. Accordingly, when angles are to be measured about a point, an initial circle reading must be made when the instrument has been pointed on the first distant signal. This initial reading is a measure of the azimuth, or direction, of the first object sighted, with respect to some reference meridian. The direction of this reference meridian is immaterial, depending entirely upon the chance position of the plate when fixed

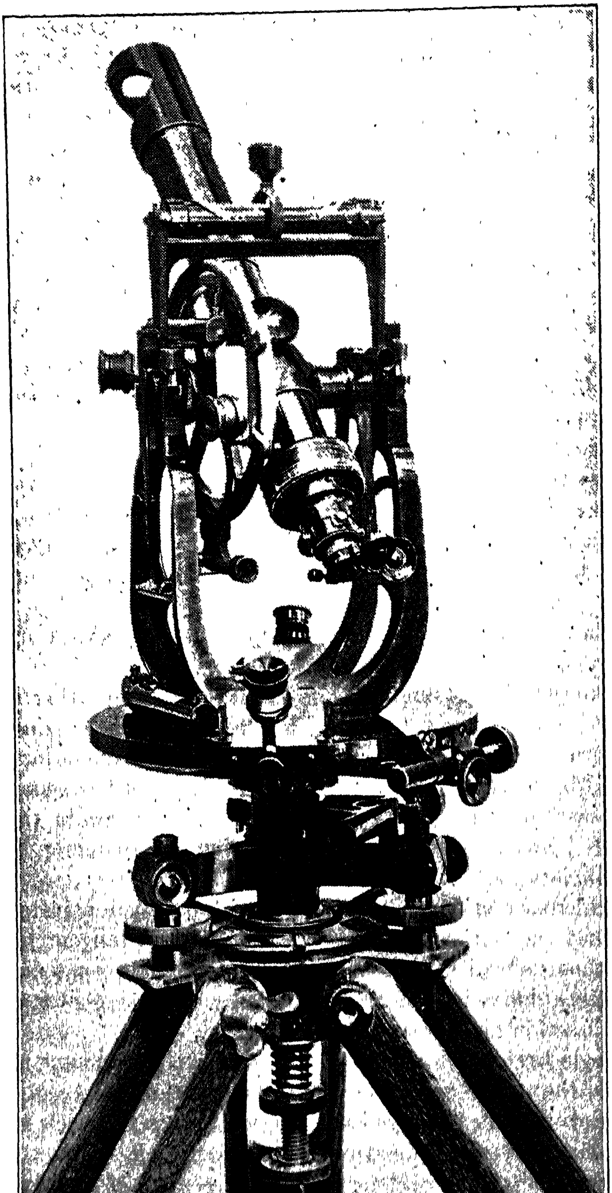


FIG. 561*a*.—Repeating instrument.

in position before making the first pointing. The directions of all distant stations are then read successively without disturbing the horizontal circle, and from these readings the values of the angles (from one station to the next) may be computed.

Precision in measuring parts of spaces of the graduated circle is secured by means of the micrometer microscope, two or three such

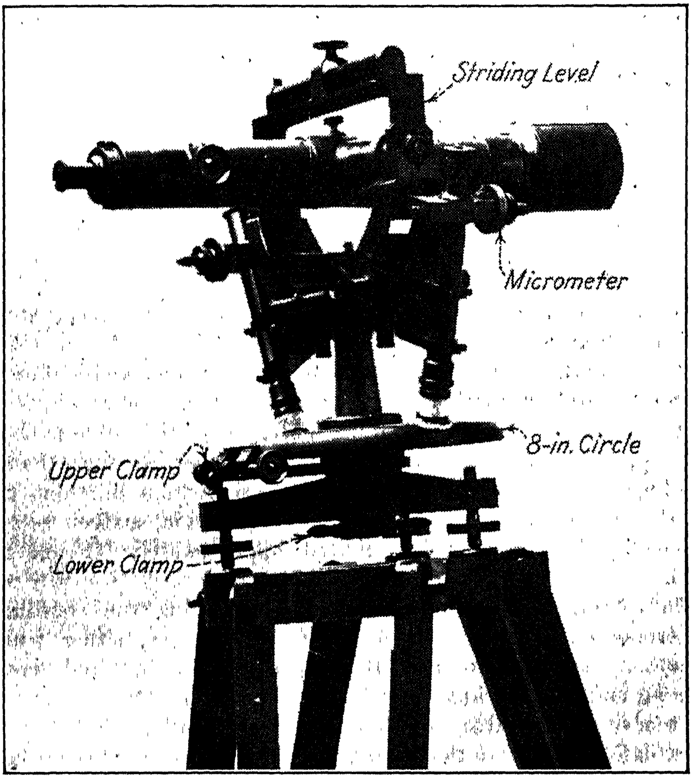


FIG. 561*b*.—Direction instrument.

microscopes being equally spaced around the circle. The device consists of a microscope focused on the graduated circle, having in the focal plane a wire or two closely spaced parallel wires mounted on a movable slide. The slide is moved by a milled thumb-screw carrying a graduated drum called the micrometer head. When the telescope has been pointed toward an object and the horizontal motion has been clamped, the index or fiducial line of the micrometer lies ordinarily between two circle graduations. To determine the

fractional part of a space, the wire is moved until it coincides with a scale division and the micrometer head is read. The direction is found by combining the micrometer reading and the scale reading. Sometimes the micrometer is read on each of the two graduations between which the index lies. The micrometer may usually be read directly to the nearest second, and by estimation to $1/10$ second.

The circle is clamped in position, and the telescope is pointed toward the initial station and clamped in that position. By means of the tangent-screw, the signal is observed precisely, and the initial reading is taken by reading the micrometers. The instrument is then turned clockwise, the cross-hairs are set on the next station, and the micrometers are read. Each station is thus observed in turn until the horizon has been closed and the initial station is again sighted. The telescope is then lifted from its Y-supports and plunged (the pivots, after plunging, resting in the same supports as before) and by revolving the telescope about the vertical axis the initial station is again sighted. The direction of each station is now observed as before, but the stations are sighted in reverse order, *i.e.*, the alidade is turned in a counter-clockwise direction from one station to the next. These two series of readings constitute one set. Before beginning a second set the circle is shifted by means of the lower clamp a number of degrees such that the readings for the several sets will be observed on different parts of the circle. In work of the highest precision, sixteen sets are observed, the circle being shifted approximately 11° between sets.

562. Azimuth Determinations.—In calculating the coordinates of triangulation stations a meridian of reference, either true or assumed, is used. For a small system of a few hundred acres or less an assumed meridian will be satisfactory; but for an extensive system it is desirable, and many times essential, that the true meridian be used. In the latter case, a determination of the azimuth of a triangle side is made at any convenient station, from which determination the azimuths of all other sides can be computed. If the system is many miles in extent, a determination is made at intervals of 20 to 30 figures as a check on the angle measurements. Solar or stellar observations may be used, depending upon the field conditions, stellar observations being by far the more accurate. The methods of determining azimuth are described in Chap. XVIII.

563. Base-line Measurement; the Tape.—The steel tape is used for measurements of ordinary precision and the invar tape, composed

of a nickel-steel alloy for which the coefficient of expansion may be as low as 0.0000002 per 1°F. (about $\frac{1}{30}$ that of steel), is used for base-line measurements of the highest precision.

The length of the tape should be accurately determined by comparison with a standard of known length. The National Bureau of Standards at Washington, D. C., for a small fee, will compare the tape with a length which has been precisely determined, and will issue a certificate showing the true length of the tape under stated conditions as regards tension, temperature, and supports (see Arts. 563*c* and 563*f*). It is desirable to have the tape standardized under the tension and supported in the manner that will be employed in the field work, so that no corrections for sag or stretch are necessary.

A tape that has been compared with the standard at Washington may itself serve to standardize other tapes in the field, but for work of the greatest accuracy all the tapes used should be compared with the standard at Washington.

If a tape is kinked in handling, its length will be appreciably changed. Invar metal is relatively soft and bends easily. Hence, tapes of this material should be handled with great care and when not in use should be kept on a reel not less than about 15 in. in diameter. In the best practice, two or three tapes are provided and when in use they are compared daily, thus to detect sudden changes in length due to whatever cause. In any case, a tape should have its length again compared with some standard upon the completion of the work.

563a. Measuring the Base Line.—Base-line measurements can be made satisfactorily over somewhat rough and uneven ground if provision is made for properly supporting and stretching the tape. The top of the rail of a railroad track or the surface of a concrete roadway of uniform grade may be used for base-line measurements, and these surfaces render unnecessary part of the special preparations required for measurements over uneven ground. These surfaces should not be used, however, when their temperatures are appreciably different from that of the surrounding air.

Where the base line is over uneven ground, supports for the tape are provided by the use of substantial posts, 2 by 4 in. or 4 by 4 in., driven firmly in the ground. These are placed on a transit line at intervals of one tape length, as nearly as may be determined by careful preliminary measurements. A strip of copper or zinc is tacked to the top of the post to provide a suitable surface on which to mark the tape lengths. Profile levels are run over the tops of the posts to determine the gradient. The tape is usually supported at one, two, or three points between the end supports.

These intermediate points must be placed accurately on the grade line between the tops of the two adjacent end posts. This may be done by driving nails at grade in 1 by 2-in. stakes placed on line at the proper intervals. These supports preferably should be provided at the same intervals as those used in the standard comparison, and the nails should be so driven that the tape will not become pinched between nail and stake.

The equipment for base-line measurement includes at least one standardized tape (on important work, two tapes are essential); two stretcher devices for applying tension (Fig. 563*a*); a spring balance,

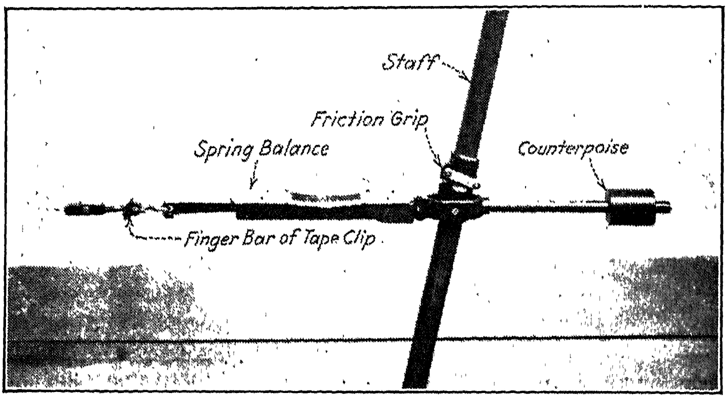


Fig. 563*a*.—Tape stretcher and spring balance.

or weight and pulley; two or three thermometers; a finely divided pocket scale; and dividers.

The party consists of four to six men whose duties are indicated by the following description of the procedure. The tape is stretched and the proper tension is applied by means of the stretcher apparatus and a spring balance (or weight and pulley) attached to the forward end of the tape beyond the end support (Fig. 563*a*). When the rear end of the tape is observed to coincide with the mark (Fig. 563*c*) and when the proper tension is applied, the position of the forward end of the tape is marked by a fine line engraved on the metal strip on the top of the stake (Fig. 563*b*); thermometers held at the same height above the ground as the tape, one at each end and sometimes one at the mid-point of the length of tape, are read at the instant that the tape length is marked on the forward post. The tape is then carried forward without allowing it to drag on the ground, and the process is repeated. After a few measurements, the end of the tape will most likely fall either beyond or short of the limits

of the metal strip of the next forward post because of variations in temperature or because of the inaccurate placement of the posts.

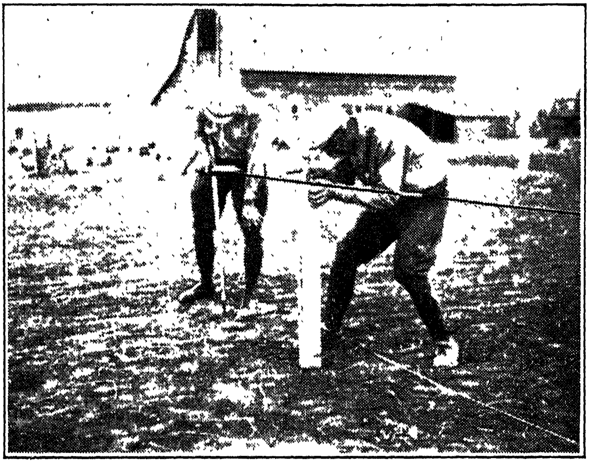


FIG. 563*b*.—Making forward contact.

Accordingly, it will be necessary occasionally to use *set forwards* or *set backs* as may be necessary to keep the tape ends on top of the posts. These are measurements of small distances made by the use of a finely divided pocket scale and a pair of dividers. A record is kept of all observations, as shown in Fig. 563*d*.

FIG. 563*c*.—Making rear contact.

563b. Errors in Base-line Measurements.—The nature of the various sources of error in tape measurements as they affect work

of ordinary precision is treated in Art. 87, p. 91, but the special procedure used in base-line measurements causes changes in sources of error which should be noted.

1. *Tape Not Standard Length.*—If the tape is carefully handled, the comparison made by the Bureau of Standards will be sufficiently precise to render the error from this source negligible for the duration of a field season. Experience has shown that a tape changes in length over a period of years even though it is not used. Hence, a tape should be standardized in the same season in which it is used.

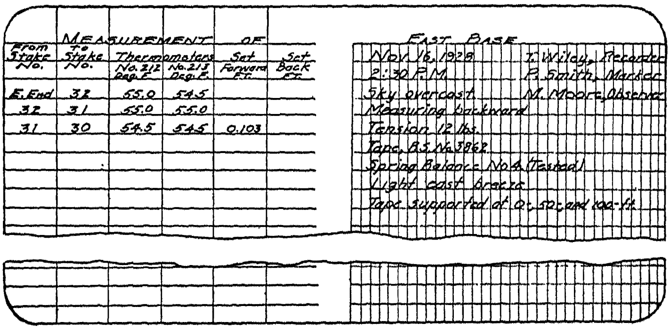


FIG. 563*d*.—Form of notes for base-line measurement.

2. *Variations in Temperature.*—It has been stated above that the effect of temperature is the most serious source of error in precise linear measurements. For example, an error of 0.5°F. in the mean temperature of a steel tape introduces an error of $\frac{1}{300,000}$ in the measured length, which alone is greater than the total probable error permitted in precise work. The magnitude of the error due to variations in temperature has been greatly reduced by the use of invar tapes.

Field conditions favorable to a small variation in the thermal expansion are those in which the air and the ground are at nearly the same temperature while the measurement is being made. These conditions are obtained on cloudy days or at night, and it is at such times that important base lines are usually measured. Measurements taken at such times have indicated probable errors as low as $\frac{1}{3,000,000}$.

A pavement or railroad track may have a temperature considerably different from that of the air; hence before a base-line measurement is made along such a surface, observations should be taken to make sure that air and surface temperatures agree closely.

3. *Tape Not Horizontal.*—This source of error is rendered negligible by determining the difference in elevation of adjacent posts by a line of profile levels run with a transit or an engineer's level. The gradient for each tape length is then readily computed and the corrections for grades up to 5 per cent may be calculated by the approximate formula (1) of Art. 86, p. 89. Steeper grades may be used if necessary, but the corrections should then be computed by the exact formula (3) of Art. 86. The same correction is applied to measurements of base lines along highway or railroad, the grade of the supporting surface being obtained by leveling.

4. *Variations in Tension; Sag in Tape.*—Variations in tension are not important if only stretch in the tape is considered, as where the tape is supported throughout its length. The effect upon the amount of sag, if the tape is supported at intervals, is much more serious. This effect is more important the less the number of supports. Other things being equal, the shortening due to sag varies inversely as the square of the tension. The amount of the resulting error for varying conditions is given in Table 90, p. 95. If the tension is determined correctly within 1 oz., the resulting error is negligible for the class of work under consideration.

A normal tension (see Art. 91, p. 96) is used where conditions are favorable.

5. *Wind.*—If a strong wind is blowing normal to the base line and if the tape is not supported throughout its length, there will be a lateral displacement of the unsupported portions of the tape, thus producing an effect similar to that of sag. Wind also sets up vibrations which render the tape unsteady. It is impossible to calculate corrections for wind effects; accordingly, precise measurements should not be attempted if the tape is unsupported and if a strong cross wind is blowing.

6. *Marking the Tape Lengths.*—The magnitude of the errors resulting from this source depends upon the fineness of the tape graduation and of the line cut on the metal strip, and also upon the precision with which these lines are made to coincide when the tape lengths are being marked. The lines marking the ends of ordinary steel tapes are relatively coarse, but makers of invar tapes use lines not exceeding 0.002 in. in width. The Bureau of Standards is careful to state which edge of the tape is used in making the comparison with the standard length. For a given tape, careful manipulation is the only means of reducing errors from this source.

563c. Corrections.—The methods of computing and applying corrections to tape measurements are given in Arts. 85 to 92, pp. 89 to 97.

Following is an example showing the corrections applied to the measured length of a base line:

Example: The length of a base line is recorded as 3,243.063 ft. and the average observed temperature is 59.7°F. The constants of the tape are given below. Corrections are to be determined.

Bureau of Standards Data for Tape No. 3862:

Length at 68°F. = 100.0214 ft.;

Tape supported at 0, 50, and 100-ft. marks;

Tension = 10 lb.;

Coefficient of expansion = 0.00000645 per degree Fahrenheit.

Corrections are as follows:

	Feet
Recorded length	3,243.063
Length correction	+0.694
Total set forwards	+0.364
Total set backs	−0.158
Temperature correction	−0.174
Total slope correction	−0.364
Length of base	3,243.425

563d. Reduction to Sea Level.—It is sometimes necessary to reduce the length of the base line to the equivalent length at mean sea level. The correction to be subtracted from the actual length is given by the equation

$$C_l = \frac{LA}{R}$$

in which L is the length of the base line, A is the mean altitude of the base line above sea level, and R is the radius of the earth (mean R = 20,890,600 ft., log R = 7.31995).

563e. Discrepancy between Bases.—Experience indicates that for first-, second-, and third-order triangulation, the precision attained in a base line computed from a measured base through a chain of approximately 20 figures will be reduced to about one fifth of that of the measured base line, provided that the angles of the system are measured with an accuracy corresponding to that of the accidental errors in the base measurement. Thus, if the probable error of a measured base is, say $\frac{1}{25,000}$, the probable error of a base computed from the measured base through a chain of 20 figures is about $\frac{1}{5,000}$. This relation makes it possible to estimate in advance the required accuracy of base-line measurements to produce

a check on base which will meet the requirements of a given specification.

563f. Specifications; *Ordinary Precision.*—The manual of instructions of the U. S. Coast and Geodetic Survey (Ref. 7, p. 864) states the requirements for the precision of base-line measurements for third-order triangulation as follows:

> "In base-line measurements, select apparatus and methods which insure that the constant error does not exceed one part in 30,000, and that the accidental errors are not greater than that represented by a probable error of one part in 100,000, in the length of the base."

In accordance with these requirements and for a base line not less than ½ mile in length measured over uneven ground, the following specifications may be relied upon to yield satisfactory results. It is assumed that all corrections will be calculated and applied.

The tape shall be of good manufacture, and shall be standardized by the Bureau of Standards; the mean temperature of the tape shall be determined within a maximum error of 3°F.; the elevations of adjacent marking posts shall be determined within a maximum error of 0.3 ft.; the tape shall be supported, if possible, under the same conditions as those existing when it was standardized; the tension of the tape shall not vary more than 2 oz. from the standard tension adopted for the work; the errors in marking shall not exceed 0.002 ft.

If the line is measured along a paved highway or a railroad track, the above conditions will apply, except that the permissible variation in tension may be increased to 1 lb.

For base lines of less precision, corresponding modifications can be made as explained in Art. 92, p. 97.

Low Precision.—For measurements of low precision the systematic errors are likely to become more important than in refined measurements, and for this reason a somewhat higher degree of accuracy in the measurements must be maintained than would otherwise be necessary. A detailed analysis of the interrelation of the errors in triangulation work is beyond the scope of this text, but the following general specifications for three degrees of low precision are given as applicable to average conditions. It should be realized that field conditions vary widely and that they appreciably influence the accuracy of results.

Case I. *Required:* Discrepancy between base lines not to exceed $\frac{1}{3{,}000}$.

Specifications: (*a*) Each base line to be not less than 2,500 ft. in length, and to be measured with a probable error not to exceed $\frac{1}{20{,}000}$; (*b*)

triangle sides to be from ½ to 3 miles in length; (*c*) the average closing error of triangles not to exceed 8″.

Case II. *Required:* Discrepancy between base lines not to exceed $\frac{1}{1,000}$.

Specifications: (*a*) Each base line to be not less than 1,500 ft. in length and to be measured with a probable error not to exceed $\frac{1}{10,000}$; (*b*) triangle sides to be from ¼ to 2 miles in length; (*c*) the average closing error of triangles not to exceed 15″.

Case III. *Required:* Discrepancy between base lines not to exceed $\frac{1}{500}$.

Specifications: (*a*) Each base line to be not less than 1,000 ft. in length and to be measured with a probable error not to exceed $\frac{1}{5,000}$; (*b*) triangle sides to be from 500 to 5,000 ft. in length; (*c*) the average closing error of triangles not to exceed 30″.

High Precision.—Following are the essential features of the procedure used by the U. S. Coast and Geodetic Survey for measurement of base lines for first-order triangulation (Ref. 2, p. 864).

The length of each base is determined by at least two complete measurements with an accuracy represented by a probable error of less than 1 part in 1,000,000. At least three tapes are used in such a way as to secure comparisons between them. The line is cleared to a width of at least 6 ft., and 4 by 4-in. stakes are set accurately on line at every tape length to support the copper strips on which the ends of the tape lengths are marked, and 2 by 4-in. stakes are set at intermediate points to support the tape.

Invar tapes 50 m. long are used, for which the following data are determined during the standardization tests:

Weight of tape in grams per meter.

Coefficient of expansion per degree Centigrade.

Length at specified temperature supported at 0, 25, and 50-m. points.

Length at specified temperature supported at 0, 12.5, 25, 37.5, and 50-m. points.

Length at specified temperature supported throughout.

Measurement of the line is made by a party of six men, a front contact man at the 50-m. mark on the tape, a rear contact man at the zero mark, a front and a rear stretcher man, a man at the middle point of support, and a recorder. A tension of 15 kg. is used. The temperature at each end of the tape is observed for each tape length. Figures 563*a* to 563*c* show the details of the stretcher, how each end of the tape is held, and how contact is made.

564. Computations; Adjustment of a Chain of Triangles.—In the case of the single chain of triangles there are two kinds of adjustments to be made: (1) the *station adjustment*, to make the

sum of the angles around each point equal 360°, and (2) the *figure adjustment*, to make the sum of the three angles in each triangle equal 180°.

In precise triangulation the station adjustment and the figure adjustment are made in one operation, by methods involving the principles of Least Squares, but the following approximate solution yields results that are sufficiently accurate for most cases of triangulation of ordinary precision.

To make the sum of the angles around each point equal to 360°, the angles are added together and the sum is subtracted from 360°. The difference resulting is divided by the number of angles around the point, and the quantity so found is algebraically added to each angle. To make the sum of the angles in each triangle equal to 180°, a similar plan is followed, using the values obtained by the first adjustment; that is, the three angles are added together and the sum is algebraically subtracted from 180°. One-third of the difference is added algebraically to each of the three angles.

This method of adjustment assumes that all tne angles were observed in the same way and with the same precision, and is only applicable when such is the case. If certain angles are measured with a higher precision than others, the method may be readily modified by weighting the observations of the several angles within

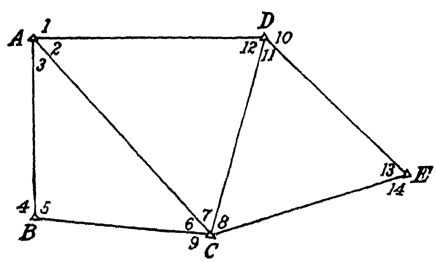


FIG. 564*a*.—Adjustment of a chain of triangles.

the system, either arbitrarily or by the method of Least Squares, as described in Chap. V.

Following is an example of the adjustment of the angles in a simple chain of three triangles, all of the observed angles being assumed to be of equal precision.

Example: Below are tabulated the observed angles in the chain of triangles shown in Fig. 564*a*. The station and figure adjustments are to be made by approximately dividing the errors equally among the angles.

The values after performing the station and figure adjustments are as shown.

Station Adjustment

Angles	Observed values	Stations	
1	240°19′00″	*A*	Sum of observed angles = 360°00′30″
2	73°31′10″		Adjusted angles
3	46°10′20″		1 .... 240°18′50″
4	267°12′20″		2 .... 73°31′00″
5	92°47′30″		3 .... 46°10′10″
6	41°02′00″		
7	63°10′40″	*B*	Sum of observed angles = 359°59′50″
8	74°43′10″		Adjusted angles
9	181°04′30″		4 .... 267°12′25″
10	260°33′00″		5 .... 92°47′35″
11	56°09′00″		
12	43°18′30″	*C*	Sum of observed angles = 360°00′20″
13	49°07′50″		Adjusted angles
14	310°52′10″		6 .... 41°01′55″
			7 .... 63°10′35″
			8 .... 74°43′05″
			9 .... 181°04′25″
		D	Sum of observed angles = 360°00′30″
			Adjusted angles
			10 .... 260°32′50″
			11 .... 56°08′50″
			12 .... 43°18′20″
		E	Sum of observed angles = 360°00′00″
			Adjusted angles
			13 .... 49°07′50″
			14 .... 310°52′10″

Figure Adjustment

Triangle *ABC*

Angles	Values from station adjustment	Values from figure adjustment
3	46°10′10″	46°10′16″
5	92°47′35″	92°47′42″
6	41°01′55″	41°02′02″
Sum....	179°59′40″	180°00′00″

Triangle *ACD*

Angles	Values from station adjustment	Values from figure adjustment
2	73°31′00″	73°31′02″
7	63°10′35″	63°10′37″
12	43°18′20″	43°18′21″
Sum....	179°59′55″	180°00′00″

	Triangle CDE	
8	74°43′05″	74°43′10″
11	56°08′50″	56°08′55″
13	49°07′50″	49°07′55″
Sum....	179°59′45″	180°00′00″

564a. Adjustment of a Quadrilateral.—In the adjustment of the angles of a quadrilateral, two conditions may be considered: the *geometric condition* that the sum of the interior angles of a rectilinear figure is equal to $(n - 2)$ 180°, in which n is the number of sides of the figure; and the *trigonometric condition* that in any triangle the sines of the angles are proportional to the lengths of the sides opposite.

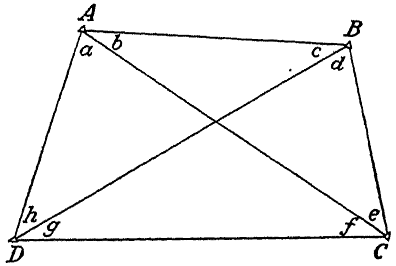


FIG. 564*b*.—Adjustment of a quadrilateral.

1. *Geometric Condition.*—When all angles in a quadrilateral are measured, there are four overlapping triangles. These are shown as triangles ABC, ACD, ABD, and BCD in Fig. 564*b*. In each of these triangles the sum of the three angles must be 180°. Hence from the figure,

$$b + c + d + e = 180° \qquad (1)$$
$$a + f + g + h = 180° \qquad (2)$$
$$a + b + c + h = 180° \qquad (3)$$
$$d + e + f + g = 180° \qquad (4)$$

Also the sum of the eight lettered angles in the figure must equal 360°, since they form the interior angles of a closed figure of four sides. This may be derived also by the addition of equations (1) and (2) or (3) and (4).

$$a + b + c + d + e + f + g + h = 360° \qquad (5)$$

Further, since the vertically opposite angles at the intersection of the diagonals must be equal, it follows that

$$b + c = f + g \qquad (6)$$
$$d + e = h + a \qquad (7)$$

Equation (6) is the equivalent of Eq. (2) minus Eq. (3), and Eq. (7) is the equivalent of Eq. (2) minus Eq. (4).

If any three of these seven equations are satisfied, the other four must of necessity be satisfied also. Equations (5), (6), and (7) are the ones most convenient to use.

The following procedure is suggested for triangulation of ordinary precision:

A. Adjust the angles around each point to make their sum equal to 360° by distributing the error equally (or approximately so) among the several angles.

B. Using the values resulting from adjustment *A*, add the eight angles *a*, *b*, *c*, *d*, *e*, *f*, *g*, and *h*, and subtract their sum from 360°. Divide the difference by 8, and algebraically add the result to each of the eight angles, thus satisfying the conditions of equation (5).

C. Using the adjusted values from *B*, find the difference between the sums $(b + c)$ and $(f + g)$ and divide that difference by four. Apply the result as a correction to each of the four angles, increasing each of the two whose sum is the smaller and decreasing each of the two whose sum is the larger, thus making these angles satisfy equation (6) without disturbing the adjustment for equation (5). Proceed in the same way with each of the four angles involved in equation (7).

2. *Trigonometric Condition.*—If the length of one line, as *AB*, is known, and the length of the opposite side *CD* is to be calculated, the computer may select one or another series of triangles for use in accomplishing this result. For example, a solution of triangle *ABC* gives the length of *AC*, then from triangle *ACD* the required length of *CD* is found; or in the triangle *ABC* the length *BC* is found, then in *BCD* the length *CD* is calculated. There are four possible choices of route through the figure and it now remains to be seen whether the angles, as so far adjusted, are so related as to make the value of the length of a computed side independent of the route used. Assume that the length of *AB* is known and the length of *CD* is to be found.

$$AD = AB\,\frac{\sin c}{\sin h}$$

$$CD = AD\frac{\sin a}{\sin f} = AB\,\frac{\sin a \sin c}{\sin f \sin h}$$

Similarly,

$$CD = AB\,\frac{\sin b \sin d}{\sin e \sin g}$$

Equating these two values of *CD*,

$$\frac{\sin a \sin c}{\sin f \sin h} = \frac{\sin b \sin d}{\sin e \sin g}$$

or

$$\frac{\sin a \sin c \sin e \sin g}{\sin b \sin d \sin f \sin h} = 1$$

Expressed in logarithmic form, this is

$$(\log \sin a + \log \sin c + \log \sin e + \log \sin g)$$
$$-(\log \sin b + \log \sin d + \log \sin f + \log \sin h) = 0 \quad (8)$$

The angles are tested for satisfaction of this equation by adding the logarithmic sines in the two groups as indicated and by finding the difference between the two sums.

Various adjustments by which this difference may be reduced to zero are possible. The Least Squares adjustment gives the most probable values to the adjusted angles, but it is somewhat more elaborate than is necessary for most surveys. A simple approximate method which gives an equal correction to each angle, and which does not disturb the geometric condition, is as follows:

(*a*) Record the log sines as shown in the following example.

(*b*) For each angle, record the tabular logarithmic sine difference for 1″ opposite each logarithm.

(*c*) Find the average required change (α) in log sin by dividing the difference between the sums by 8.

(*d*) Find the average difference (β) for 1″.

(*e*) The ratio $\frac{\alpha}{\beta}$ gives the number of seconds of arc to be applied as a correction to each angle.

(*f*) Add this correction to each of the four angles the sum of whose log sines is the smaller, and subtract it from each of the angles the sum of whose log sines is the larger, and thus the corrected values are obtained.

It must be noted that the sine of an angle greater than 90° decreases as the angle increases; in this case the size of the angle is changed in the reverse order, and the corrections must be interpreted accordingly.

Example: Given the angles as measured in the quadrilateral of Fig. 564*b*, for which the station adjustment, indicated as adjustment *A* under the suggested procedure (p. 852), has been made. Find the adjusted angles for both the geometric and the trigonometric conditions.

Adjustment B.—The sum of the angles *a*, *b*, *c*, etc., resulting from the station adjustment *A*, is found to differ from 360° by the amount of 08″. This amount divided by the number of angles gives the amount of the correction 01″, to be subtracted from each angle as shown in the second column in the table on p. 854.

Adjustment C.—The sum of the angles $b + c$ = 66°03′45″
The sum of the angles $f + g$ = 66°03′37″

	Geometric Condition		Trigonometric Condition
Adjustment A	Adjustment B	Adjustment C	Adjustment D
a 38°44′06″	38°44′05″	38°44′06″	38°44′08″
b 23 44 38	37	35	33
c 42 19 09	08	06	08
d 44 52 01	00	51 59	51 57
e 69 04 21	20	19	21
f 39 37 48	47	49	47
g 26 25 51	50	52	54
h 75 12 14	13	14	12
360 00 08	360 00 00	360 00 00	360 00 00

Dividing this difference by 4, the correction to each angle is found to be 02″, to be subtracted from the angles b and c, and added to the angles f and g. In like manner, the corrections to each of the angles d, e, h, and a is found to be 01″, to be added to h and a, and subtracted from d and e. The resultant angles are shown in the third column of the above table.

Adjustment D. Trigonometric Condition.—The sums of the logarithmic sines of the angles as given by adjustment C and as indicated in equation (8) are recorded as shown below, and the tabular difference for 1″ is recorded for each angle.

		Difference for 1″	
log sin a	9.796380	2.6	
log sin c	9.828176	2.3	
log sin e	9.970361	0.8	
log sin g	9.648479	4.2	
	9.243396		
log sin b	9.604912	4.8	
log sin d	9.848470	2.1	
log sin f	9.804706	2.6	
log sin h	9.985355	0.6	
	9.243443	20.0	8)20.0
	9.243396		2.5 = β
Difference	47		8)47
			5.9 = α

The difference between the two sums is 47 units of the six places of logarithms used. This value, divided by 8, gives the average required change in log sine, 5.9 = α. The average tabular difference for 1″

is $2.5 = \beta$. Hence $\frac{\alpha}{\beta} = \frac{5.9}{2.5} = 2''$ (nearly), which is the correction to be applied to the angles. Obviously, it will be added to angles a, c, e, g, and subtracted from angles b, d, f, and h.

Since this adjustment is applied with opposite sign to alternate angles it does not disturb the geometric condition. The final adjustment is given in the fourth column of the above table.

564b. Reduction to Center.—At certain triangulation stations it is difficult, if not impossible, to place the instrument vertically beneath the object which has been observed from adjacent stations. At such a place, the instrument is set over any convenient point near the principal station and angles to the adjacent stations are measured with the same precision as other angles in the system. These angles

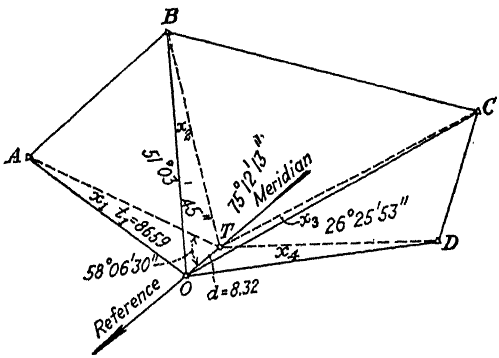


FIG. 564*c*.—Reduction to center.

will not be the same as those which would be observed if the instrument were occupying the exact location of the station; to obtain the corresponding values for the main station, corrections are computed and applied to the measured angles. This procedure of correcting the observed angles is termed *reduction to center*.

In addition to the measurement of the angles to adjacent stations, measurements are made of (1) the distance from the main station to the occupied station, and (2) the (clockwise) angle at the occupied station between the main station and an adjacent station in the system. The situation is illustrated by Fig. 564*c*, where O represents a main station in the system $OABCD$; T represents the point occupied by the instrument; the distance d and the angle ATO are measured. The lengths of all sides in the main system, as for example $t_1 = 8{,}659$ ft., are known approximately from the angles which have been measured at the stations A, B, C, D, and from the known sides AB, BC, etc.

In the triangle AOT, the angle T and the two sides t_1 and d are known, hence the angle x_1 may be computed. This angle is seen to be the difference in direction at station A between the lines AT and AO. Therefore, if the direction (azimuth) of AT is known with respect to any reference meridian, the direction of AO with respect to the same meridian can now be calculated. Likewise, the directions of the lines BO, CO, and DO can be found, and since these directions are referred to the same meridian, the correct angles at O between these stations can be determined.

The value of x_1 in triangle AOT is given by the equation

$$\sin x_1 = \frac{d \sin T}{t_1} \tag{9}$$

and since the angle x_1 is a small angle for which the sine is nearly equal to the arc, the value of x_1 will be given in seconds of arc if we divide both members of the equation by the sine of 1″, or

$$x_1'' = \frac{d \sin T}{\sin 1'' t_1} \text{ (approximate)} \tag{10}$$

It will be noted that $\dfrac{d}{\sin 1''}$ is a constant for a given station, so that once its value has been determined the successive correction angles x_1, x_2, etc. can be computed by a single multiplication. Since the correction angles are usually small, the slide rule will ordinarily render values correct to seconds.

564c. Computations of Triangles and Coordinates.—In computing the lengths of the sides and the coordinates of the stations in a triangulation system, it is desirable to follow an orderly procedure to expedite the work and to avoid mistakes. Convenient arrangements for these computations for plane triangulation are given below.

Triangles.—A sketch of the figure is drawn (Fig. 564*d*) and the vertices are lettered A, B, and C in a clockwise direction, beginning with the side whose length is known. The sides opposite the vertices are indicated by the corresponding lower-case letters, as a, b, and c. The sine relation states that

$$b = c\frac{\sin B}{\sin C}, \text{ or } \log b = \log c - \log \sin C + \log \sin B \tag{11}$$

Accordingly, if the logarithms are recorded in the column of logarithms in the order, log c, colog sin C, log sin A, and log sin B, then log a is found by covering log sin B with a narrow strip of paper and by adding the other three values. Also, to find log b, cover log sin A, and add the remaining three values. Finally, the distances

a and b are found as the numbers corresponding to their respective logarithms.

Following is an example of calculations:

Example 1: It is desired to calculate lengths of the sides a and b, the known data being given in the figure of the triangle, Fig. 564*d*.

The logarithmic calculations are shown in the tabulation of Fig. 564*d*:

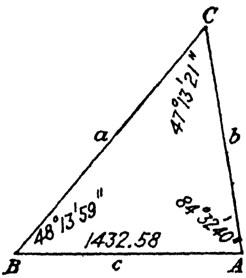
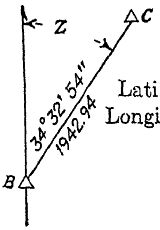
Station or line	Angle or distance	Logarithm	Figure
c	1,432.58 ft.	3.156119	
C	47°13′21″	0.134306 (colog)	
A	84°32′40″	9.998028	
B	48°13′59″	9.872657	
a	1,942.94 ft.	3.288453	
b	1,455.74 ft.	3.163082	

Fig. 564*d*.

Coordinates.—Example 2 (Fig. 564*e*) shows a form for calculating the coordinates of station C from each of the stations B and A. For each calculation a sketch is drawn and the known data are recorded, the azimuth of the side being changed to the bearing angle Z. The computation is then carried out as indicated, due regard being paid to the signs of the latitude and longitude values. Beginning with log L in the tabulation, computations for latitude are made reading upward, and computations for longitude are made reading downward.

Example 2: It is desired to compute the coordinates of C, the coordinates of A and B and the lengths a and b being given. The logarithmic calculations are given in Fig. 564*e*.

564d. Three-point Problem.—The purpose of the three-point problem has been stated and a graphical solution has been given in Chap. XXIII. In triangulation work the position of an instrument station as O (Fig. 564*f*) is determined by measuring each of the two angles subtended by three visible stations, as A, B, and C, and by solving the triangles involved. Thus, in Fig. 564*f*, all parts of the triangle formed by the stations A, B, and C are known. The angles α and β are measured at the station O. The problem is solved

From Station B	From Station A
 $Z = 34°32'54''$; $BC = 1{,}942.94$; Latitude $B = +661.36$; Longitude $B = -1{,}590.94$	$Z = 12°40'27''$; $AC = 1{,}455.74$; Latitude $A = +841.37$; Longitude $A = -169.71$ 

Mean Latitude C = +2261.64

(Check)

Latitude C	+2,261.64	Latitude C	+2,261.63
Latitude B	+ 661.36	Latitude A	+ 841.37
$L \cos Z$	+1,600.28	$L \cos Z$	+1,420.26
Log $L \cos Z$	3.204195	Log $L \cos Z$	3.152369
Log $\cos Z$	9.915742	Log $\cos Z$	9.989287
Log L	3.288453	Log L	3.163082
Log $\sin Z$	9.753661	Log $\sin Z$	9.341249
Log $L \sin Z$	3.042114	Log $L \sin Z$	2.504331
$L \sin Z$	+1,101.83	$L \sin Z$	−319.40
Longitude B	−1,590.94	Longitude A	−169.71
Longitude C	− 489.11	Longitude C	−489.11

(Check)

Mean Longitude C = −489.11

FIG. 564e.

when the values of the angles x and y have been determined, for the remaining parts in each of the triangles ABO and ACO can then be computed. A check is afforded if the same value for the side AO results from each of these triangles.

The problem is indeterminate if the station O lies on or near the great circle passing through the stations A, B, and C. This condition will be evidenced by the condition that $\alpha + \beta + A = 180°$.

There are many solutions for this problem. The one presented here follows that given by the U. S. Coast and Geodetic Survey (Ref. 7, p. 864).

Solution: Given the sides b and c and the angle A, also the observed angles α and β (Fig. 564f).

Let

$$S = 180° - \tfrac{1}{2}(A + \alpha + \beta) = \tfrac{1}{2}(x + y)$$

If the stations A and O lie on the same side of the side a, and if the station O is outside the triangle ABC, then

$$S = \tfrac{1}{2}(A - \alpha - \beta) = \tfrac{1}{2}(x + y) \qquad (12)$$

for which case the solution by this method is impossible when $\alpha + \beta = A$. Let

$$\tan \phi = \frac{c \sin \beta}{b \sin \alpha}, \text{ and let } \Delta = \frac{1}{2}(x - y)$$

then

$$\tan \Delta = \cot (\phi + 45°) \tan S \qquad (13)$$

If $\tan \Delta$ is positive, $x = S + \Delta$ and $y = S - \Delta$.
If $\tan \Delta$ is negative, $x = S - \Delta$ and $y = S + \Delta$.

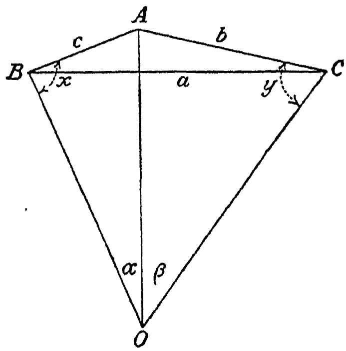


Fig. 564*f*.—Three-point problem.

Example:
Given

$$\begin{aligned} c &= 6{,}672.5 \text{ ft.} \\ b &= 12{,}481.7 \text{ ft.} \\ A &= 152°23'22'' \\ \alpha &= 20°05'53'' \\ \beta &= 35°06'08'' \end{aligned}$$

To find the angles x and y.

$$S = 180° - \tfrac{1}{2}(A + \alpha + \beta) = \tfrac{1}{2}(x + y)$$

$$\begin{aligned} A &= 152°23'22'' \\ \alpha &= \ \ 20°05'53'' \\ \beta &= \ \ \underline{35°06'08''} \\ & \ \ 2)\underline{207°35'23''} \\ & \ \ \ \ 103°47'42'' \\ S &= 76°12'18'' \end{aligned}$$

Let

$$\Delta = \tfrac{1}{2}(x - y)$$

$$\tan \phi = \frac{c \sin \beta}{b \sin \alpha}$$

$$\tan \Delta = \cot (\phi + 45°) \tan S$$

Then

$$
\begin{aligned}
\log c &= 3.824288 \\
\log \sin \beta &= \underline{9.759696} \\
\log (c \sin \beta) = \text{sum} &= 3.583984 \\
\log b &= 4.096274 \\
\log \sin \alpha &= \underline{9.536088} \\
\log (b \sin \alpha) = \text{sum} &= 3.632362
\end{aligned}
$$

$$\log \tan \phi = \log (c \sin \beta) - \log (b \sin \alpha) = 3.583984 - 3.632362 = 9.951622$$

$$
\begin{aligned}
\phi &= 41°48'55'' \\
& \underline{45°00'00''} \\
\phi + 45° &= 86°48'55'' \\
\log \cot (\phi + 45°) &= 8.745396 \\
\log \tan S &= \underline{0.609894} \\
\log \tan \Delta &= 9.355290 \\
\Delta &= 12°46'06''
\end{aligned}
$$

$$
\begin{aligned}
S &= 76°12'18'' \\
\Delta &= \underline{+12°46'06''} \\
x &= 88°58'24''
\end{aligned}
\qquad\qquad
\begin{aligned}
S &= 76°12'18'' \\
\Delta &= \underline{-12°46'06''} \\
y &= 63°26'12''
\end{aligned}
$$

564e. Computation of Geodetic Position.—Geodetic position is computed only for triangulation of high precision. The adjustment of the observations is accomplished by the method of Least Squares and is too elaborate for treatment here. The angles of the system are adjusted, and the lengths of the sides are computed. From these data are computed the geodetic coordinates, *i.e.*, the latitude and longitude of the stations included in the system.

The geodetic position of a station is calculated from that of a station of known latitude and longitude, having given the length and the azimuth (at the known station) of the connecting line. Due to the convergency of meridians, the azimuth of the connecting line at the unknown station will not differ by exactly 180° from that at the known station. In the following formulas, azimuths are measured clockwise from south.

Let

ϕ = latitude of the known station,

λ = longitude of the known station,

α = azimuth of the connecting line at the known station,

ϕ', λ', α' = corresponding quantities for the unknown station,

s = length of line, in meters,

N' = length of the normal at the unknown station, produced to the earth's polar axis, in meters,

a = semidiameter of equatorial axis = 6,378,206.4 meters, and

b = semidiameter of polar axis = 6,356,583.8 meters, both for the Clarke spheroid of 1866, upon which the published tables are based.

$$e^2 = \text{square of eccentricity} = \frac{a^2 - b^2}{a^2} = 0.006{,}768{,}658$$

From these quantities are computed the factors A', B, C, and D, using the following formulas:

$$A' = \frac{1}{N' \text{ arc } 1''} = \frac{(1 - e^2 \sin^2 \phi')^{1/2}}{a \sin 1''} \tag{14}$$

$$B = \frac{(1 - e^2 \sin^2 \phi)^{3/2}}{a(1 - e^2) \sin 1''} \tag{15}$$

$$C = \frac{(1 - e^2 \sin^2 \phi)^2 \tan \phi}{2a^2(1 - e^2) \sin 1''} \tag{16}$$

$$D = \frac{\frac{3}{2} e^2 \sin \phi \cos \phi \sin 1''}{1 - e^2 \sin^2 \phi} \tag{17}$$

Then

$$\phi - \phi' \text{ (in seconds)} = s \cos \alpha \cdot B + s^2 \sin^2 \alpha \cdot C + (s \cos \alpha \cdot B)^2 \cdot D \tag{18}$$

$$\lambda' - \lambda \text{ (in seconds)} = \frac{s \sin \alpha \cdot A'}{\cos \phi'} \tag{19}$$

$$(\alpha - \alpha' + 180°) \text{ (in seconds)} = (\lambda' - \lambda) \sin \tfrac{1}{2}(\phi' + \phi) \tag{20}$$

Special care must be taken with regard to the signs of the azimuth functions.

These formulas give sufficiently accurate results for a distance s not greater than 25 kilometers, or about 15 miles. For the more precise formulas used on longer lines and for examples of the use of these formulas, see Ref. 8, p. 864. For derivation of the formulas, see Ref. 9. For tables of values of the factors used, which factors vary according to the latitude, see Refs. 8 and 10.

Inverse Solution.—Sometimes the latitudes and longitudes of the two stations are given and the problem is to find the length and direction of the connecting line.

Knowing $(\phi - \phi')$ and $(\lambda' - \lambda)$, find the value of the term $s \cos \alpha \cdot B$ from Eq. (18) by subtracting the values of the small C and D terms from the known value of $(\phi - \phi')$. Divide $s \cos \alpha \cdot B$ by B to find the value of $s \cos \alpha$. Then

$$s \sin \alpha = \frac{(\lambda' - \lambda) \cos \phi'}{A'}$$

$$\tan \alpha = \frac{(\lambda' - \lambda) \cos \phi'}{A'(s \cos \alpha)} = \frac{s \sin \alpha}{s \cos \alpha}.$$

Knowing α, s is found from $s \cos \alpha$ or from $s \sin \alpha$.

Finally, $(\alpha - \alpha')$ is found as before, from Eq. (20).

564f. Correction for Spherical Excess.—Since the measured angles are spherical angles, each triangle will contain more than 180°. The amount greater than 180° is termed the *spherical excess* and is about one second for each 75 square miles of area of triangle. More exactly,

$$E = \frac{a}{C}(1 - e^2 \sin^2 \phi)^2$$

in which E is the spherical excess in seconds,
a is the area in square miles,
ϕ is the latitude at center of triangle,
$\log e^2 = 7.8305026 - 10$, and
$\log C = 1.8787228$.

It is clear that no correction for spherical excess will be necessary unless the triangles are very large, and then only in the most precise work.

565. Problems.

1. In the triangulation system illustrated in Fig. 558a, estimate the values of the angles in each of the two base nets; and by use of Fig. 21a, p. 19, or by an examination of the tabular differences for the sines of the angles, determine which base net is the stronger.

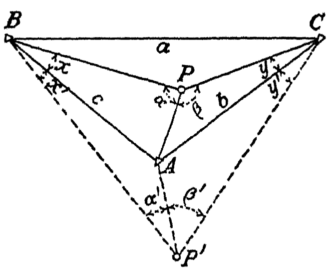


FIG. 565.

2. For the measurement of a base line the following data are given: The Bureau of Standards certificate states that the tape has a length of 99.946 ft. at 68°F., when supported at the 0 and 100-ft. points and under a tension of 10 lb.; the coefficient of expansion of the tape is 0.00000645 per degree Fahrenheit; the tape weighs 1½ lb. The field records give the measured length as 1,418.314 ft.; the average temperature was 63.6°F.; the stakes 0 to 8 were set on a 1 per cent grade,

the stakes 8 to end were set on a 3 per cent grade; the sum of the "set forwards" was 0.234 ft., the sum of the "set backs" was 0.114 ft. The interval and tension used were the same as those used for the standard comparison.

Calculate the length of the line.

3. For the conditions given in problem 2, calculate the normal tension.

4. For the conditions given in problem 2, the interval between supports in the field is assumed to be 50 ft. instead of 100 ft.

Calculate the corresponding change in the length of the tape.

5. The angles in a quadrilateral $ABCD$ resulting from the station adjustments, are as follows: $CAD = 45°30'55''$, $CAB = 42°11'39''$, $ABD = 41°54'40''$, $DBC = 62°40'53''$, $ACB = 33°12'51''$, $ACD = 28°05'30''$, $CDB = 56°00'50''$, $BDA = 50°22'51''$. The length of the side AD is 2,910.63 ft.

(*a*) Calculate the adjustment for the geometric condition.

(*b*) Calculate the adjustment for the trigonometric condition.

6. In measuring the angles at a triangulation station O, it was necessary to set the transit over another point T, at a distance of 13.25 ft. from O. The angle measured at T from O to the first distant station W, was $95°10'30''$. The angles between the distant stations W, X, Y, and Z were as follows: $WTX = 39°37'48''$; $XTY = 69°04'20''$; $YTZ = 83°16'08''$. The distances to the stations are found to be: $OW = 8{,}949$ ft.; $OX = 14{,}334$ ft.; $OY = 5{,}647$ ft.; and $OZ = 7{,}326$ ft.

Correct the angles measured at station T, to those which would have been measured if the transit had been set at station O.

7. In the quadrilateral $ABCD$, of Art. 564*a*, it is assumed that the side AB has a length of 13,100.3 ft. Use the finally adjusted angles and calculate the length of the side CD by two independent series of computations.

8. For the quadrilateral of problem 7, the azimuth of AB is assumed to be $102°35'18''$, and the coordinates of station A are latitude $= +50{,}000$ ft.; longitude $= +40{,}000$ ft. Calculate the coordinates of stations B, C, and D.

9. Given the data shown below (see Fig. 565): Assume the instrument at station P, within the triangle ABC, and solve the three-point problem for the angles x and y.

$$\begin{aligned} A &= 102°45'20'' \\ b &= 6{,}883.4 \\ c &= 6{,}605.3 \\ \alpha &= 89°15'30'' \\ \beta &= 128°20'10'' \end{aligned}$$

10. Given the data of problem 9 and the additional data shown below. Assume the instrument at station P' (Fig. 565), outside the triangle ABC, and solve the three-point problem for the angles x' and y'.

$$\begin{aligned} \alpha' &= 26°34'50'' \\ \beta' &= 44°15'15'' \end{aligned}$$

References

1. Arthur, N. N., and Randall, R. H., "The Geodetic and Topographic Survey of Pittsburgh and Allegheny County," *Proc. Eng. Soc. Western Pennsylvania*, Vol. 42, No. 3, pp. 119–154, April 1926.
2. Hodgson, C. V., "Manual of First-order Triangulation," U. S. Coast and Geodetic Survey, Sp. Pub. 120, Government Printing Office, Washington, D. C.
3. Hosmer, G. L., "Geodesy," Second Edition, John Wiley and Sons, New York, 1930.
4. Mitchell, Hugh C., "Use of Control for City Surveys," U. S. Coast and Geodetic Survey, Sp. Pub. 91, Government Printing Office, Washington, D. C.
5. Wright, T. W., and Hayford, J. F., "The Adjustment of Observations," D. Van Nostrand Co., New York, 1906.
6. "Topographic Instructions of the U. S. Geological Survey," Government Printing Office, Washington, D. C.
7. "General Instructions for the Field Work of the U. S. Coast and Geodetic Survey," Sp. Pub. 26, Government Printing Office, Washington, D. C.
8. "Formulas and Tables for Computation of Geodetic Positions," U. S. Coast and Geodetic Survey, Sp. Pub. 8, Government Printing Office, Washington, D. C.
9. U. S. Coast and Geodetic Survey Report for 1894, Appendix 9.
10. "Geographic Tables and Formulas," U. S. Geological Survey, Bull. 650, Government Printing Office, Washington, D. C.

See also references under Topographic Surveying, page 697.

CHAPTER XXIX

MAP PROJECTIONS

566. Maps of Small Areas.—In plotting a map of a small area the curvature of the earth need not be considered. A level surface is assumed to be a plane, and points are usually plotted on the map by linear coordinates, or given distances respectively from selected north-south and east-west coordinate axes.

567. Maps of Large Areas.—For maps of larger areas this simple method is not satisfactory because of the sphericity[1] of the earth. It is impossible to represent the surface of a sphere on a plane surface without distortion, just as it is impossible to flatten a section of orange peel without tearing. In consequence, any plane map of a relatively large area of spherical surface must be distorted to some degree. For example, a great circle on the surface of the earth should appear as a straight line on the map, but on most plane maps a great circle is represented by a curved line in order to avoid undue distortions in shape and area of the territory represented. Again, it has been shown in Chapter XX that meridians are not parallel but that they converge toward the poles, the angular convergency varying with the latitude; nevertheless the horizontal projections of all meridians are straight lines and the curved parallels of latitude are always perpendicular to them.

568. Map Projection Defined.—In maps of large areas where curvature becomes important, it is necessary to locate points by coordinates which are the geographical latitudes and longitudes expressed in angular units; for example, New York is at a latitude 40°45′ north of the equator and at a longitude 74°00′ west of Greenwich. Points are plotted with respect to a series of lines representing the earth's parallels and meridians. Any system of representing these parallels and meridians on a plane surface is called a *map projection.*

569. Ideal vs. Practicable Projection.—On a theoretically perfect map, without distortion, the following conditions would be satisfied: (1) all distances and areas would have correct relative magnitudes, (2) all azimuths and angles would be correctly shown, (3) all great

[1] The earth is very nearly a sphere, since the polar diameter is only one third of one per cent shorter than the equatorial diameter.

circles would appear as straight lines, and (4) geographic latitudes and longitudes of all points would be correctly shown. While in a plane map not all of these requirements can be satisfied at the same time, one or more conditions may be satisfied, as follows:

1. An *equal-area* projection results in a map showing all areas in proper relative *size*, although these areas may be much out of shape and the map may have other defects.

2. A *conformal* or *orthomorphic* projection results in a map showing the correct angle between any pair of short intersecting lines, thus making small areas appear in correct *shape*. As the scale varies from point to point, the shapes of larger areas are incorrect.

3. An *azimuthal* projection results in a map showing the correct *direction* or azimuth of any point from one central point.

570. Types of Projections.—In the following paragraphs a few of the more important types of projections are briefly described. These descriptions are of a general nature, involving little mathematical treatment, and no attempt has been made to make the list complete. For more detailed treatment the student is referred to the references listed at the end of this chapter, particularly to Publication No. 68, "Elements of Map Projection," of Ref. 8, p. 872. Some of the projections are true projections in the geometrical sense; others are map projections in the sense that they represent the parallels and meridians on a plane surface, although they can not be obtained by any perspective or geometric projecting process. Some of the more useful map projections are of this second kind.

571. Gnomonic Projection.—This is a geometric projection to a plane, which is tangent to the sphere at any point A, Fig. 571.

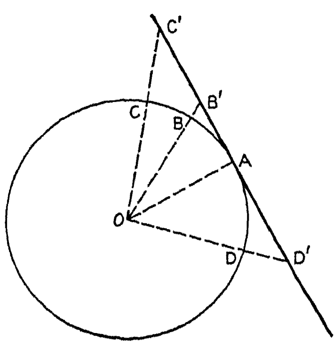


FIG. 571.

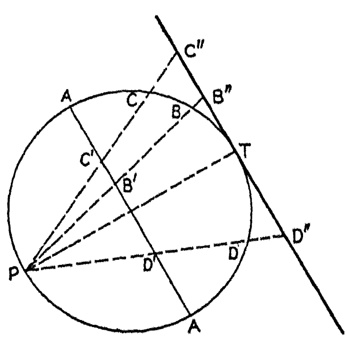


FIG. 572.

Radiating lines from the earth's center O through such points (B, C, etc.) on the surface of the earth as are to be shown on the map, are produced to an intersection with the tangent plane, where the

point in question is plotted. In the figure, B is plotted at B', C at C', and so on. These plottings in the tangent plane give in that plane a map constructed on the gnomonic projection. The important property of maps made on this projection is that they show great circles as straight lines, which renders them useful for navigational purposes; otherwise they are not particularly useful except for the part of the map near the point of tangency. The shapes and sizes of areas are much distorted except near the tangent point.

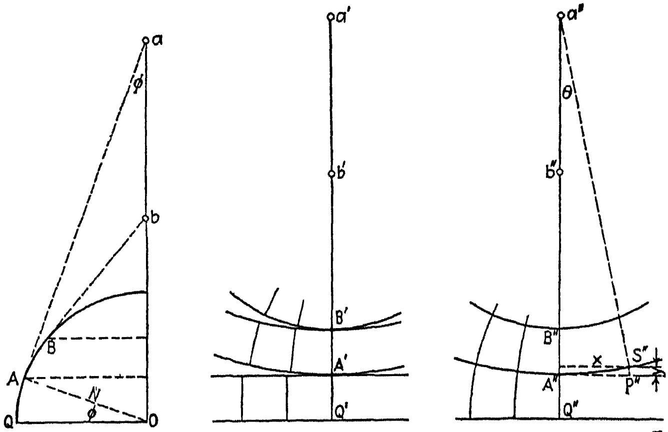
572. Stereographic Projection.—This is a geometric projection to a plane. Let A–A Fig. 572, represent any circle of the earth (in practice, generally a great circle), and let P represent the pole of that circle. Then by lines radiating from P to points B, C, and so on, these points are projected to form a map in the plane A–A, and are represented in that map by the points B', C', etc. If the plane of projection is tangent to the sphere, as at T (Fig. 572), the points are plotted at B'', C'', etc. This is a conformal projection, and is an excellent one for general maps showing a hemisphere: its main defect is that areas are not correctly shown.

573. Orthographic Projection.—This is a geometric projection to a plane tangent to the sphere at any point; the projecting lines are parallel and are perpendicular to the tangent plane in which the map is constructed. If the central tangent point of the map is at one of the poles of the earth, each parallel of latitude is shown correctly to scale, but the distance between parallels becomes rapidly smaller as we depart farther from the center of the map. The map is true to scale along the parallels but not along the meridians. Different but comparable results are obtained if the tangent plane touches the earth's surface at some point other than the pole. Maps of the surface of the moon are usually constructed on this projection.

574. Geometric Projections to a Cylinder.—The surface of a cylinder is curved in one direction only, and can be developed into a plane. Advantage is taken of this fact in the so-called cylindrical projections. The cylinder used may cut the sphere, but is usually tangent along a great circle, generally the equator. The projecting lines used may radiate from the center of the sphere or may all be parallel to the equatorial plane. These particular projections are little used, but a modification of the cylindrical projection, called the *Mercator projection,* possesses some valuable properties (see Art. 576).

575. Geometric Projections to a Cone.—Like the surface of a cylinder, the surface of a cone is capable of development, without distortion, into a plane. On this account a number of conical projections have been devised, using sometimes a cone tangent along

a parallel of latitude and sometimes a cone cutting the sphere along two parallels. The lines of projection may emanate from a central point or may be parallel to each other and to the plane of the chosen parallel. But here again, the more important and valuable projections are not of the geometric-projection type.



(*a*) Section (*b*) Development (*c*) Map

Fig. 575.—Polyconic projection.

575a. Polyconic Projection.—Instead of using a single cone, a series of conical surfaces may be used, points on the surface of the earth being considered as projected to a series of frustums of cones which are fitted together. These conical surfaces are then developed each way from a central meridian. Due to differences in radii, the resulting strips would when laid flat not exactly fit together, but spaces would appear between them, such spaces increasing in width as the distance from the central meridian increases (Fig. 575*b*). To avoid such spaces, the north-south scale must be modified along the various meridians. Upon such a system of lines points are plotted by latitude and longitude.

Referring to Fig. 575, it is seen that each parallel of latitude appears on the map as the arc of a circle having as radius the corresponding tangent distance; the parallel through A has a radius Aa, that through B has a radius Bb, and so on. The centers of these circles all lie on the central meridian of the map. The length of each tangent distance Aa, etc., is $N \cot \phi$ in which N is the length of the normal or vertical at latitude ϕ extended to its intersection with the earth's axis. For the assumption that the earth is a sphere, N is equal to the radius of the sphere. More exactly

$$N = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \tag{1}$$

in which a is the earth's equatorial radius and e is the eccentricity of the ellipse in the meridian section ($e^2 = 0.00676866$). The length of the tangent distance $N \cot \phi$ varies with the latitude.

The distances $Q'A'$, $A'B'$, etc. along the central meridian on the map (Fig. 575*b*) are true scale representations of the corresponding arc distances QA, AB, etc. in the meridian section (Fig. 575*a*). The parallels drawn on the map may be selected with as small a difference in latitude as may be desired, and each one is drawn with its own particular radius as shown, with the center of the arc on the central meridian the proper tangent distance (to scale) above the point where the parallel cuts the central meridian.

It should be observed that the method of drawing these arcs of the parallels of latitude on the map is such that each parallel is separately developed as the circumference of the base of its own distinct cone, and also that the spacing between them increases with increasing differences of longitude from the central meridian, thereby changing the north-south scale of the map from place to place as the longitude difference increases.

The arc distance $A''S''$ in the map, Fig. 575*c*, represents to true scale the difference in longitude between the points A'' and S''. The angle $A''a''S''$ is the angle θ of Fig. 352*a*, p. 535, and from Eq. (1) on page 533

$$\theta = \lambda \sin \phi$$

in which λ is the arc $A''S''$. The rectangular coordinates of the point S'' referred to A'' as origin are

$$x = A''P'' = a''S'' \sin \theta = N \cot \phi \sin \theta \tag{2}$$
$$y = P''S'' = a''S'' \operatorname{vers} \theta = N \cot \phi \operatorname{vers} \theta \tag{3}$$

and also if the chord $A''S''$ be drawn, in the triangle $A''S''P''$, $S''P'' = A''P'' \tan P''A''S''$, or $y = x \tan \frac{\theta}{2}$.

Values of x and y have been computed and are tabulated in Special Publication No. 5 of the U. S. Coast and Geodetic Survey.

As many points as desired along the parallels on the map, like point S'', are plotted by the use of the table. Then the meridians are drawn through such points. These meridians are curved, concave toward the central meridian, but if the parallels are drawn close enough together each meridian may be drawn as a series of straight lines from parallel to parallel. On the network of parallels and meridians so prepared, points are plotted by latitude and longitude. Near the central meridian there is little error in such a map, but

the error increases in proportion to the square of the difference in longitude along any one parallel. The variation with difference in latitude is not in direct proportion.

It is to be noted that along the central meridian and along every parallel the map is true to scale; that along the other meridians the scale is somewhat changed; that near the central meridian the parallels and meridians intersect nearly at right angles; and that areas of great extent north and south may be mapped with a very small distortion.

While better adapted to mapping an area of great extent in latitude than for an area of great extent east and west, the polyconic projection is sufficiently accurate for maps of considerable areas, and it is widely used by the U. S. Geological Survey and the U. S. Coast and Geodetic Survey.

575b. Lambert Conformal Conic Projection.—Attention was called to this excellent projection by its use for the French battle maps during the World War. It has since been fully investigated by the U. S. Coast and Geodetic Survey, and tables for its construction have been published.

This is a simple conic projection, the cone used being imagined to cut the surface of the earth along two parallels of latitude, called *standard parallels*. When points on the earth's surface are projected to such a cone, there is a slight compression or decrease of scale between the standard parallels, and a stretching or increase of scale outside the standard parallels. Only a slight adjustment of scales is necessary to make the map conformal. It has been shown that for a map of the United States, scale errors need not exceed 2 per cent at any point. For details the reader is referred to publications of the United States Coast and Geodetic Survey dealing with this projection.

575c. Albers Equal-area Conic Projection.—The 1933 National Geographic Society map of the United States is on the Albers equal-area conic projection, with 29½°N and 45½°N as the standard parallels which are intersected by the cone. This projection has the advantages that it is an equal-area projection and that (for a map of the United States) any distance may be measured with a maximum scale error of less than 1¼ per cent and an average scale error of less than ½ per cent, as against a possible maximum error of 7 per cent on a polyconic projection. The meridians are straight lines meeting in a common point, and the parallels are concentric circles intersecting the meridians at right angles. The arcs of longitude along any given parallel are of equal length, and on the two selected parallels these arcs are represented in their true length.

The projection is especially suited for maps of great east-and-west extent, such as that of the United States; but the scale errors vary increasingly with the distance north or south from the standard parallels. The projection is simple to construct and use.

576. Mercator Projection.—As previously stated, this projection is cylindrical, but it can not be constructed as a geometrical projection.

In a cylindrical projection, formed by means of a cylinder touching the earth along the equator, all meridians appear as straight parallel lines. But on the sphere any two such meridians are a maximum distance apart at the equator and converge toward the poles. Showing them parallel therefore results in a systematically increasing scale along the parallels of latitude as we pass from the equator toward the pole, with resulting distortion of all areas shown on the map. The change of scale along the parallel, varying with the latitude, is readily calculated (still assuming the earth to be spherical) by means of the formula

$$S' = S \cos \phi \tag{4}$$

in which S is the scale at the equator and S' is the scale at any latitude ϕ. For example, if the equatorial scale of the map is 1000 miles per inch, then at latitude 60° (since the cosine of 60° is ½) the scale is 500 miles to the inch.

The particular feature of the Mercator projection is that the scale along the meridian is varied to agree with the scale along the parallel, so that while the scale varies from point to point on the map, at any given point the scale is the same in all directions. The map is therefore conformal. It has also the important property that a line of constant true bearing, or *rhumb line*, appears straight, which property renders it invaluable for purposes of navigation. The shortest course between two points is determined by drawing on a gnomonic chart a great circle, which there appears as a straight line. Selected points, at convenient distances apart, of this great circle are then plotted on the Mercator chart, after making any necessary corrections on account of shoals, wind, currents, etc. The rhumb line connecting any two adjacent points indicates the true bearing of the course, which is read by means of a protractor. This true bearing, corrected for magnetic declination, gives the compass bearing to be used in steering.

Due to the rapid variation of scale, maps constructed on the Mercator projection give very inaccurate information as to relative sizes of areas in widely different latitudes. For example, on the map Greenland appears larger than South America, whereas in fact South America is nine times as large as Greenland. Consequently, such

a map is not suited to general use, although because of its many other advantages it is widely published.

577. The Earth a Spheroid.—In the foregoing discussion it has been assumed that the earth is spherical, as it is very nearly. Actually, however, the meridian section of the earth is an ellipse, the polar diameter being some twenty-seven miles shorter than the equatorial diameter. This fact is recognized in the mathematical solutions of the problems involved and in the preparation of tables for the various map projections. The radius of curvature in the meridian is different for different latitudes, as is also the length of the normal from the surface terminating in the polar axis. Other dimensions depart correspondingly from those of a true sphere.

It is evident that this variation from the truly spherical shape does not change the nature of the various map projections that have been discussed, although it does make necessary certain corrections to and changes in the numerical values of the quantities used for plotting the different projections. A discussion of these refinements is beyond the scope of this volume, and the reader is referred to the publications dealing with map projections and with geodesy in general. A few of these publications are listed below.

References

1. Fowler, G. H., "Charts, Their Use and Meaning," J. D. Potter, London, England, 1916.
2. Hinks, A. R., "Map Projections," University Press, Cambridge, England, 1912.
3. Holway, R. S., "Maps Illustrating the Special Properties of a Few Types of Projections," University of California Map Series No. 3, 1916.
4. Morrison, G. J., "Maps, Their Uses and Construction," E. Stanford, London, England, 1901.
5. Reeves, E. A., "Maps and Map Making," Royal Geographic Society, London, England, 1910.
6. Steers, J. S., "An Introduction to the Study of Map Projections," London, England, 1929.
7. Sweitzer, N. B., and discussion by others, "Field Procedure of Adjusting the Great Circle Line to the Rhumb Line," Trans. Am. Soc. Civ. Engr., Vol. 90, 1927, pp. 1005–1034.
8. United States Coast and Geodetic Survey, Special Publications No. 5, 47, 49, 52, 53, 57, 59, 60, 68, 112, 130, 153, Superintendent of Documents, Washington, D. C.

TABLE I.—CORRECTION FOR REFRACTION AND PARALLAX, TO BE SUBTRACTED FROM THE OBSERVED ALTITUDE OF THE SUN

App't alt.	Temperature										App't alt.
	−10° C. +14° F.	−5° C. +23° F.	0° C. +32° F.	+5° C. +41° F.	+10° C. +50° F.	+15° C. +59° F.	+20° C. +68° F.	+25° C. +77° F.	+30° C. +86° F.	+35° C. +95° F.	
°	′	′	′	′	′	′	′	′	′	′	°
10	5.52	5.42	5.30	5.20	5.10	5.00	4.92	4.83	4.75	4.67	10
11	5.02	4.92	4.82	4.73	4.63	4.55	4.47	4.38	4.32	4.23	11
12	4.60	4.50	4.42	4.33	4.25	4.17	4.10	4.03	3.97	3.88	12
13	4.23	4.15	4.07	4.00	3.92	3.85	3.78	3.72	3.65	3.58	13
14	3.92	3.83	3.77	3.70	3.62	3.55	3.50	3.45	3.37	3.32	14
15	3.65	3.58	3.50	3.43	3.37	3.32	3.25	3.20	3.13	3.08	15
16	3.43	3.35	3.30	3.23	3.17	3.12	3.07	3.00	2.95	2.90	16
17	3.22	3.15	3.10	3.03	2.98	2.92	2.88	2.82	2.77	2.72	17
18	3.02	2.95	2.90	2.85	2.80	2.75	2.70	2.65	2.60	2.55	18
19	2.83	2.78	2.73	2.68	2.63	2.58	2.53	2.48	2.43	2.40	19
20	2.68	2.63	2.58	2.53	2.48	2.43	2.38	2.33	2.30	2.27	20
21	2.53	2.48	2.43	2.38	2.35	2.30	2.27	2.22	2.17	2.13	21
22	2.38	2.35	2.30	2.25	2.22	2.18	2.13	2.08	2.05	2.02	22
23	2.28	2.25	2.20	2.15	2.12	2.08	2.03	1.98	1.95	1.93	23
24	2.17	2.13	2.08	2.05	2.02	1.98	1.93	1.88	1.87	1.83	24
25	2.07	2.03	1.98	1.95	1.92	1.88	1.83	1.80	1.77	1.75	25
26	1.99	1.95	1.90	1.87	1.83	1.80	1.75	1.72	1.70	1.67	26
27	1.88	1.85	1.82	1.78	1.75	1.72	1.68	1.63	1.62	1.60	27
28	1.80	1.77	1.72	1.70	1.67	1.63	1.60	1.57	1.53	1.52	28
29	1.72	1.68	1.65	1.63	1.60	1.57	1.53	1.50	1.47	1.46	29
30	1.65	1.62	1.58	1.57	1.53	1.50	1.47	1.45	1.42	1.40	30
32	1.53	1.50	1.47	1.45	1.42	1.38	1.35	1.33	1.30	1.28	32
34	1.41	1.37	1.35	1.32	1.30	1.27	1.25	1.23	1.20	1.18	34
36	1.30	1.27	1.25	1.22	1.20	1.18	1.15	1.13	1.10	1.08	36
38	1.20	1.18	1.15	1.13	1.12	1.10	1.07	1.05	1.02	1.02	38
40	1.11	1.10	1.07	1.05	1.03	1.02	0.98	0.97	0.95	0.93	40
42	1.03	1.00	0.98	0.97	0.95	0.93	0.90	0.88	0.87	0.87	42
44	0.96	0.93	0.92	0.90	0.88	0.87	0.85	0.83	0.82	0.80	44
46	0.89	0.88	0.87	0.85	0.83	0.82	0.80	0.78	0.77	0.75	46
48	0.83	0.82	0.80	0.78	0.77	0.75	0.73	0.72	0.70	0.68	48
50	0.77	0.75	0.73	0.72	0.70	0.68	0.67	0.67	0.65	0.63	50
55	0.63	0.62	0.60	0.60	0.58	0.57	0.57	0.55	0.53	0.52	55
60	0.52	0.52	0.50	0.50	0.48	0.47	0.47	0.45	0.45	0.43	60
65	0.42	0.40	0.40	0.40	0.38	0.38	0.37	0.37	0.35	0.33	65
70	0.32	0.32	0.32	0.30	0.30	0.30	0.28	0.28	0.28	0.27	70
75	0.23	0.23	0.23	0.22	0.22	0.22	0.20	0.20	0.20	0.18	75
80	0.15	0.15	0.13	0.13	0.13	0.13	0.13	0.12	0.12	0.12	80
85	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.05	0.05	0.05	85
90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	90

TABLE II.—CORRECTION FOR REFRACTION, TO BE SUBTRACTED FROM THE OBSERVED ALTITUDE OF A STAR

App't alt.	Temperature										App't alt.
	−10° C. +14° F.	−5° C. +23° F.	0° C. +32° F.	+5° C. +41° F.	+10° C. +50° F.	+15° C. +59° F.	+20° C. +68° F.	+25° C. +77° F.	+30° C. +86° F.	+35° C. +95° F.	
°	′	′	′	′	′	′	′	′	′	′	°
10	5.67	5.57	5.45	5.35	5.25	5.15	5.07	4.98	4.90	4.82	10
11	5.17	5.07	4.97	4.88	4.78	4.70	4.62	4.53	4.47	4.38	11
12	4.75	4.65	4.57	4.48	4.40	4.32	4.25	4.18	4.12	4.03	12
13	4.38	4.30	4.22	4.15	4.07	4.00	3.93	3.87	3.80	3.73	13
14	4.06	3.97	3.91	3.84	3.76	3.69	3.64	3.59	3.51	3.46	14
15	3.79	3.72	3.64	3.57	3.51	3.46	3.39	3.34	3.27	3.22	15
16	3.57	3.49	3.44	3.37	3.31	3.26	3.21	3.14	3.09	3.04	16
17	3.36	3.29	3.24	3.17	3.12	3.06	3.02	2.96	2.91	2.86	17
18	3.16	3.09	3.04	2.99	2.94	2.89	2.84	2.79	2.74	2.69	18
19	2.97	2.92	2.87	2.82	2.77	2.72	2.67	2.62	2.57	2.54	19
20	2.82	2.77	2.72	2.67	2.62	2.57	2.52	2.47	2.44	2.41	20
21	2.67	2.62	2.57	2.52	2.49	2.44	2.41	2.36	2.31	2.27	21
22	2.52	2.49	2.44	2.39	2.36	2.32	2.27	2.22	2.19	2.16	22
23	2.42	2.39	2.34	2.29	2.26	2.22	2.17	2.12	2.09	2.07	23
24	2.31	2.27	2.22	2.19	2.16	2.12	2.07	2.02	2.01	1.97	24
25	2.21	2.17	2.12	2.09	2.06	2.02	1.97	1.94	1.91	1.89	25
26	2.12	2.08	2.03	2.00	1.96	1.93	1.88	1.85	1.83	1.80	26
27	2.01	1.98	1.95	1.91	1.88	1.85	1.81	1.76	1.75	1.73	27
28	1.93	1.90	1.85	1.83	1.80	1.76	1.73	1.70	1.66	1.65	28
29	1.85	1.81	1.78	1.76	1.73	1.70	1.66	1.63	1.60	1.59	29
30	1.78	1.75	1.71	1.70	1.66	1.63	1.60	1.58	1.55	1.53	30
32	1.65	1.62	1.59	1.57	1.54	1.50	1.47	1.45	1.42	1.40	32
34	1.53	1.49	1.47	1.44	1.42	1.39	1.35	1.35	1.32	1.30	34
36	1.42	1.39	1.37	1.34	1.32	1.30	1.27	1.25	1.22	1.20	36
38	1.32	1.30	1.27	1.25	1.24	1.22	1.19	1.17	1.14	1.14	38
40	1.22	1.21	1.18	1.16	1.14	1.13	1.09	1.08	1.06	1.04	40
42	1.14	1.11	1.09	1.08	1.06	1.04	1.01	0.99	0.98	0.98	42
44	1.07	1.04	1.03	1.01	0.99	0.98	0.96	0.94	0.93	0.91	44
46	0.99	0.98	0.97	0.95	0.93	0.92	0.90	0.88	0.87	0.85	46
48	0.93	0.92	0.90	0.88	0.87	0.85	0.83	0.82	0.80	0.78	48
50	0.86	0.84	0.82	0.81	0.79	0.77	0.76	0.76	0.74	0.72	50
55	0.72	0.71	0.69	0.69	0.67	0.66	0.66	0.64	0.62	0.61	55
60	0.59	0.59	0.57	0.57	0.55	0.54	0.54	0.52	0.52	0.50	60
65	0.48	0.46	0.46	0.46	0.44	0.44	0.43	0.43	0.41	0.39	65
70	0.37	0.37	0.37	0.35	0.35	0.35	0.33	0.33	0.33	0.32	70
75	0.27	0.27	0.27	0.26	0.26	0.26	0.24	0.24	0.24	0.22	75
80	0.18	0.18	0.16	0.16	0.16	0.16	0.16	0.15	0.15	0.15	80
85	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.06	0.06	0.06	85
90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	90

TABLE III.—REFRACTION CORRECTIONS TO BE APPLIED TO APPARENT DECLINATIONS

(To be used with solar attachment)

January	Hour angle	Refraction correction lat. 40°	February	Hour angle	Refraction correction lat. 40°	March	Hour angle	Refraction correction lat. 40°	April	Hour angle	Refraction correction lat. 40°	May	Hour angle	Refraction correction lat. 40°	June	Hour angle	Refraction correction lat. 40°
		′ ″		See Jan. 30	′ ″			′ ″			′ ″			′ ″			′ ″
1	1	1 58	1			1	1	1 03	1	3	0 57	1	3	0 39	1	4	0 44
2	2	2 16	2	3	2 13	2	2	1 10	2	4	1 19	2	4	0 55	2	5	1 11
3	3	3 04	3	4	3 41	3	3	1 27	3	5	2 18	3	5	1 30	3		
4	4	6 23	4			4	4	2 06	4			4			4	1	0 19
5			5	1	1 26	5			5	1	0 39	5	1	0 26	5	2	0 23
	1	1 54		2	1 37		1	0 59		2	0 44		2	0 30		3	0 30
6	2	2 11	6	3	2 04	6	2	1 06	6	3	0 54	6	3	0 37	6	4	0 43
7	3	2 59	7	4	3 21	7	3	1 21	7	4	1 14	7	4	0 53	7	5	1 10
8	4	6 01	8			8	4	1 56	8	5	2 08	8	5	1 26	8		
9			9	1	1 21	9	5	4 04	9			9			9	1	0 18
10			10	2	1 31	10			10	1	0 36	10	1	0 25	10	2	0 22
	1	1 50		3	1 56		1	0 55		2	0 41		2	0 29		3	0 29
11	2	2 06	11	4	3 04	11	2	1 02	11	3	0 51	11	3	0 36	11	4	0 43
12	3	2 49	12			12	3	1 15	12	4	1 10	12	4	0 51	12	5	1 09
13	4	5 33	13			13	4	1 47	13	5	1 58	13	5	1 22	13		
14			14	1	1 16	14	5	3 34	14			14			14	1	0 18
15			15	2	1 25	15			15	1	0 34	15	1	0 23	15	2	0 22
	1	1 46		3	1 48		1	0 52		2	0 38		2	0 27		3	0 29
16	2	2 01	16	4	2 47	16	2	0 58	16	3	0 48	16	3	0 34	16	4	0 42
17	3	2 40	17	5	8 39	17	3	1 10	17	4	1 06	17	4	0 49	17	5	1 08
18	4	5 00	18			18	4	1 39	18	5	1 49	18	5	1 18	18		
19			19	1	1 12	19	5	3 08	19			19			19	1	0 18
20			20	2	1 20	20			20	1	0 32	20	1	0 22	20	2	0 22
	1	1 42		3	1 40		1	0 48		2	0 36		2	0 26		3	0 29
21	2	1 56	21	4	2 31	21	2	0 54	21	3	0 45	21	3	0 33	21	4	0 42
22	3	2 31	22	5	6 49	22	3	1 05	22	4	1 02	22	4	0 47	22	5	1 08
23	4	4 35	23			23	4	1 32	23	5	1 42	23	5	1 15	23		
24			24	1	1 07	24	5	2 51	24			24			24	1	0 18
25	1	1 37	25	2	1 15	25			25	1	0 30	25	1	0 21	25	2	0 22
	2	1 50		3	1 33		1	0 45		2	0 34		2	0 25		3	0 29
26	3	2 22	26	4	2 18	26	2	0 50	26	3	0 42	26	3	0 32	26	4	0 42
27	4	4 07	27	5	5 29	27	3	1 01	27	4	0 58	27	4	0 46	27	5	1 08
28			28			28	4	1 25	28	5	1 36	28	5	1 13	28		
29			29	See Mar. 1		29	5	2 34	29			29			29	1	0 18
30	1	1 32				30			30	1	0 28	30	1	0 20	30	2	0 22
31	2	1 44				31	1	0 42		2	0 32	31	2	0 24		3	0 29
							2	0 47					3	0 31			

For latitudes other than 40° multiply by latitude coefficient, Table III(a).

TABLE III (*Cont'd*).—REFRACTION CORRECTIONS TO BE APPLIED TO APPARENT DECLINATIONS

(*To be used with solar attachment*)

July	Hour angle	Refraction correction lat. 40°	August	Hour angle	Refraction correction lat. 40°	September	Hour angle	Refraction correction lat. 40°	October	Hour angle	Refraction correction lat. 40°	November	Hour angle	Refraction correction lat. 40°	December	Hour angle	Refraction correction lat. 40°
		' "			' "			' "			' "			' "			' "
1	4	0 43	1	5	1 22	1	1	0 39	1	1	0 56	1			1	1	1 54
2	5	1 09	2			2	2	0 44	2	2	1 06	2	1	1 26	2	2	2 11
3			3	1	0 26	3	3	0 54	3	3	1 21	3	2	1 37	3	3	2 59
4	1	0 19	4	2	0 30	4	4	1 14	4	4	1 56	4	3	2 04	4	4	6 01
5	2	0 23	5	3	0 37	5	5	2 08	5	5	4 04	5	4	3 21	5		
	3	0 30		4	0 53												
6	4	0 43	6	5	1 26	6	1	0 42	6	1	1 03	6	1	1 32	6	1	1 58
7	5	1 10	7			7	2	0 47	7	2	1 10	7	2	1 44	7	2	2 16
8			8	1	0 28	8	3	0 57	8	3	1 27	8	3	2 13	8	3	3 04
9	1	0 20	9	2	0 32	9	4	1 19	9	4	2 06	9	4	3 41	9	4	6 23
10	2	0 24	10	3	0 39	10	5	2 18	10	5	4 39	10			10		
	3	0 31		4	0 55												
11	4	0 44	11	5	1 30	11	1	0 45	11	1	1 07	11	1	1 37	11	1	2 00
12	5	1 11	12			12	2	0 50	12	2	1 15	12	2	1 50	12	2	2 19
13			13	1	0 30	13	3	1 01	13	3	1 33	13	3	2 22	13	3	3 09
14	1	0 21	14	2	0 34	14	4	1 25	14	4	2 18	14	4	4 07	14	4	6 43
15	2	0 25	15	3	0 42	15	5	2 34	15	5	5 29	15			15		
	3	0 32		4	0 58												
16	4	0 46	16	5	1 36	16	1	0 48	16	1	1 12	16	1	1 42	16	1	2 01
17	5	1 13	17			17	2	0 54	17	2	1 20	17	2	1 56	17	2	2 20
18			18	1	0 32	18	3	1 05	18	3	1 40	18	3	2 31	18	3	3 11
19	1	0 22	19	2	0 36	19	4	1 32	19	4	2 31	19	4	4 35	19	4	6 47
20	2	0 26	20	3	0 45	20	5	2 51	20	5	6 49	20			20		
	3	0 33		4	1 02												
21	4	0 47	21	5	1 42	21	1	0 52	21	1	1 16	21	1	1 46	21	1	2 01
22	5	1 15	22			22	2	0 58	22	2	1 25	22	2	2 01	22	2	2 20
23			23	1	0 34	23	3	1 10	23	3	1 48	23	3	2 40	23	3	3 11
24	1	0 23	24	2	0 38	24	4	1 39	24	4	2 47	24	4	4 59	24	4	6 49
25	2	0 27	25	3	0 48	25	5	3 08	25	5	8 39	25			25		
	3	0 34		4	1 06												
26	4	0 49	26	5	1 49	26	1	0 55	26	1	1 21	26	1	1 50	26	1	2 00
27	5	1 18	27			27	2	1 02	27	2	1 31	27	2	2 06	27	2	2 19
28			28	1	0 36	28	3	1 15	28	3	1 56	28	3	2 49	28	3	3 09
29	1	0 25	29	2	0 41	29	4	1 47	29	4	3 04	29	4	5 33	29	4	6 43
30	2	0 29	30	3	0 51	30	5	3 34	30	5	11 01	30			30		
31	3	0 36	31	4	1 10				31	See Nov. 1					31	See Jan. 1	
	4	0 51		5	1 58												

For latitudes other than 40° multiply by latitude coefficient, Table III(a).

TABLE III(a).—LATITUDE COEFFICIENTS

Latitude	Coefficient	Latitude	Coefficient	Latitude	Coefficient
15°	0.30	30°	0.65	45°	1.20
16	0.32	31	0.68	46	1.24
17	0.34	32	0.71	47	1.29
18	0.36	33	0.75	48	1.33
19	0.38	34	0.78	49	1.38
20	0.40	35	0.82	50	1.42
21	0.42	36	0.85	51	1.47
22	0.44	37	0.89	52	1.53
23	0.46	38	0.92	53	1.58
24	0.48	39	0.96	54	1.64
25	0.50	40	1.00	55	1.70
26	0.53	41	1.04	56	1.76
27	0.56	42	1.08	57	1.82
28	0.59	43	1.12	58	1.88
29	0.62	44	1.16	59	1.94

To obtain the refraction correction (to be applied to declination) for any other latitude than 40° multiply the refraction correction for latitude 40° (Table III) by the coefficient corresponding to the latitude of observation.

TABLE IV.—LOCAL CIVIL TIME OF UPPER CULMINATION OF POLARIS IN THE YEAR 1931

(Computed for 90°, or 6 hours west of Greenwich)

Date, 1931	Civil time of upper culmination			Variation per day		Date, 1931	Civil time of upper culmination			Variation per day	
	h	m	s	m	s		h	m	s	m	s
Jan. 1	18	54	26	−3	57	July 10	6	27	12	−3	55
Jan. 11	18	14	56	−3	57	July 20	5	48	05	−3	55
Jan. 21	17	35	25	−3	57	July 30	5	08	57	−3	55
Jan. 31	16	55	54	−3	57	Aug. 9	4	29	50	−3	55
Feb. 10	16	16	24	−3	57	Aug. 19	3	50	41	−3	55
Feb. 20	15	36	55	−3	57	Aug. 29	3	11	31	−3	55
Mar. 2	14	57	27	−3	57	Sept. 8	2	32	21	−3	55
Mar. 12	14	18	00	−3	57	Sept. 18	1	53	10	−3	55
Mar. 22	13	38	36	−3	56	Sept. 28	1	13	57	−3	55
Apr. 1	12	59	14	−3	56	Oct. 8	0	34	42	−3	56
Apr. 11	12	19	54	−3	56	Oct. 17	23	55	25	−3	56
Apr. 21	11	40	36	−3	56	Oct. 27	23	16	07	−3	56
May 1	11	01	21	−3	56	Nov. 6	22	36	47	−3	56
May 11	10	22	06	−3	55	Nov. 16	21	57	25	−3	56
May 21	9	42	54	−3	55	Nov. 26	21	18	01	−3	56
May 31	9	03	44	−3	55	Dec. 6	20	38	35	−3	57
June 10	8	24	34	−3	55	Dec. 16	19	59	08	−3	57
June 20	7	45	27	−3	55	Dec. 26	19	19	40	−3	57
June 30	7	06	19	−3	55	Jan. 5, 1932	18	40	10	−3	57

TABLE IV(a).—MEAN TIME INTERVAL BETWEEN UPPER CULMINATION AND ELONGATION

Latitude	Time interval		Latitude	Time interval		Latitude	Time interval		Latitude	Time interval	
°	h	m	°	h	m	°	h	m	°	h	m
10	5	58.2	35	5	56.1	48	5	54.3	58	5	52.3
15	5	57.9	40	5	55.4	50	5	53.9	60	5	51.7
20	5	57.4	42	5	55.2	52	5	53.6	62	5	51.1
25	5	57.0	44	5	54.9	54	5	53.2	64	5	50.4
30	5	56.5	46	5	54.6	56	5	52.8			

Eastern elongation precedes and western elongation follows upper culmination by the time interval given in Table IV(a). Lower culmination precedes or follows upper culmination by $11^h\ 58.0^m$. It should be noted that there are two upper culminations on one day in October (15th in 1931) and two lower culminations in April (16th in 1931). There are also two western elongations on one day in January and two eastern elongations on one day in July.

A. To refer the times in Table IV to other years:

FOR YEAR		m	
1932	add	1.5	up to Mar. 1
1932	subtract	2.4	on and after Mar. 1
1933	subtract	0.8	
1934	add	0.8	
1935	add	2.4	
1936	add	4.0	up to Mar. 1
1936	add	0.1	on and after Mar. 1
1937	add	1.8	
1938	add	3.5	
1939	add	5.2	
1940	add	6.8	up to Mar. 1
1940	add	2.8	on and after Mar. 1

B. *To refer to other than the tabular days:* SUBTRACT from the time for the preceding tabular day the product of the variation per day and the days elapsed, as given below:

Days elapsed	Variation per day						Days elapsed	Variation per day					
	3^m 57^s		3^m 56^s		3^m 55^s			3^m 57^s		3^m 56^s		3^m 55^s	
	m	s	m	s	m	s		m	s	m	s	m	s
1	3	57	3	56	3	55	6	23	42	23	36	23	30
2	7	54	7	52	7	50	7	27	39	27	32	27	25
3	11	51	11	48	11	45	8	31	36	31	28	31	20
4	15	48	15	44	15	40	9	35	33	35	24	35	15
5	19	45	19	40	19	35							

C. *To refer to any other than the tabular longitude* (90°)*:* ADD 0.1^m for each 10° east of the ninetieth meridian or SUBTRACT 0.1^m for each 10° west of the ninetieth meridian.

D. *To refer to standard time:* ADD to the quantities in Table IV four minutes for every degree of longitude the place of observation is west of the standard meridian (60°, 75°, 90°, etc.). SUBTRACT when the place is east of the standard meridian.

TABLE V.—AZIMUTH OF POLARIS AT ELONGATION, 1931 TO 1940

Latitude	1931	1932	1933	1934	1935	1936	1937	1938	1939	1940
	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
10°	1 05.0	1 04.7	1 04.4	1 04.1	1 03.8	1 03.4	1 03.1	1 02.8	1 02.5	1 02.2
11	1 05.2	1 04.9	1 04.6	1 04.3	1 04.0	1 03.6	1 03.3	1 03.0	1 02.7	1 02.4
12	1 05.4	1 05.1	1 04.8	1 04.5	1 04.2	1 03.9	1 03.6	1 03.2	1 02.9	1 02.6
13	1 05.7	1 05.3	1 05.0	1 04.7	1 04.4	1 04.1	1 03.8	1 03.5	1 03.2	1 02.9
14	1 05.9	1 05.6	1 05.3	1 05.0	1 04.7	1 04.4	1 04.1	1 03.8	1 03.5	1 03.2
15	1 06.2	1 05.9	1 05.6	1 05.3	1 05.0	1 04.7	1 04.4	1 04.1	1 03.8	1 03.5
16	1 06.6	1 06.2	1 05.9	1 05.6	1 05.3	1 05.0	1 04.7	1 04.4	1 04.1	1 03.8
17	1 06.9	1 06.6	1 06.3	1 06.0	1 05.7	1 05.3	1 05.0	1 04.7	1 04.4	1 04.1
18	1 07.3	1 07.0	1 06.6	1 06.3	1 06.0	1 05.7	1 05.4	1 05.0	1 04.7	1 04.4
19	1 07.7	1 07.3	1 07.0	1 06.7	1 06.4	1 06.1	1 05.7	1 05.4	1 05.0	1 04.7
20	1 08.1	1 07.8	1 07.4	1 07.1	1 06.8	1 06.5	1 06.2	1 05.8	1 05.4	1 05.1
21	1 08.5	1 08.2	1 07.9	1 07.6	1 07.2	1 06.9	1 06.6	1 06.3	1 05.9	1 05.6
22	1 09.0	1 08.7	1 08.4	1 08.0	1 07.7	1 07.4	1 07.0	1 06.7	1 06.3	1 06.0
23	1 09.5	1 09.2	1 08.9	1 08.5	1 08.2	1 07.9	1 07.5	1 07.2	1 06.8	1 06.5
24	1 10.0	1 09.7	1 09.4	1 09.0	1 08.7	1 08.4	1 08.1	1 07.7	1 07.3	1 07.0
25	1 10.6	1 10.3	1 09.9	1 09.6	1 09.3	1 08.9	1 08.6	1 08.3	1 07.9	1 07.5
26	1 11.2	1 10.9	1 10.5	1 10.2	1 09.9	1 09.5	1 09.2	1 08.8	1 08.4	1 08.0
27	1 11.8	1 11.5	1 11.1	1 10.8	1 10.5	1 10.1	1 09.8	1 09.4	1 09.0	1 08.6
28	1 12.5	1 12.1	1 11.8	1 11.4	1 11.1	1 10.8	1 10.4	1 10.1	1 09.7	1 09.3
29	1 13.2	1 12.8	1 12.5	1 12.1	1 11.8	1 11.4	1 11.1	1 10.7	1 10.3	1 09.9
30	1 13.9	1 13.5	1 13.2	1 12.8	1 12.5	1 12.1	1 11.8	1 11.4	1 11.0	1 10.6
31	1 14.6	1 14.3	1 13.9	1 13.6	1 13.2	1 12.9	1 12.5	1 12.2	1 11.8	1 11.4
32	1 15.4	1 15.1	1 14.7	1 14.4	1 14.0	1 13.7	1 13.3	1 12.9	1 12.6	1 12.2
33	1 16.3	1 15.9	1 15.6	1 15.2	1 14.9	1 14.5	1 14.1	1 13.8	1 13.4	1 13.0
34	1 17.2	1 16.8	1 16.4	1 16.1	1 15.7	1 15.4	1 15.0	1 14.6	1 14.2	1 13.8
35	1 18.1	1 17.7	1 17.4	1 17.0	1 16.6	1 16.3	1 15.9	1 15.5	1 15.1	1 14.7
36	1 19.1	1 18.7	1 18.3	1 18.0	1 17.6	1 17.2	1 16.8	1 16.5	1 16.1	1 15.7
37	1 20.1	1 19.7	1 19.4	1 19.0	1 18.6	1 18.2	1 17.8	1 17.5	1 17.1	1 16.7
38	1 21.2	1 20.8	1 20.4	1 20.0	1 19.7	1 19.3	1 18.9	1 18.5	1 18.1	1 17.7
39	1 22.3	1 21.9	1 21.6	1 21.2	1 20.8	1 20.4	1 20.0	1 19.6	1 19.2	1 18.8
40	1 23.5	1 23.1	1 22.7	1 22.3	1 22.0	1 21.6	1 21.2	1 20.8	1 20.4	1 20.0
41	1 24.8	1 24.4	1 24.0	1 23.6	1 23.2	1 22.8	1 22.4	1 22.0	1 21.6	1 21.2
42	1 26.1	1 25.7	1 25.3	1 24.9	1 24.5	1 24.1	1 23.7	1 23.2	1 22.8	1 22.4
43	1 27.5	1 27.1	1 26.7	1 26.3	1 25.8	1 25.4	1 25.0	1 24.6	1 24.2	1 23.7
44	1 29.0	1 28.5	1 28.1	1 27.7	1 27.3	1 26.8	1 26.4	1 26.0	1 25.6	1 25.1
45	1 30.5	1 30.1	1 29.6	1 29.2	1 28.8	1 28.4	1 27.9	1 27.5	1 27.1	1 26.6
46	1 32.1	1 31.7	1 31.2	1 30.8	1 30.4	1 29.9	1 29.5	1 29.1	1 28.7	1 28.2
47	1 33.8	1 33.4	1 32.9	1 32.5	1 32.0	1 31.6	1 31.2	1 30.7	1 30.3	1 29.8
48	1 35.6	1 35.2	1 34.7	1 34.3	1 33.8	1 33.4	1 32.9	1 32.5	1 32.0	1 31.5
49	1 37.5	1 37.1	1 36.6	1 36.1	1 35.7	1 35.2	1 34.8	1 34.3	1 33.8	1 33.3
50	1 39.5	1 39.1	1 38.6	1 38.1	1 37.7	1 37.2	1 36.7	1 36.2	1 35.7	1 35.2

Table V was computed using the mean declination of Polaris for the beginning of each year. A more accurate result will be obtained by applying to the tabular values the following corrections, which depend on the difference between the mean and apparent place of the star:

Month	Correction	Month	Correction	Month	Correction
	′		′		′
January.........	−0.6	May..........	0.0	September.....	−0.2
February........	−0.5	June..........	+0.1	October........	−0.4
March..........	−0.4	July...........	+0.1	November.....	−0.7
April...........	−0.2	August...... .	0.0	December......	−0.9

TABLE V(a).—FOR REDUCING TO ELONGATION OBSERVATIONS MADE NEAR ELONGATION

Azimuth at elongation / * Time	1° 0′	1° 10′	1° 20′	1° 30′	1° 40′	1° 50′	2° 0′	2° 10′	Azimuth at elongation / * Time
m	″	″	″	″	″	″	″	″	m
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0
1	0.0	0.0	0.0	+ 0.1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	1
2	+ 0.1	+ 0.2	+ 0.2	0.2	0.2	0.3	0.3	0.3	2
3	0.3	0.4	0.4	0.5	0.5	0.6	0.6	0.7	3
4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	4
5	+ 0.9	+ 1.0	+ 1.1	+ 1.3	+ 1.4	+ 1.6	+ 1.7	+ 1.9	5
6	1.2	1.4	1.6	1.8	2.1	2.3	2.5	2.7	6
7	1.7	2.0	2.2	2.5	2.8	3.1	3.4	3.7	7
8	2.2	2.6	2.9	3.3	3.7	4.0	4.4	4.8	8
9	2.8	3.2	3.7	4.2	4.6	5.1	5.6	6.0	9
10	+ 3.4	+ 4.0	+ 4.6	+ 5.1	+ 5.7	+ 6.3	+ 6.9	+ 7.4	10
11	4.1	4.8	5.5	6.2	6.9	7.6	8.3	9.0	11
12	4.9	5.8	6.6	7.4	8.2	9.0	9.9	10.7	12
13	5.8	6.8	7.7	8.7	9.7	10.6	11.6	12.6	13
14	6.7	7.8	9.0	10.1	11.2	12.3	13.4	14.6	14
15	+ 7.7	+ 9.0	+10.3	+11.6	+12.8	+14.1	+15.4	+16.7	15
16	8.8	10.2	11.7	13.2	14.6	16.1	17.5	19.0	16
17	9.9	11.5	13.2	14.9	16.5	18.2	19.8	21.5	17
18	11.1	12.9	14.8	16.7	18.5	20.4	22.2	24.1	18
19	12.4	14.4	16.5	18.6	20.6	22.7	24.7	26.8	19
20	+13.7	+16.0	+18.3	+20.6	+22.8	+25.1	+27.4	+29.7	20
21	15.1	17.6	20.1	22.7	25.2	27.7	30.2	32.7	21
22	16.6	19.3	22.1	24.9	27.6	30.4	33.2	35.9	22
23	18.1	21.1	24.2	27.2	30.2	33.2	36.2	39.3	23
24	19.7	23.0	26.3	29.6	32.9	36.2	39.5	42.8	24
25	+21.4	+25.0	+28.5	+32.1	+35.7	+39.2	+42.8	+46.4	25

Azimuth at elongation / * Time	2° 10′	2° 20′	2° 30′	2° 40′	2° 50′	3° 0′	3° 10′	3° 20′	Azimuth at elongation / * Time
m	″	″	″	″	″	″	″	″	m
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0
1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	+ 0.1	1
2	0.3	0.3	0.4	0.4	0.4	0.4	0.4	0.5	2
3	0.7	0.7	0.8	0.8	0.9	0.9	1.0	1.0	3
4	1.2	1.3	1.4	1.5	1.6	1.6	1.7	1.8	4
5	+ 1.9	+ 2.0	+ 2.1	+ 2.3	+ 2.4	+ 2.6	+ 2.7	+ 2.9	5
6	2.7	2.9	3.1	3.3	3.5	3.7	3.9	4.1	6
7	3.7	3.9	4.2	4.5	4.8	5.0	5.3	5.6	7
8	4.8	5.1	5.5	5.9	6.2	6.6	7.0	7.3	8
9	6.0	6.5	7.0	7.4	7.9	8.3	8.8	9.3	9
10	+ 7.4	+ 8.0	+ 8.6	+ 9.2	+ 9.7	+10.3	+10.9	+11.4	10
11	9.0	9.7	10.4	11.1	11.8	12.4	13.1	13.8	11
12	10.7	11.5	12.3	13.2	14.0	14.8	15.6	16.5	12
13	12.6	13.5	14.5	15.4	16.4	17.4	18.4	19.3	13
14	14.6	15.7	16.8	17.9	19.0	20.2	21.3	22.4	14
15	+16.7	+18.0	+19.3	+20.6	+21.9	+23.1	+24.4	+25.7	15
16	19.0	20.5	21.9	23.4	24.9	26.3	27.8	29.3	16
17	21.5	23.1	24.8	26.4	28.1	29.7	31.4	33.0	17
18	24.1	25.9	27.8	29.6	31.5	33.3	35.2	37.0	18
19	26.8	28.9	30.9	33.0	35.1	37.1	39.2	41.3	19
20	+29.7	+32.0	+34.3	+36.6	+38.8	+41.1	+43.4	+45.7	20
21	32.7	35.3	37.8	40.3	42.8	45.3	47.9	50.4	21
22	35.9	38.7	41.5	44.2	47.0	49.8	52.5	55.3	22
23	39.3	42.3	45.3	48.3	51.4	54.4	57.4	60.4	23
24	42.8	46.0	49.3	52.6	55.9	59.2	62.5	65.8	24
25	+46.4	+49.9	+53.5	+57.1	+60.7	+64.2	+67.8	+71.4	25

* Sidereal time from elongation.

TABLE VI.—TOTAL SOLAR-DIURNAL VARIATION OF THE MAGNETIC DECLINATION, ON THE YEARLY AVERAGE, AT PROMINENT PLACES IN NORTH AMERICA

[*A + sign indicates a deflection of the north-seeking end of the magnet toward the east, a − sign the contrary direction*]

Local mean time	1 Key West, Fla.	2 Los Angeles, Cal.	3 Washington, D. C.	4 Philadelphia, Pa.	5 Madison, Wis.	6 Toronto, Canada	7 Sitka, Alaska	8 Uglaamie, Point Barrow	9 Plover Point, Point Barrow	10 Fort Rae, Great Slave Lake	11 Kingua Fjord, Cumberland Sound	12 Fort Conger, Grinnell Land	Average values, stations 1 to 6, inclusive
	′	′	′	′	′	′	′	′	′	′	′	′	′
1 a.m. ..	+0.0	+0.0	+0.7	+0.6	+0.1	+0.6	+0.2	−12.8	− 8.0	−11.0	+11.7	+43.2	+0.35
2 a.m. ..	−0.0	+0.1	+0.7	+0.5	0.0	+0.5	+1.0	− 4.9	− 1.9	− 6.6	+15.8	+45.1	+0.05
3 a.m. ..	+0.1	+0.2	+0.9	+0.6	+0.2	+0.8	+1.4	+ 3.3	+ 3.6	+ 0.8	+18.0	+41.2	+0.07
4 a.m. ..	+0.2	+0.3	+1.2	+1.0	+0.5	+1.2	+2.0	+ 6.2	+10.9	+ 7.4	+19.1	+25.7	+0.75
5 a.m. ..	+0.4	+0.6	+1.7	+1.5	+1.0	+1.8	+2.9	+14.3	+16.6	+13.6	+19.3	+31.6	+1.19
6 a.m. ..	+1.0	+1.3	+2.1	+2.1	+1.4	+2.7	+4.2	+21.6	+19.3	+21.0	+20.1	+19.7	+1.79
7 a.m. ..	+2.1	+2.4	+2.8	+3.3	+2.6	+3.5	+5.3	+26.1	+27.1	+26.2	+19.9	+26.6	+2.80
8 a.m. ..	+2.5	+3.1	+3.2	+3.5	+3.2	+3.8	+6.0	+26.7	+27.0	+29.4	+17.4	+18.7	+3.24
9 a.m. ..	+2.2	+2.6	+2.3	+2.8	+3.0	+3.0	+5.3	+26.1	+19.9	+25.5	+10.8	+ 1.2	+2.67
10 a.m. ..	+1.1	+1.1	+0.9	+0.8	+1.7	−0.8	+3.0	+ 9.9	+ 9.3	+16.8	+ 3.7	−12.7	+1.09
11 a.m. ..	−0.2	−0.8	−1.3	−1.6	−0.7	−2.0	+0.6	+ 1.4	− 0.4	+ 8.0	+ 1.3	−21.4	−1.08
Noon.....	−1.4	−2.2	−3.2	−3.4	−2.5	−4.2	−2.1	− 5.9	− 8.2	− 0.9	− 9.0	−40.7	−2.80
1 p.m. ..	−2.1	−2.7	−4.3	−4.3	−3.5	−5.0	−3.2	− 7.3	−10.7	− 4.0	−15.1	−45.6	−3.63
2 p.m. ..	−2.2	−2.6	−4.3	−4.1	−3.5	−4.8	−4.2	− 7.7	− 9.8	− 8.1	−21.2	−49.2	−3.56
3 p.m. ..	−1.9	−2.0	−3.5	−3.1	−2.6	−3.8	−4.6	− 7.3	− 9.9	−10.6	−20.4	−45.8	−2.80
4 p.m. ..	−1.3	−1.1	−2.5	−2.2	−1.6	−2.5	−4.6	− 9.1	− 9.8	−11.3	−20.6	−53.7	−1.85
5 p.m. ..	−0.8	−0.5	−1.5	−1.0	−0.7	−1.3	−3.8	− 9.9	−10.2	−12.1	−23.6	−23.7	−0.95
6 p.m. ..	−0.4	−0.2	−0.8	−0.4	−0.2	−0.3	−3.2	− 9.9	− 9.7	−12.9	−19.4	−17.3	−0.36
7 p.m. ..	−0.2	−0.0	0.0	+0.0	+0.2	+0.2	−2.4	− 8.4	− 8.4	−12.5	−16.1	−27.2	+0.05
8 p.m. ..	+0.1	+0.1	+0.6	+0.8	+0.2	+0.7	−1.4	− 6.0	− 9.0	−11.0	−15.5	− 3.5	+0.44
9 p.m. ..	+0.2	+0.1	+1.0	+0.6	+0.6	+1.2	−0.8	− 8.1	− 7.5	−12.0	− 8.8	+ 3.5	+0.64
10 p.m. ..	+0.2	+0.1	+1.1	+1.2	+0.7	+1.3	−0.4	−10.9	− 7.9	−11.9	− 0.6	+22.4	+0.79
11 p.m. ..	+0.2	+0.1	+1.1	+0.7	+0.2	+1.2	−0.6	− 9.1	−11.5	−11.9	+ 3.9	−30.0	+0.60
Midnight.	+0.1	+0.0	+1.0	+0.6	+0.1	+0.8	−0.6	−13.4	−10.8	−12.0	+ 9.2	+32.6	+0.45
Range..	4.7	5.8	7.5	7.8	6.7	8.8	10.6	40.1	38.6	41.4	43.7	98.8	6.9

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 10°	Latitude 11°	Latitude 12°	Latitude 13°	Latitude 14°	Latitude 15°	Latitude 16°	Latitude 17°	Latitude 18°	Latitude 19°	Latitude 20°	Latitude 10°	Latitude 20°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0 15	0 04 36	0 04 37	0 04 38	0 04 39	0 04 41	0 04 42	0 04 43	0 04 45	0 04 47	0 04 48	0 04 50	− 4	− 4
0 30	0 09 11	0 09 13	0 09 15	0 09 17	0 09 20	0 09 23	0 09 25	0 09 29	0 09 32	0 09 36	0 09 39	− 8	− 8
0 45	0 13 43	0 13 46	0 13 49	0 13 53	0 13 57	0 14 01	0 14 05	0 14 10	0 14 15	0 14 20	0 14 26	−12	−13
1 00	0 18 12	0 18 16	0 18 20	0 18 25	0 18 30	0 18 35	0 18 41	0 18 47	0 18 54	0 19 01	0 19 09	−16	−17
1 15	0 22 36	0 22 41	0 22 46	0 22 52	0 22 58	0 23 05	0 23 12	0 23 20	0 23 28	0 23 37	0 23 46	−20	−21
1 30	0 26 54	0 27 00	0 27 06	0 27 13	0 27 21	0 27 29	0 27 37	0 27 46	0 27 56	0 28 07	0 28 18	−24	−25
1 45	0 31 05	0 31 12	0 31 19	0 31 27	0 31 36	0 31 45	0 31 55	0 32 06	0 32 17	0 32 29	0 32 42	−27	−29
2 00	0 35 09	0 35 16	0 35 24	0 35 33	0 35 43	0 35 53	0 36 04	0 36 16	0 36 29	0 36 43	0 36 57	−31	−32
2 15	0 39 03	0 39 11	0 39 20	0 39 30	0 39 41	0 39 52	0 40 04	0 40 18	0 40 32	0 40 47	0 41 03	−34	−36
2 30	0 42 47	0 42 56	0 43 06	0 43 17	0 43 28	0 43 41	0 43 54	0 44 09	0 44 24	0 44 40	0 44 58	−37	−39
2 45	0 46 19	0 46 29	0 46 40	0 46 52	0 47 04	0 47 18	0 47 32	0 47 48	0 48 04	0 48 22	0 48 41	−40	−42
3 00	0 49 40	0 49 51	0 50 02	0 50 15	0 50 28	0 50 42	0 50 58	0 51 15	0 51 33	0 51 52	0 52 12	−43	−45
3 15	0 52 48	0 52 59	0 53 11	0 53 24	0 53 39	0 53 54	0 54 11	0 54 28	0 54 47	0 55 07	0 55 29	−46	−48
3 30	0 55 43	0 55 54	0 56 07	0 56 21	0 56 36	0 56 52	0 57 09	0 57 28	0 57 47	0 58 09	0 58 31	−49	−51
3 45	0 58 22	0 58 34	0 58 48	0 59 02	0 59 18	0 59 35	0 59 53	1 00 12	1 00 33	1 00 55	1 01 18	−51	−53
4 00	1 00 47	1 01 00	1 01 13	1 01 28	1 01 44	1 02 02	1 02 21	1 02 41	1 03 02	1 03 25	1 03 49	−53	−55
4 15	1 02 56	1 03 09	1 03 23	1 03 38	1 03 55	1 04 13	1 04 33	1 04 53	1 05 15	1 05 39	1 06 04	−55	−57
4 30	1 04 49	1 05 02	1 05 17	1 05 33	1 05 50	1 06 08	1 06 28	1 06 49	1 07 12	1 07 36	1 08 02	−56	−59
4 45	1 06 25	1 06 39	1 06 53	1 07 10	1 07 27	1 07 46	1 08 06	1 08 28	1 08 51	1 09 15	1 09 42	−58	−61
5 00	1 07 44	1 07 58	1 08 13	1 08 29	1 08 47	1 09 06	1 09 26	1 09 48	1 10 12	1 10 37	1 11 03	−59	−62
5 15	1 08 46	1 08 59	1 09 15	1 09 31	1 09 49	1 10 08	1 10 29	1 10 51	1 11 15	1 11 40	1 12 07	−60	−62
5 30	1 09 30	1 09 43	1 09 59	1 10 15	1 10 33	1 10 52	1 11 13	1 11 35	1 11 59	1 12 25	1 12 52	−60	−63
5 45	1 09 56	1 10 09	1 10 25	1 10 41	1 10 59	1 11 18	1 11 39	1 12 01	1 12 25	1 12 51	1 13 18	−61	−63
6 00	1 10 04	1 10 17	1 10 32	1 10 49	1 11 07	1 11 26	1 11 47	1 12 09	1 12 33	1 12 58	1 13 26	−61	−64
6 15	1 09 54	1 10 07	1 10 22	1 10 39	1 10 55	1 11 15	1 11 36	1 11 58	1 12 22	1 12 47	1 13 14	−61	−63
6 30	1 09 26	1 09 39	1 09 54	1 10 10	1 10 28	1 10 46	1 11 07	1 11 29	1 11 52	1 12 17	1 12 44	−60	−63
6 45	1 08 40	1 08 53	1 09 08	1 09 23	1 09 41	1 09 59	1 10 19	1 10 41	1 11 04	1 11 29	1 11 55	−60	−62

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 10°	Latitude 11°	Latitude 12°	Latitude 13°	Latitude 14°	Latitude 15°	Latitude 16°	Latitude 17°	Latitude 18°	Latitude 19°	Latitude 20°	Latitude 10°	Latitude 20°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
7 00	1 07 37	1 07 50	1 08 05	1 08 19	1 08 36	1 08 54	1 09 14	1 09 35	1 09 58	1 10 22	1 10 47	−59	−61
7 15	1 06 16	1 06 29	1 06 42	1 06 57	1 07 14	1 07 32	1 07 51	1 08 11	1 08 33	1 08 57	1 09 22	−58	−60
7 30	1 04 39	1 04 51	1 05 04	1 05 19	1 05 35	1 05 52	1 06 10	1 06 30	1 06 52	1 07 15	1 07 39	−56	−58
7 45	1 02 44	1 02 56	1 03 09	1 03 23	1 03 38	1 03 55	1 04 13	1 04 32	1 04 53	1 05 15	1 05 39	−54	−56
8 00	1 00 34	1 00 45	1 00 58	1 01 11	1 01 26	1 01 42	1 01 59	1 02 18	1 02 38	1 02 59	1 03 22	−52	−55
8 15	0 58 09	0 58 19	0 58 31	0 58 44	0 58 58	0 59 13	0 59 30	0 59 47	1 00 06	1 00 27	1 00 49	−50	−53
8 30	0 55 28	0 55 38	0 55 49	0 56 02	0 56 15	0 56 29	0 56 45	0 57 02	0 57 20	0 57 39	0 58 00	−48	−50
8 45	0 52 33	0 52 43	0 52 53	0 53 05	0 53 18	0 53 31	0 53 46	0 54 02	0 54 19	0 54 37	0 54 57	−45	−48
9 00	0 49 25	0 49 33	0 49 44	0 49 55	0 50 07	0 50 19	0 50 33	0 50 48	0 51 04	0 51 21	0 51 40	−42	−45
9 15	0 46 05	0 46 13	0 46 22	0 46 32	0 46 43	0 46 55	0 47 08	0 47 21	0 47 36	0 47 52	0 48 09	−40	−42
9 30	0 42 32	0 42 40	0 42 48	0 42 57	0 43 07	0 43 18	0 43 30	0 43 43	0 43 57	0 44 11	0 44 27	−37	−38
9 45	0 38 49	0 38 56	0 39 03	0 39 12	0 39 21	0 39 31	0 39 42	0 39 53	0 40 06	0 40 19	0 40 33	−34	−35
10 00	0 34 56	0 35 02	0 35 09	0 35 16	0 35 24	0 35 33	0 35 43	0 35 53	0 36 04	0 36 16	0 36 29	−31	−31
10 15	0 30 54	0 30 59	0 31 05	0 31 12	0 31 19	0 31 27	0 31 35	0 31 44	0 31 54	0 32 05	0 32 16	−28	−28
10 30	0 26 44	0 26 48	0 26 54	0 26 59	0 27 06	0 27 12	0 27 20	0 27 28	0 27 36	0 27 45	0 27 55	−24	−24
10 45	0 22 27	0 22 31	0 22 35	0 22 40	0 22 45	0 22 51	0 22 57	0 23 04	0 23 11	0 23 18	0 23 27	−20	−20
11 00	0 18 04	0 18 08	0 18 11	0 18 15	0 18 19	0 18 24	0 18 29	0 18 34	0 18 40	0 18 46	0 18 52	−16	−16
11 15	0 13 37	0 13 40	0 13 42	0 13 45	0 13 48	0 13 52	0 13 56	0 14 00	0 14 04	0 14 09	0 14 13	−12	−12
11 30	0 09 07	0 09 08	0 09 10	0 09 12	0 09 14	0 09 16	0 09 19	0 09 22	0 09 25	0 09 28	0 09 31	− 8	− 8
11 45	0 04 34	0 04 35	0 04 36	0 04 37	0 04 38	0 04 39	0 04 40	0 04 41	0 04 43	0 04 44	0 04 46	− 4	− 4
Elongation: Azimuth	1 10 04	1 10 18	1 10 33	1 10 49	1 11 07	1 11 26	1 11 47	1 12 09	1 12 33	1 12 58	1 13 26	−61	−64
	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	s	s
Hour angle	5 59 11	5 59 06	5 59 02	5 58 56	5 58 51	5 58 46	5 58 41	5 58 36	5 58 30	5 58 25	5 58 19	+ 2	+ 2

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 20°	Latitude 21°	Latitude 22°	Latitude 23°	Latitude 24°	Latitude 25°	Latitude 26°	Latitude 27°	Latitude 28°	Latitude 29°	Latitude 30°	Latitude 20°	Latitude 30°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
0 15	0 04 50	0 04 52	0 04 55	0 04 57	0 04 59	0 05 02	0 05 04	0 05 07	0 05 10	0 05 13	0 05 17	− 4	− 5
0 30	0 09 39	0 09 43	0 09 48	0 09 52	0 09 57	0 10 02	0 10 07	0 10 13	0 10 19	0 10 25	0 10 31	− 8	− 9
0 45	0 14 26	0 14 32	0 14 38	0 14 45	0 14 52	0 15 00	0 15 07	0 15 16	0 15 25	0 15 34	0 15 44	−13	−14
1 00	0 19 09	0 19 16	0 19 25	0 19 34	0 19 43	0 19 53	0 20 04	0 20 15	0 20 26	0 20 39	0 20 51	−17	−18
1 15	0 23 46	0 23 56	0 24 07	0 24 18	0 24 29	0 24 42	0 24 55	0 25 08	0 25 23	0 25 38	0 25 54	−21	−23
1 30	0 28 18	0 28 29	0 28 42	0 28 55	0 29 09	0 29 24	0 29 39	0 29 55	0 30 12	0 30 30	0 30 49	−25	−27
1 45	0 32 42	0 32 55	0 33 10	0 33 25	0 33 41	0 33 58	0 34 16	0 34 34	0 34 54	0 35 15	0 35 37	−29	−31
2 00	0 36 57	0 37 12	0 37 29	0 37 46	0 38 04	0 38 23	0 38 43	0 39 04	0 39 26	0 39 50	0 40 15	−32	−35
2 15	0 41 03	0 41 20	0 41 38	0 41 57	0 42 17	0 42 38	0 43 00	0 43 24	0 43 49	0 44 14	0 44 42	−36	−39
2 30	0 44 58	0 45 16	0 45 36	0 45 57	0 46 19	0 46 42	0 47 06	0 47 32	0 47 59	0 48 27	0 48 57	−39	−43
2 45	0 48 41	0 49 01	0 49 22	0 49 45	0 50 08	0 50 33	0 51 00	0 51 27	0 51 57	0 52 27	0 53 00	−42	−46
3 00	0 52 12	0 52 33	0 52 56	0 53 20	0 53 45	0 54 12	0 54 40	0 55 10	0 55 41	0 56 14	0 56 48	−45	−50
3 15	0 55 29	0 55 51	0 56 15	0 56 41	0 57 08	0 57 36	0 58 06	0 58 37	0 59 10	0 59 45	1 00 22	−48	−53
3 30	0 58 31	0 58 55	0 59 20	0 59 47	1 00 15	1 00 45	1 01 16	1 01 49	1 02 24	1 03 01	1 03 40	−51	−56
3 45	1 01 18	1 01 43	1 02 09	1 02 37	1 03 07	1 03 38	1 04 11	1 04 46	1 05 22	1 06 00	1 06 41	−53	−58
4 00	1 03 49	1 04 15	1 04 43	1 05 12	1 05 42	1 06 15	1 06 49	1 07 25	1 08 02	1 08 42	1 09 24	−55	−61
4 15	1 06 04	1 06 31	1 06 59	1 07 29	1 08 01	1 08 34	1 09 09	1 09 46	1 10 25	1 11 06	1 11 49	−57	−63
4 30	1 08 02	1 08 29	1 08 58	1 09 29	1 10 01	1 10 35	1 11 12	1 11 50	1 12 30	1 13 12	1 13 56	−59	−64
4 45	1 09 42	1 10 10	1 10 39	1 11 10	1 11 44	1 12 19	1 12 55	1 13 34	1 14 15	1 14 58	1 15 44	−61	−66
5 00	1 11 03	1 11 32	1 12 02	1 12 34	1 13 07	1 13 43	1 14 20	1 15 00	1 15 41	1 16 25	1 17 11	−62	−68
5 15	1 12 07	1 12 36	1 13 06	1 13 38	1 14 12	1 14 48	1 15 26	1 16 06	1 16 48	1 17 32	1 18 19	−62	−69
5 30	1 12 52	1 13 21	1 13 51	1 14 24	1 14 58	1 15 34	1 16 13	1 16 53	1 17 35	1 18 20	1 19 07	−63	−69
5 45	1 13 18	1 13 47	1 14 18	1 14 50	1 15 25	1 16 01	1 16 39	1 17 19	1 18 02	1 18 47	1 19 34	−63	−69
6 00	1 13 26	1 13 54	1 14 25	1 14 57	1 15 32	1 16 08	1 16 46	1 17 26	1 18 09	1 18 53	1 19 41	−64	−70
6 15	1 13 14	1 13 43	1 14 13	1 14 45	1 15 19	1 15 55	1 16 33	1 17 13	1 17 55	1 18 40	1 19 26	−63	−69
6 30	1 12 44	1 13 12	1 13 42	1 14 14	1 14 48	1 15 23	1 16 01	1 16 40	1 17 22	1 18 06	1 18 52	−63	−68
6 45	1 11 55	1 12 23	1 12 52	1 13 24	1 13 57	1 14 32	1 15 09	1 15 48	1 16 29	1 17 12	1 17 58	−62	−67

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 20°	Latitude 21°	Latitude 22°	Latitude 23°	Latitude 24°	Latitude 25°	Latitude 26°	Latitude 27°	Latitude 28°	Latitude 29°	Latitude 30°	Latitude 20°	Latitude 30°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
7 00	1 10 47	1 11 15	1 11 44	1 12 15	1 12 47	1 13 22	1 13 58	1 14 36	1 15 16	1 15 59	1 16 44	−61	−66
7 15	1 09 22	1 09 49	1 10 17	1 10 47	1 11 19	1 11 53	1 12 28	1 13 05	1 13 45	1 14 26	1 15 10	−60	−65
7 30	1 07 39	1 08 05	1 08 32	1 09 02	1 09 33	1 10 05	1 10 40	1 11 16	1 11 54	1 12 35	1 13 17	−58	−64
7 45	1 05 39	1 06 04	1 06 30	1 06 59	1 07 28	1 08 00	1 08 33	1 09 08	1 09 46	1 10 24	1 11 06	−56	−62
8 00	1 03 22	1 03 46	1 04 11	1 04 38	1 05 07	1 05 38	1 06 10	1 06 43	1 07 19	1 07 57	1 08 36	−55	−60
8 15	1 00 49	1 01 11	1 01 36	1 02 02	1 02 30	1 02 59	1 03 29	1 04 02	1 04 36	1 05 12	1 05 49	−53	−57
8 30	0 58 00	0 58 22	0 58 45	0 59 10	0 59 36	1 00 04	1 00 33	1 01 04	1 01 36	1 02 10	1 02 46	−50	−54
8 45	0 54 57	0 55 17	0 55 40	0 56 03	0 56 28	0 56 54	0 57 22	0 57 51	0 58 21	0 58 53	0 59 27	−48	−51
9 00	0 51 40	0 51 59	0 52 20	0 52 41	0 53 04	0 53 29	0 53 55	0 54 22	0 54 51	0 55 21	0 55 53	−45	−48
9 15	0 48 09	0 48 27	0 48 46	0 49 07	0 49 28	0 49 51	0 50 15	0 50 40	0 51 07	0 51 35	0 52 05	−42	−45
9 30	0 44 27	0 44 43	0 45 01	0 45 20	0 45 40	0 46 01	0 46 22	0 46 46	0 47 11	0 47 37	0 48 04	−38	−42
9 45	0 40 33	0 40 48	0 41 04	0 41 21	0 41 39	0 41 58	0 42 19	0 42 40	0 43 02	0 43 26	0 43 51	−35	−38
10 00	0 36 29	0 36 43	0 36 57	0 37 12	0 37 29	0 37 46	0 38 04	0 38 23	0 38 43	0 39 04	0 39 27	−31	−34
10 15	0 32 16	0 32 28	0 32 41	0 32 54	0 33 08	0 33 24	0 33 40	0 33 57	0 34 14	0 34 34	0 34 53	−28	−30
10 30	0 27 55	0 28 05	0 28 16	0 28 28	0 28 40	0 28 53	0 29 07	0 29 22	0 29 37	0 29 53	0 30 10	−24	−26
10 45	0 23 27	0 23 35	0 23 44	0 23 54	0 24 05	0 24 16	0 24 27	0 24 39	0 24 52	0 25 06	0 25 20	−20	−22
11 00	0 18 52	0 18 59	0 19 07	0 19 15	0 19 23	0 19 32	0 19 41	0 19 51	0 20 01	0 20 12	0 20 24	−16	−18
11 15	0 14 13	0 14 19	0 14 24	0 14 30	0 14 37	0 14 43	0 14 50	0 14 58	0 15 05	0 15 14	0 15 22	−12	−13
11 30	0 09 31	0 09 34	0 09 38	0 09 42	0 09 46	0 09 51	0 09 56	0 10 01	0 10 06	0 10 11	0 10 17	− 8	− 9
11 45	0 04 46	0 04 48	0 04 50	0 04 52	0 04 54	0 04 56	0 04 58	0 05 00	0 05 03	0 05 06	0 05 09	− 4	− 4
Elongation: Azimuth	1 13 26	1 13 55	1 14 25	1 14 58	1 15 32	1 16 08	1 16 46	1 17 27	1 18 09	1 18 54	1 19 41	−64	−69
	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	s	s
Hour angle	5 58 19	5 58 14	5 58 08	5 58 03	5 57 57	5 57 51	5 57 45	5 57 39	5 57 33	5 57 27	5 57 21	+ 2	+ 2

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 30°	Latitude 31°	Latitude 32°	Latitude 33°	Latitude 34°	Latitude 35°	Latitude 36°	Latitude 37°	Latitude 38°	Latitude 39°	Latitude 40°	Latitude 30°	Latitude 40°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
0 15	0 05 17	0 05 20	0 05 23	0 05 27	0 05 31	0 05 35	0 05 40	0 05 44	0 05 49	0 05 54	0 06 00	− 5	− 5
0 30	0 10 31	0 10 38	0 10 45	0 10 53	0 11 01	0 11 09	0 11 18	0 11 27	0 11 37	0 11 47	0 11 57	− 9	−10
0 45	0 15 44	0 15 54	0 16 04	0 16 16	0 16 27	0 16 40	0 16 53	0 17 07	0 17 21	0 17 36	0 17 52	−14	−16
1 00	0 20 51	0 21 05	0 21 19	0 21 34	0 21 50	0 22 06	0 22 24	0 22 42	0 23 01	0 23 21	0 23 42	−18	−21
1 15	0 25 54	0 26 11	0 26 28	0 26 47	0 27 06	0 27 27	0 27 48	0 28 11	0 28 34	0 28 59	0 29 26	−23	−26
1 30	0 30 49	0 31 09	0 31 30	0 31 52	0 32 15	0 32 40	0 33 05	0 33 32	0 34 00	0 34 30	0 35 01	−27	−31
1 45	0 35 37	0 36 00	0 36 24	0 36 49	0 37 16	0 37 44	0 38 14	0 38 44	0 39 17	0 39 51	0 40 27	−31	−36
2 00	0 40 15	0 40 41	0 41 08	0 41 37	0 42 07	0 42 38	0 43 12	0 43 47	0 44 23	0 45 02	0 45 42	−35	−40
2 15	0 44 42	0 45 11	0 45 41	0 46 13	0 46 46	0 47 21	0 47 58	0 48 37	0 49 18	0 50 00	0 50 45	−39	−45
2 30	0 48 57	0 49 29	0 50 02	0 50 37	0 51 13	0 51 52	0 52 32	0 53 14	0 53 59	0 54 46	0 55 35	−43	−49
2 45	0 53 00	0 53 34	0 54 10	0 54 47	0 55 27	0 56 08	0 56 52	0 57 37	0 58 25	0 59 16	1 00 09	−46	−53
3 00	0 56 48	0 57 25	0 58 03	0 58 43	0 59 25	1 00 10	1 00 56	1 01 45	1 02 37	1 03 31	1 04 28	−50	−57
3 15	1 00 22	1 01 01	1 01 41	1 02 24	1 03 08	1 03 55	1 04 45	1 05 37	1 06 31	1 07 29	1 08 29	−53	−60
3 30	1 03 40	1 04 20	1 05 03	1 05 48	1 06 35	1 07 24	1 08 16	1 09 11	1 10 08	1 11 09	1 12 12	−56	−63
3 45	1 06 41	1 07 23	1 08 08	1 08 54	1 09 44	1 10 35	1 11 30	1 12 27	1 13 27	1 14 30	1 15 36	−58	−66
4 00	1 09 24	1 10 08	1 10 54	1 11 43	1 12 34	1 13 28	1 14 24	1 15 23	1 16 26	1 17 31	1 18 40	−61	−69
4 15	1 11 49	1 12 35	1 13 23	1 14 13	1 15 06	1 16 01	1 16 59	1 18 00	1 19 05	1 20 12	1 21 23	−63	−72
4 30	1 13 56	1 14 43	1 15 32	1 16 23	1 17 18	1 18 14	1 19 14	1 20 17	1 21 23	1 22 32	1 23 45	−64	−74
4 45	1 15 44	1 16 31	1 17 21	1 18 14	1 19 09	1 20 07	1 21 08	1 22 12	1 23 20	1 24 31	1 25 45	−66	−75
5 00	1 17 11	1 18 00	1 18 51	1 19 44	1 20 40	1 21 39	1 22 41	1 23 46	1 24 55	1 26 07	1 27 23	−68	−76
5 15	1 18 19	1 19 08	1 19 59	1 20 54	1 21 50	1 22 50	1 23 53	1 24 59	1 26 08	1 27 21	1 28 38	−69	−77
5 30	1 19 07	1 19 56	1 20 48	1 21 42	1 22 40	1 23 40	1 24 43	1 25 49	1 26 59	1 28 12	1 29 30	−69	−78
5 45	1 19 34	1 20 23	1 21 15	1 22 10	1 23 07	1 24 08	1 25 11	1 26 17	1 27 27	1 28 41	1 29 58	−69	−78
6 00	1 19 41	1 20 30	1 21 22	1 22 16	1 23 13	1 24 14	1 25 17	1 26 23	1 27 33	1 28 47	1 30 04	−70	−78
6 15	1 19 26	1 20 15	1 21 07	1 22 01	1 22 58	1 23 58	1 25 01	1 26 07	1 27 17	1 28 30	1 29 46	−69	−78
6 30	1 18 52	1 19 41	1 20 32	1 21 26	1 22 22	1 23 21	1 24 24	1 25 29	1 26 38	1 27 50	1 29 06	−68	−77
6 45	1 17 58	1 18 46	1 19 36	1 20 29	1 21 25	1 22 23	1 23 24	1 24 29	1 25 37	1 26 48	1 28 03	−67	−76

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 30°	Latitude 31°	Latitude 32°	Latitude 33°	Latitude 34°	Latitude 35°	Latitude 36°	Latitude 37°	Latitude 38°	Latitude 39°	Latitude 40°	Latitude 30°	Latitude 40°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
7 00	1 16 44	1 17 31	1 18 20	1 19 12	1 20 06	1 21 04	1 22 04	1 23 07	1 24 14	1 25 24	1 26 37	−66	−75
7 15	1 15 10	1 15 56	1 16 44	1 17 35	1 18 28	1 19 24	1 20 23	1 21 25	1 22 30	1 23 38	1 24 50	−65	−73
7 30	1 13 17	1 14 02	1 14 49	1 15 38	1 16 30	1 17 24	1 18 21	1 19 21	1 20 25	1 21 31	1 22 41	−64	−72
7 45	1 11 06	1 11 49	1 12 34	1 13 22	1 14 12	1 15 05	1 16 00	1 16 58	1 17 59	1 19 04	1 20 11	−62	−69
8 00	1 08 36	1 09 18	1 10 01	1 10 47	1 11 36	1 12 26	1 13 20	1 14 16	1 15 14	1 16 16	1 17 21	−60	−66
8 15	1 05 49	1 06 29	1 07 11	1 07 55	1 08 41	1 09 30	1 10 21	1 11 14	1 12 11	1 13 10	1 14 12	−57	−64
8 30	1 02 46	1 03 24	1 04 04	1 04 46	1 05 29	1 06 16	1 07 04	1 07 55	1 08 49	1 09 45	1 10 44	−54	−61
8 45	0 59 27	1 00 03	1 00 40	1 01 20	1 02 01	1 02 45	1 03 31	1 04 19	1 05 10	1 06 03	1 06 59	−51	−58
9 00	0 55 53	0 56 26	0 57 02	0 57 39	0 58 18	0 58 59	0 59 43	1 00 27	1 01 14	1 02 04	1 02 57	−48	−54
9 15	0 52 05	0 52 36	0 53 09	0 53 43	0 54 20	0 54 58	0 55 38	0 56 20	0 57 04	0 57 50	0 58 39	−45	−50
9 30	0 48 04	0 48 33	0 49 03	0 49 35	0 50 08	0 50 43	0 51 20	0 51 58	0 52 39	0 53 22	0 54 07	−42	−46
9 45	0 43 51	0 44 17	0 44 44	0 45 13	0 45 44	0 46 16	0 46 49	0 47 24	0 48 01	0 48 40	0 49 21	−38	−42
10 00	0 39 27	0 39 50	0 40 15	0 40 41	0 41 08	0 41 37	0 42 07	0 42 39	0 43 12	0 43 47	0 44 24	−34	−38
10 15	0 34 53	0 35 14	0 35 35	0 35 58	0 36 22	0 36 48	0 37 14	0 37 42	0 38 12	0 38 43	0 39 15	−30	−34
10 30	0 30 10	0 30 28	0 30 47	0 31 07	0 31 28	0 31 50	0 32 12	0 32 37	0 33 02	0 33 29	0 33 57	−26	−29
10 45	0 25 20	0 25 35	0 25 51	0 26 08	0 26 25	0 26 43	0 27 03	0 27 23	0 27 44	0 28 07	0 28 30	−22	−24
11 00	0 20 24	0 20 36	0 20 49	0 21 02	0 21 16	0 21 31	0 21 46	0 22 03	0 22 20	0 22 38	0 22 57	−18	−20
11 15	0 15 22	0 15 31	0 15 41	0 15 51	0 16 02	0 16 13	0 16 24	0 16 37	0 16 50	0 17 03	0 17 17	−13	−15
11 30	0 10 17	0 10 23	0 10 29	0 10 36	0 10 43	0 10 51	0 10 58	0 11 07	0 11 15	0 11 24	0 11 34	− 9	−10
11 45	0 05 09	0 05 12	0 05 15	0 05 19	0 05 22	0 05 26	0 05 30	0 05 34	0 05 38	0 05 43	0 05 47	− 4	− 5
Elongation: Azimuth.	1 19 41	1 20 30	1 21 22	1 22 17	1 23 14	1 24 14	1 25 18	1 26 24	1 27 34	1 28 48	1 30 05	−69	−78
	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	s	s
Hour angle.	5 57 21	5 57 14	5 57 08	5 57 01	5 56 54	5 56 47	5 56 39	5 56 32	5 56 24	5 56 16	5 56 08	+ 2	+ 3

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Continued*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of Polaris	
	Latitude 40°	Latitude 41°	Latitude 42°	Latitude 43°	Latitude 44°	Latitude 45°	Latitude 46°	Latitude 47°	Latitude 48°	Latitude 49°	Latitude 50°	Latitude 40°	Latitude 50°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
0 15	0 06 00	0 06 05	0 06 11	0 06 17	0 06 24	0 06 31	0 06 38	0 06 46	0 06 54	0 07 03	0 07 12	−5	−6
0 30	0 11 57	0 12 09	0 12 21	0 12 33	0 12 46	0 13 00	0 13 14	0 13 30	0 13 46	0 14 03	0 14 21	−10	−13
0 45	0 17 52	0 18 09	0 18 27	0 18 45	0 19 05	0 19 25	0 19 47	0 20 10	0 20 34	0 21 00	0 21 27	−16	−19
1 00	0 23 42	0 24 04	0 24 28	0 24 52	0 25 18	0 25 45	0 26 14	0 26 45	0 27 17	0 27 51	0 28 27	−21	−25
1 15	0 29 26	0 29 53	0 30 22	0 30 53	0 31 25	0 31 59	0 32 34	0 33 12	0 33 52	0 34 34	0 35 18	−26	−32
1 30	0 35 01	0 35 34	0 36 08	0 36 45	0 37 23	0 38 03	0 38 46	0 39 30	0 40 18	0 41 08	0 42 01	−31	−38
1 45	0 40 27	0 41 05	0 41 45	0 42 27	0 43 11	0 43 57	0 44 46	0 45 38	0 46 33	0 47 30	0 48 31	−36	−43
2 00	0 45 42	0 46 25	0 47 10	0 47 57	0 48 47	0 49 39	0 50 35	0 51 33	0 52 35	0 53 40	0 54 49	−40	−49
2 15	0 50 45	0 51 33	0 52 22	0 53 15	0 54 10	0 55 08	0 56 09	0 57 14	0 58 22	0 59 35	1 00 51	−45	−54
2 30	0 55 35	0 56 26	0 57 21	0 58 18	0 59 18	1 00 22	1 01 29	1 02 40	1 03 54	1 05 13	1 06 37	−49	−59
2 45	1 00 09	1 01 05	1 02 04	1 03 06	1 04 11	1 05 20	1 06 32	1 07 48	1 09 09	1 10 34	1 12 04	−53	−64
3 00	1 04 28	1 05 28	1 06 30	1 07 36	1 08 46	1 10 00	1 11 17	1 12 39	1 14 05	1 15 36	1 17 12	−57	−68
3 15	1 08 29	1 09 32	1 10 39	1 11 49	1 13 03	1 14 21	1 15 43	1 17 10	1 18 41	1 20 18	1 22 00	−60	−72
3 30	1 12 12	1 13 19	1 14 29	1 15 43	1 17 00	1 18 22	1 19 49	1 21 20	1 22 56	1 24 37	1 26 25	−63	−76
3 45	1 15 36	1 16 46	1 17 59	1 19 16	1 20 37	1 22 03	1 23 33	1 25 08	1 26 49	1 28 35	1 30 27	−66	−80
4 00	1 18 40	1 19 52	1 21 08	1 22 28	1 23 53	1 25 21	1 26 54	1 28 34	1 30 18	1 32 08	1 34 05	−69	−83
4 15	1 21 23	1 22 38	1 23 56	1 25 19	1 26 46	1 28 17	1 29 54	1 31 33	1 33 24	1 35 17	1 37 17	−72	−86
4 30	1 23 45	1 25 02	1 26 22	1 27 47	1 29 16	1 30 50	1 32 30	1 34 14	1 36 06	1 38 01	1 40 04	−74	−88
4 45	1 25 45	1 27 03	1 28 26	1 29 52	1 31 23	1 32 59	1 34 41	1 36 28	1 38 20	1 40 19	1 42 25	−75	−90
5 00	1 27 23	1 28 42	1 30 06	1 31 34	1 33 06	1 34 44	1 36 27	1 38 16	1 40 10	1 42 11	1 44 19	−76	−91
5 15	1 28 38	1 29 58	1 31 23	1 32 52	1 34 25	1 36 04	1 37 48	1 39 38	1 41 34	1 43 36	1 45 45	−77	−92
5 30	1 29 30	1 30 50	1 32 16	1 33 45	1 35 20	1 36 59	1 38 44	1 40 34	1 42 31	1 44 34	1 46 44	−78	−93
5 45	1 29 58	1 31 20	1 32 45	1 34 15	1 35 50	1 37 29	1 39 14	1 41 05	1 43 02	1 45 05	1 47 16	−78	−94
6 00	1 30 04	1 31 25	1 32 50	1 34 20	1 35 55	1 37 34	1 39 19	1 41 09	1 43 06	1 45 09	1 47 19	−78	−93
6 15	1 29 46	1 31 07	1 32 32	1 34 01	1 35 35	1 37 14	1 38 58	1 40 48	1 42 44	1 44 46	1 46 56	−78	−93
6 30	1 29 06	1 30 26	1 31 50	1 33 18	1 34 51	1 36 29	1 38 12	1 40 01	1 41 56	1 43 57	1 46 04	−77	−92
6 45	1 28 03	1 29 20	1 30 44	1 32 11	1 33 43	1 35 19	1 37 01	1 38 48	1 40 41	1 42 40	1 44 46	−76	−91

TABLE VII.—AZIMUTH OF POLARIS AT ANY HOUR ANGLE.—*Concluded*

Hour angle before or after upper culmination	Azimuth of Polaris computed for declination 88° 51′											Correction for 1′ increase in declination of polaris	
	Latitude 40°	Latitude 41°	Latitude 42°	Latitude 43°	Latitude 44°	Latitude 45°	Latitude 46°	Latitude 47°	Latitude 48°	Latitude 49°	Latitude 50°	Latitude 40°	Latitude 50°
h m	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″	″
7 00	1 26 37	1 27 54	1 29 15	1 30 41	1 32 11	1 33 45	1 35 25	1 37 10	1 39 01	1 40 58	1 43 02	−75	−89
7 15	1 24 50	1 26 05	1 27 24	1 28 48	1 30 16	1 31 48	1 33 25	1 35 08	1 36 56	1 38 51	1 40 51	−73	−87
7 30	1 22 41	1 23 54	1 25 11	1 26 32	1 27 58	1 29 27	1 31 02	1 32 42	1 34 27	1 36 18	1 38 16	−72	−85
7 45	1 20 11	1 21 22	1 22 36	1 23 55	1 25 17	1 26 44	1 28 16	1 29 52	1 31 34	1 33 22	1 35 15	−69	−82
8 00	1 17 21	1 18 29	1 19 41	1 20 57	1 22 16	1 23 40	1 25 08	1 26 41	1 28 19	1 30 02	1 31 51	−66	−79
8 15	1 14 12	1 15 17	1 16 26	1 17 38	1 18 54	1 20 14	1 21 39	1 23 07	1 24 41	1 26 20	1 28 05	−64	−76
8 30	1 10 44	1 11 46	1 12 52	1 14 00	1 15 12	1 16 29	1 17 49	1 19 14	1 20 43	1 22 17	1 23 56	−61	−72
8 45	1 06 59	1 07 57	1 08 59	1 10 04	1 11 12	1 12 23	1 13 40	1 15 00	1 16 24	1 17 53	1 19 27	−58	−68
9 00	1 02 57	1 03 52	1 04 50	1 05 50	1 06 55	1 08 02	1 09 13	1 10 28	1 11 47	1 13 10	1 14 38	−54	−64
9 15	0 58 39	0 59 30	1 00 24	1 01 21	1 02 20	1 03 23	1 04 29	1 05 39	1 06 52	1 08 10	1 09 32	−50	−59
9 30	0 54 07	0 54 54	0 55 44	0 56 36	0 57 31	0 58 28	0 59 29	1 00 34	1 01 41	1 02 53	1 04 08	−46	−55
9 45	0 49 21	0 50 04	0 50 49	0 51 37	0 52 27	0 53 20	0 54 15	0 55 13	0 56 15	0 57 20	0 58 29	−42	−50
10 00	0 44 24	0 45 02	0 45 43	0 46 25	0 47 10	0 47 58	0 48 47	0 49 40	0 50 35	0 51 34	0 52 35	−38	−45
10 15	0 39 15	0 39 49	0 40 25	0 41 03	0 41 42	0 42 24	0 43 08	0 43 54	0 44 43	0 45 35	0 46 29	−34	−40
10 30	0 33 57	0 34 26	0 34 57	0 35 30	0 36 04	0 36 40	0 37 18	0 37 58	0 38 40	0 39 24	0 40 12	−29	−34
10 45	0 28 30	0 28 55	0 29 21	0 29 48	0 30 17	0 30 47	0 31 19	0 31 53	0 32 28	0 33 05	0 33 45	−24	−29
11 00	0 22 57	0 23 16	0 23 37	0 23 59	0 24 22	0 24 47	0 25 12	0 25 39	0 26 08	0 26 38	0 27 09	−20	−23
11 15	0 17 17	0 17 32	0 17 48	0 18 05	0 18 22	0 18 40	0 19 00	0 19 20	0 19 42	0 20 04	0 20 28	−15	−18
11 30	0 11 34	0 11 44	0 11 54	0 12 06	0 12 17	0 12 29	0 12 42	0 12 56	0 13 10	0 13 25	0 13 43	−10	−12
11 45	0 05 47	0 05 52	0 05 58	0 06 04	0 06 09	0 06 15	0 06 22	0 06 29	0 06 36	0 06 44	0 06 51	− 5	− 6
Elongation: Azimuth.	1 30 05	1 31 26	1 32 51	1 34 21	1 35 56	1 37 35	1 39 20	1 41 11	1 43 08	1 45 11	1 47 21	−78	−93
	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	h m s	s	h m s
Hour angle.	5 56 08	5 56 00	5 55 51	5 55 43	5 55 33	5 55 24	5 55 14	5 55 04	5 54 53	5 54 42	5 54 31	+ 3	+ 3

TABLE VIII.—MEAN POLAR DISTANCE OF POLARIS FOR THE BEGINNING OF EACH YEAR, 1931 TO 1940

Year	Mean polar distance	Year	Mean polar distance
1931	1° 03′ 59″	1936	1° 02′ 28″
1932	1° 03′ 41″	1937	1° 02′ 10″
1933	1° 03′ 22″	1938	1° 01′ 51″
1934	1° 03′ 04″	1939	1° 01′ 33″
1935	1° 02′ 46″	1940	1° 01′ 15″

Mean polar distances of Polaris are found by subtracting from 90° the corresponding *mean* declinations. For precise observations the *apparent* declination (to be found in the American Ephemeris) should be used since the variation between mean and apparent declinations is constantly changing, and has as a maximum value about ½′.

TABLE IX.[1]—HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS

	0°		1°		2°		3°	
Minutes	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	100.00	0.00	99.97	1.74	99.88	3.49	99.73	5.23
2	100.00	0.06	99.97	1.80	99.87	3.55	99.72	5.28
4	100.00	0.12	99.97	1.86	99.87	3.60	99.71	5.34
6	100.00	0.17	99.96	1.92	99.87	3.66	99.71	5.40
8	100.00	0.23	99.96	1.98	99.86	3.72	99.70	5.46
10	100.00	0.29	99.96	2.04	99.86	3.78	99.69	5.52
12	100.00	0.35	99.96	2.09	99.85	3.84	99.69	5.57
14	100.00	0.41	99.95	2.15	99.85	3.90	99.68	5.63
16	100.00	0.47	99.95	2.21	99.84	3.95	99.68	5.69
18	100.00	0.52	99.95	2.27	99.84	4.01	99.67	5.75
20	100.00	0.58	99.95	2.33	99.83	4.07	99.66	5.80
22	100.00	0.64	99.94	2.38	99.83	4.13	99.66	5.86
24	100.00	0.70	99.94	2.44	99.82	4.18	99.65	5.92
26	99.99	0.76	99.94	2.50	99.82	4.24	99.64	5.98
28	99.99	0.81	99.93	2.56	99.81	4.30	99.63	6.04
30	99.99	0.87	99.93	2.62	99.81	4.36	99.63	6.09
32	99.99	0.93	99.93	2.67	99.80	4.42	99.62	6.15
34	99.99	0.99	99.93	2.73	99.80	4.48	99.62	6.21
36	99.99	1.05	99.92	2.79	99.79	4.53	99.61	6.27
38	99.99	1.11	99.92	2.85	99.79	4.59	99.60	6.33
40	99.99	1.16	99.92	2.91	99.78	4.65	99.59	6.38
42	99.99	1.22	99.91	2.97	99.78	4.71	99.59	6.44
44	99.98	1.28	99.91	3.02	99.77	4.76	99.58	6.50
46	99.98	1.34	99.90	3.08	99.77	4.82	99.57	6.56
48	99.98	1.40	99.90	3.14	99.76	4.88	99.56	6.61
50	99.98	1.45	99.90	3.20	99.76	4.94	99.56	6.67
52	99.98	1.51	99.89	3.26	99.75	4.99	99.55	6.73
54	99.98	1.57	99.89	3.31	99.74	5.05	99.54	6.78
56	99.97	1.63	99.89	3.37	99.74	5.11	99.53	6.84
58	99.97	1.69	99.88	3.43	99.73	5.17	99.52	6.90
60	99.97	1.74	99.88	3.49	99.73	5.23	99.51	6.96
c = 0.75	0.75	0.01	0.75	0.02	0.75	0.03	0.75	0.05
c = 1.00	1.00	0.01	1.00	0.03	1.00	0.04	1.00	0.06
c = 1.25	1.25	0.02	1.25	0.03	1.25	0.05	1.25	0.08

[1] From "Theory and Practice of Surveying," by Prof. J. B. Johnson, New York; John Wiley & Sons.

TABLE IX.—HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS.—*Continued*

	4°		5°		6°		7°	
Minutes	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	99.51	6.96	99.24	8.68	98.91	10.40	98.51	12.10
2	99.51	7.02	99.23	8.74	98.90	10.45	98.50	12.15
4	99.50	7.07	99.22	8.80	98.88	10.51	98.48	12.21
6	99.49	7.13	99.21	8.85	98.87	10.57	98.47	12.26
8	99.48	7.19	99.20	8.91	98.86	10.62	98.46	12.32
10	99.47	7.25	99.19	8.97	98.85	10.68	98.44	12.38
12	99.46	7.30	99.18	9.03	98.83	10.74	98.43	12.43
14	99.46	7.36	99.17	9.08	98.82	10.79	98.41	12.49
16	99.45	7.42	99.16	9.14	98.81	10.85	98.40	12.55
18	99.44	7.48	99.15	9.20	98.80	10.91	98.39	12.60
20	99.43	7.53	99.14	9.25	98.78	10.96	98.37	12.66
22	99.42	7.59	99.13	9.31	98.77	11.02	98.36	12.72
24	99.41	7.65	99.11	9.37	98.76	11.08	98.34	12.77
26	99.40	7.71	99.10	9.43	98.74	11.13	98.33	12.83
28	99.39	7.76	99.09	9.48	98.73	11.19	98.31	12.88
30	99.38	7.82	99.08	9.54	98.72	11.25	98.29	12.94
32	99.38	7.88	99.07	9.60	98.71	11.30	98.28	13.00
34	99.37	7.94	99.06	9.65	98.69	11.36	98.27	13.05
36	99.36	7.99	99.05	9.71	98.68	11.42	98.25	13.11
38	99.35	8.05	99.04	9.77	98.67	11.47	98.24	13.17
40	99.34	8.11	99.03	9.83	98.65	11.53	98.22	13.22
42	99.33	8.17	99.01	9.88	98.64	11.59	98.20	13.28
44	99.32	8.22	99.00	9.94	98.63	11.64	98.19	13.33
46	99.31	8.28	98.99	10.00	98.61	11.70	98.17	13.39
48	99.30	8.34	98.98	10.05	98.60	11.76	98.16	13.45
50	99.29	8.40	98.97	10.11	98.58	11.81	98.14	13.50
52	99.28	8.45	98.96	10.17	98.57	11.87	98.13	13.56
54	99.27	8.51	98.94	10.22	98.56	11.93	98.11	13.61
56	99.26	8.57	98.93	10.28	98.54	11.98	98.10	13.67
58	99.25	8.63	98.92	10.34	98.53	12.04	98.08	13.73
60	99.24	8.68	98.91	10.40	98.51	12.10	98.06	13.78
c = 0.75	0.75	0.06	0.75	0.07	0.75	0.08	0.74	0.10
c = 1.00	1.00	0.08	0.99	0.09	0.99	0.11	0.99	0.13
c = 1.25	1.25	0.10	1.24	0.11	1.24	0.14	1.24	0.16

TABLE IX.—HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Continued*

	8°		9°		10°		11°	
Minutes	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	98.06	13.78	97.55	15.45	96.98	17.10	96.36	18.73
2	98.05	13.84	97.53	15.51	96.96	17.16	96.34	18.78
4	98.03	13.89	97.52	15.56	96.94	17.21	96.32	18.84
6	98.01	13.95	97.50	15.62	96.92	17.26	96.29	18.89
8	98.00	14.01	97.48	15.67	96.90	17.32	96.27	18.95
10	97.98	14.06	97.46	15.73	96.88	17.37	96.25	19.00
12	97.97	14.12	97.44	15.78	96.86	17.43	96.23	19.05
14	97.95	14.17	97.43	15.84	96.84	17.48	96.21	19.11
16	97.93	14.23	97.41	15.89	96.82	17.54	96.18	19.16
18	97.92	14.28	97.39	15.95	96.80	17.59	96.16	19.21
20	97.90	14.34	97.37	16.00	96.78	17.65	96.14	19.27
22	97.88	14.40	97.35	16.06	96.76	17.70	96.12	19.32
24	97.87	14.45	97.33	16.11	96.74	17.76	96.09	19.38
26	97.85	14.51	97.31	16.17	96.72	17.81	96.07	19.43
28	97.83	14.56	97.29	16.22	96.70	17.86	96.05	19.48
30	97.82	14.62	97.28	16.28	96.68	17.92	96.03	19.54
32	97.80	14.67	97.26	16.33	96.66	17.97	96.00	19.59
34	97.78	14.73	97.24	16.39	96.64	18.03	95.98	19.64
36	97.76	14.79	97.22	16.44	96.62	18.08	95.96	19.70
38	97.75	14.84	97.20	16.50	96.60	18.14	95.93	19.75
40	97.73	14.90	97.18	16.55	96.57	18.19	95.91	19.80
42	97.71	14.95	97.16	16.61	96.55	18.24	95.89	19.86
44	97.69	15.01	97.14	16.66	96.53	18.30	95.86	19.91
46	97.68	15.06	97.12	16.72	96.51	18.35	95.84	19.96
48	97.66	15.12	97.10	16.77	96.49	18.41	95.82	20.02
50	97.64	15.17	97.08	16.83	96.47	18.46	95.79	20.07
52	97.62	15.23	97.06	16.88	96.45	18.51	95.77	20.12
54	97.61	15.28	97.04	16.94	96.42	18.57	95.75	20.18
56	97.59	15.34	97.02	16.99	96.40	18.62	95.72	20.23
58	97.57	15.40	97.00	17.05	96.38	18.68	95.70	20.28
60	97.55	15.45	96.98	17.10	96.36	18.73	95.68	20.34
c = 0.75.	0.74	0.11	0.74	0.12	0.74	0.14	0.73	0.15
c = 1.00.	0.99	0.15	0.99	0.16	0.98	0.18	0.98	0.20
c = 1.25.	1.23	0.18	1.23	0.21	1.23	0.23	1.22	0.25

TABLE IX.—HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Continued*

	12°		13°		14°		15°	
Minutes	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	95.68	20.34	94.94	21.92	94.15	23.47	93.30	25.00
2	95.65	20.39	94.91	21.97	94.12	23.52	93.27	25.05
4	95.63	20.44	94.89	22.02	94.09	23.58	93.24	25.10
6	95.61	20.50	94.86	22.08	94.07	23.63	93.21	25.15
8	95.58	20.55	94.84	22.13	94.04	23.68	93.18	25.20
10	95.56	20.60	94.81	22.18	94.01	23.73	93.16	25.25
12	95.53	20.66	94.79	22.23	93.98	23.78	93.13	25.30
14	95.51	20.71	94.76	22.28	93.95	23.83	93.10	25.35
16	95.49	20.76	94.73	22.34	93.93	23.88	93.07	25.40
18	95.46	20.81	94.71	22.39	93.90	23.93	93.04	25.45
20	95.44	20.87	94.68	22.44	93.87	23.99	93.01	25.50
22	95.41	20.92	94.66	22.49	93.84	24.04	92.98	25.55
24	95.39	20.97	94.63	22.54	93.81	24.09	92.95	25.60
26	95.36	21.03	94.60	22.60	93.79	24.14	92.92	25.65
28	95.34	21.08	94.58	22.65	93.76	24.19	92.89	25.70
30	95.32	21.13	94.55	22.70	93.73	24.24	92.86	25.75
32	95.29	21.18	94.52	22.75	93.70	24.29	92.83	25.80
34	95.27	21.24	94.50	22.80	93.67	24.34	92.80	25.85
36	95.24	21.29	94.47	22.85	93.65	24.39	92.77	25.90
38	95.22	21.34	94.44	22.91	93.62	24.44	92.74	25.95
40	95.19	21.39	94.42	22.96	93.59	24.49	92.71	26.00
42	95.17	21.45	94.39	23.01	93.56	24.55	92.68	26.05
44	95.14	21.50	94.36	23.06	93.53	24.60	92.65	26.10
46	95.12	21.55	94.34	23.11	93.50	24.65	92.62	26.15
48	95.09	21.60	94.31	23.16	93.47	24.70	92.59	26.20
50	95.07	21.66	94.28	23.22	93.45	24.75	92.56	26.25
52	95.04	21.71	94.26	23.27	93.42	24.80	92.53	26.30
54	95.02	21.76	94.23	23.32	93.39	24.85	92.49	26.35
56	94.99	21.81	94.20	23.37	93.36	24.90	92.46	26.40
58	94.97	21.87	94.17	23.42	93.33	24.95	92.43	26.45
60	94.94	21.92	94.15	23.47	93.30	25.00	92.40	26.50
c = 0.75	0.73	0.16	0.73	0.17	0.73	0.19	0.72	0.20
c = 1.00	0.98	0.22	0.97	0.23	0.97	0.25	0.96	0.27
c = 1.25	1.22	0.27	1.21	0.29	1.21	0.31	1.20	0.34

TABLE IX.—HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Continued*

	16°		17°		18°		19°	
Minutes	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	92.40	26.50	91.45	27.96	90.45	29.39	89.40	30.78
2	92.37	26.55	91.42	28.01	90.42	29.44	89.36	30.83
4	92.34	26.59	91.39	28.06	90.38	29.48	89.33	30.87
6	92.31	26.64	91.35	28.10	90.35	29.53	89.29	30.92
8	92.28	26.69	91.32	28.15	90.31	29.58	89.26	30.97
10	92.25	26.74	91.29	28.20	90.28	29.62	89.22	31.01
12	92.22	26.79	91.26	28.25	90.24	29.67	89.18	31.06
14	92.19	26.84	91.22	28.30	90.21	29.72	89.15	31.10
16	92.15	26.89	91.19	28.34	90.18	29.76	89.11	31.15
18	92.12	26.94	91.16	28.39	90.14	29.81	89.08	31.19
20	92.09	26.99	91.12	28.44	90.11	29.86	89.04	31.24
22	92.06	27.04	91.09	28.49	90.07	29.90	89.00	31.28
24	92.03	27.09	91.06	28.54	90.04	29.95	88.96	31.33
26	92.00	27.13	91.02	28.58	90.00	30.00	88.93	31.38
28	91.97	27.18	90.99	28.63	89.97	30.04	88.89	31.42
30	91.93	27.23	90.96	28.68	89.93	30.09	88.86	31.47
32	91.90	27.28	90.92	28.73	89.90	30.14	88.82	31.51
34	91.87	27.33	90.89	28.77	89.86	30.19	88.78	31.56
36	91.84	27.38	90.86	28.82	89.83	30.23	88.75	31.60
38	91.81	27.43	90.82	28.87	89.79	30.28	88.71	31.65
40	91.77	27.48	90.79	28.92	89.76	30.32	88.67	31.69
42	91.74	27.52	90.76	28.96	89.72	30.37	88.64	31.74
44	91.71	27.57	90.72	29.01	89.69	30.41	88.60	31.78
46	91.68	27.62	90.69	29.06	89.65	30.46	88.56	31.83
48	91.65	27.67	90.66	29.11	89.61	30.51	88.53	31.87
50	91.61	27.72	90.62	29.15	89.58	30.55	88.49	31.92
52	91.58	27.77	90.59	29.20	89.54	30.60	88.45	31.96
54	91.55	27.81	90.55	29.25	89.51	30.65	88.41	32.01
56	91.52	27.86	90.52	29.30	89.47	30.69	88.38	32.05
58	91.48	27.91	90.48	29.34	89.44	30.74	88.34	32.09
60	91.45	27.96	90.45	29.39	89.40	30.78	88.30	32.14
c = 0.75.	0.72	0.21	0.72	0.23	0.71	0.24	0.71	0.25
c = 1.00.	0.96	0.28	0.95	0.30	0.95	0.32	0.94	0.33
c = 1.25.	1.20	0.35	1.19	0.38	1.19	0.40	1.18	0.42

TABLE IX.—HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Continued*

	20°		21°		22°		23°	
Minutes.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	88.30	32.14	87.16	33.46	85.97	34.73	84.73	35.97
2	88.26	32.18	87.12	33.50	85.93	34.77	84.69	36.01
4	88.23	32.23	87.08	33.54	85.89	34.82	84.65	36.05
6	88.19	32.27	87.04	33.59	85.85	34.86	84.61	36.09
8	88.15	32.32	87.00	33.63	85.80	34.90	84.57	36.13
10	88.11	32.36	86.96	33.67	85.76	34.94	84.52	36.17
12	88.08	32.41	86.92	33.72	85.72	34.98	84.48	36.21
14	88.04	32.45	86.88	33.76	85.68	35.02	84.44	36.25
16	88.00	32.49	86.84	33.80	85.64	35.07	84.40	36.29
18	87.96	32.54	86.80	33.84	85.60	35.11	84.35	36.33
20	87.93	32.58	86.77	33.89	85.56	35.15	84.31	36.37
22	87.89	32.63	86.73	33.93	85.52	35.19	84.27	36.41
24	87.85	32.67	86.69	33.97	85.48	35.23	84.23	36.45
26	87.81	32.72	86.65	34.01	85.44	35.27	84.18	36.49
28	87.77	32.76	86.61	34.06	85.40	35.31	84.14	36.53
30	87.74	32.80	86.57	34.10	85.36	35.36	84.10	36.57
32	87.70	32.85	86.53	34.14	85.31	35.40	84.06	36.61
34	87.66	32.89	86.49	34.18	85.27	35.44	84.01	36.65
36	87.62	32.93	86.45	34.23	85.23	35.48	83.97	36.69
38	87.58	32.98	86.41	34.27	85.19	35.52	83.93	36.73
40	87.54	33.02	86.37	34.31	85.15	35.56	83.89	36.77
42	87.51	33.07	86.33	34.35	85.11	35.60	83.84	36.80
44	87.47	33.11	86.29	34.40	85.07	35.64	83.80	36.84
46	87.43	33.15	86.25	34.44	85.02	35.68	83.76	36.88
48	87.39	33.20	86.21	34.48	84.98	35.72	83.72	36.92
50	87.35	33.24	86.17	34.52	84.94	35.76	83.67	36.96
52	87.31	33.28	86.13	34.57	84.90	35.80	83.63	37.00
54	87.27	33.33	86.09	34.61	84.86	35.85	83.59	37.04
56	87.24	33.37	86.05	34.65	84.82	35.89	83.54	37.08
58	87.20	33.41	86.01	34.69	84.77	35.93	83.50	37.12
60	87.16	33.46	85.97	34.73	84.73	35.97	83.46	37.16
c = 0.75.	0.70	0.26	0.70	0.27	0.69	0.29	0.69	0.30
c = 1.00.	0.94	0.35	0.93	0.37	0.92	0.38	0.92	0.40
c = 1.25.	1.17	0.44	1.16	0.46	1.15	0.48	1.15	0.50

TABLE IX.—HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Continued*

	24°		25°		26°		27°	
Minutes	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff Elev.	Hor. Dist.	Diff. Elev.
0	83.46	37.16	82.14	38.30	80.78	39.40	79.39	40.45
2	83.41	37.20	82.09	38.34	80.74	39.44	79.34	40.49
4	83.37	37.23	82.05	38.38	80.69	39.47	79.30	40.52
6	83.33	37.27	82.01	38.41	80.65	39.51	79.25	40.55
8	83.28	37.31	81.96	38.45	80.60	39.54	79.20	40.59
10	83.24	37.35	81.92	38.49	80.55	39.58	79.15	40.62
12	83.20	37.39	81.87	38.53	80.51	39.61	79.11	40.66
14	83.15	37.43	81.83	38.56	80.46	39.65	79.06	40.69
16	83.11	37.47	81.78	38.60	80.41	39.69	79.01	40.72
18	83.07	37.51	81.74	38.64	80.37	39.72	78.96	40.76
20	83.02	37.54	81.69	38.67	80.32	39.76	78.92	40.79
22	82.98	37.58	81.65	38.71	80.28	39.79	78.87	40.82
24	82.93	37.62	81.60	38.75	80.23	39.83	78.82	40.86
26	82.89	37.66	81.56	38.78	80.18	39.86	78.77	40.89
28	82.85	37.70	81.51	38.82	80.14	39.90	78.73	40.92
30	82.80	37.74	81.47	38.86	80.09	39.93	78.68	40.96
32	82.76	37.77	81.42	38.89	80.04	39.97	78.63	40.99
34	82.72	37.81	81.38	38.93	80.00	40.00	78.58	41.02
36	82.67	37.85	81.33	38.97	79.95	40.04	78.54	41.06
38	82.63	37.89	81.28	39.00	79.90	40.07	78.49	41.09
40	82.58	37.93	81.24	39.04	79.86	40.11	78.44	41.12
42	82.54	37.96	81.19	39.08	79.81	40.14	78.39	41.16
44	82.49	38.00	81.15	39.11	79.76	40.18	78.34	41.19
46	82.45	38.04	81.10	39.15	79.72	40.21	78.30	41.22
48	82.41	38.08	81.06	39.18	79.67	40.24	78.25	41.26
50	82.36	38.11	81.01	39.22	79.62	40.28	78.20	41.29
52	82.32	38.15	80.97	39.26	79.58	40.31	78.15	41.32
54	82.27	38.19	80.92	39.29	79.53	40.35	78.10	41.35
56	82.23	38.23	80.87	39.33	79.48	40.38	78.06	41.39
58	82.18	38.26	80.83	39.36	79.44	40.42	78.01	41.42
60	82.14	38.30	80.78	39.40	79.39	40.45	77.96	41.45
c = 0.75 .	0.68	0.31	0.68	0.32	0.67	0.33	0.66	0.35
c = 1.00 .	0.91	0.41	0.90	0.43	0.89	0.45	0.89	0.46
c = 1.25 .	1.14	0.52	1.13	0.54	1.12	0.56	1.11	0.58

TABLE IX.—HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. — *Concluded*

	28°		29°		30°	
Minutes	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	77.96	41.45	76.50	42.40	75.00	43.30
2	77.91	41.48	76.45	42.43	74.95	43.33
4	77.86	41.52	76.40	42.46	74.90	43.36
6	77.81	41.55	76.35	42.49	74.85	43.39
8	77.77	41.58	76.30	42.53	74.80	43.42
10	77.72	41.61	76.25	42.56	74.75	43.45
12	77.67	41.65	76.20	42.59	74.70	43.47
14	77.62	41.68	76.15	42.62	74.65	43.50
16	77.57	41.71	76.10	42.65	74.60	43.53
18	77.52	41.74	76.05	42.68	74.55	43.56
20	77.48	41.77	76.00	42.71	74.49	43.59
22	77.42	41.81	75.95	42.74	74.44	43.62
24	77.38	41.84	75.90	42.77	74.39	43.65
26	77.33	41.87	75.85	42.80	74.34	43.67
28	77.28	41.90	75.80	42.83	74.29	43.70
30	77.23	41.93	75.75	42.86	74.24	43.73
32	77.18	41.97	75.70	42.89	74.19	43.76
34	77.13	42.00	75.65	42.92	74.14	43.79
36	77.09	42.03	75.60	42.95	74.09	43.82
38	77.04	42.06	75.55	42.98	74.04	43.84
40	76.99	42.09	75.50	43.01	73.99	43.87
42	76.94	42.12	75.45	43.04	73.93	43.90
44	76.89	42.15	75.40	43.07	73.88	43.93
46	76.84	42.19	75.35	43.10	73.83	43.95
48	76.79	42.22	75.30	43.13	73.78	43.98
50	76.74	42.25	75.25	43.16	73.73	44.01
52	76.69	42.28	75.20	43.18	73.68	44.04
54	76.64	42.31	75.15	43.21	73.63	44.07
56	76.59	42.34	75.10	43.24	73.58	44.09
58	76.55	42.37	75.05	43.27	73.52	44.12
60	76.50	42.40	75.00	43.30	73.47	44.15
c = 0.75..	0.66	0.36	0.65	0.37	0.65	0.38
c = 1.00..	0.88	0.48	0.87	0.49	0.86	0.51
c = 1.25..	1.10	0.60	1.09	0.62	1.08	0.64

TABLE X.*—MINUTES IN DECIMALS OF A DEGREE

′	0″	10″	15″	20″	30″	40″	45″	50″	′
0	.00000	.00278	.00417	.00556	.00833	.01111	.01250	.01389	0
1	.01667	.01944	.02083	.02222	.02500	.02778	.02917	.03055	1
2	.03333	.03611	.03750	.03889	.04167	.04444	.04583	.04722	2
3	.05000	.05278	.05417	.05556	.05833	.06111	.06250	.06389	3
4	.06667	.06944	.07083	.07222	.07500	.07778	.07917	.08056	4
5	.08333	.08611	.08750	.08889	.09167	.09444	.09583	.09722	5
6	.10000	.10278	.10417	.10556	.10833	.11111	.11250	.11389	6
7	.11667	.11944	.12083	.12222	.12500	.12778	.12917	.13056	7
8	.13333	.13611	.13750	.13889	.14167	.14444	.14583	.14722	8
9	.15000	.15278	.15417	.15556	.15833	.16111	.16250	.16389	9
10	.16667	.16944	.17083	.17222	.17500	.17778	.17917	.18056	10
11	.18333	.18611	.18750	.18889	.19167	.19444	.19583	.19722	11
12	.20000	.20278	.20417	.20556	.20833	.21111	.21250	.21389	12
13	.21667	.21944	.22083	.22222	.22500	.22778	.22917	.23056	13
14	.23333	.23611	.23750	.23889	.24167	.24444	.24583	.24722	14
15	.25000	.25278	.25417	.25556	.25833	.26111	.26250	.26389	15
16	.26667	.26944	.27083	.27222	.27500	.27778	.27917	.28056	16
17	.28333	.28611	.28750	.28889	.29167	.29444	.29583	.29722	17
18	.30000	.30278	.30417	.30556	.30833	.31111	.31250	.31389	18
19	.31667	.31944	.32083	.32222	.32500	.32778	.32917	.33056	19
20	.33333	.33611	.33750	.33889	.34167	.34444	.34583	.34722	20
21	.35000	.35278	.35417	.35556	.35833	.36111	.36250	.36389	21
22	.36667	.36944	.37083	.37222	.37500	.37778	.37917	.38056	22
23	.38333	.38611	.38750	.38889	.39167	.39444	.39583	.39722	23
24	.40000	.40278	.40417	.40556	.40833	.41111	.41250	.41389	24
25	.41667	.41944	.42083	.42222	.42500	.42778	.42917	.43056	25
26	.43333	.43611	.43750	.43889	.44167	.44444	.44583	.44722	26
27	.45000	.45278	.45417	.45556	.45833	.46111	.46250	.46389	27
28	.46667	.46944	.47083	.47222	.47500	.47778	.47917	.48056	28
29	.48333	.48611	.48750	.48889	.49167	.49444	.49583	.49722	29
30	.50000	.50278	.50417	.50556	.50833	.51111	.51250	.51389	30
31	.51667	.51944	.52083	.52222	.52500	.52778	.52917	.53056	31
32	.53333	.53611	.53750	.53889	.54167	.54444	.54583	.54722	32
33	.55000	.55278	.55417	.55556	.55833	.56111	.56250	.56389	33
34	.56667	.56944	.57083	.57222	.57500	.57778	.57917	.58056	34
35	.58333	.58611	.58750	.58889	.59167	.59444	.59583	.59722	35
36	.60000	.60278	.60417	.60556	.60833	.61111	.61250	.61389	36
37	.61667	.61944	.62083	.62222	.62500	.62778	.62917	.63056	37
38	.63333	.63611	.63750	.63889	.64167	.64444	.64583	.64722	38
39	.65000	.65278	.65417	.65556	.65833	.66111	.66250	.66389	39
40	.66667	.66944	.67083	.67222	.67500	.67778	.67917	.68056	40
41	.68333	.68611	.68750	.68889	.69167	.69444	.69583	.69722	41
42	.70000	.70278	.70417	.70556	.70833	.71111	.71250	.71389	42
43	.71667	.71944	.72083	.72222	.72500	.72778	.72917	.73056	43
44	.73333	.73611	.73750	.73889	.74167	.74444	.74583	.74722	44
45	.75000	.75278	.75417	.75556	.75833	.76111	.76250	.76389	45
46	.76667	.76944	.77083	.77222	.77500	.77778	.77917	.78056	46
47	.78333	.78611	.78750	.78889	.79167	.79444	.79583	.79722	47
48	.80000	.80278	.80417	.80556	.80833	.81111	.81250	.81389	48
49	.81667	.81944	.82083	.82222	.82500	.82778	.82917	.83056	49
50	.83333	.83611	.83750	.83889	.84167	.84444	.84583	.84722	50
51	.85000	.85278	.85417	.85556	.85833	.86111	.86250	.86389	51
52	.86667	.86944	.87083	.87222	.87500	.87778	.87917	.88056	52
53	.88333	.88611	.88750	.88889	.89167	.89444	.89583	.89722	53
54	.90000	.90278	.90417	.90556	.90833	.91111	.91250	.91389	54
55	.91667	.91944	.92083	.92222	.92500	.92778	.92917	.93056	55
56	.93333	.93611	.93750	.93889	.94167	.94444	.94583	.94722	56
57	.95000	.95278	.95417	.95556	.95833	.96111	.96250	.96389	57
58	.96667	.96944	.97083	.97222	.97500	.97778	.97917	.98056	58
59	.98333	.98611	.98750	.98889	.99167	.99444	.99583	.99722	59
′	0″	10″	15″	20″	30″	40″	45″	50″	′

* From "Field Engineering" by Searles and Ives. By permission of the publishers, John Wiley & Sons, Inc., New York.

TABLE XI.—CONVERGENCY OF MERIDIANS, SIX MILES LONG AND SIX MILES APART, AND DIFFERENCES OF LATITUDE AND LONGITUDE

Lat.	Convergency		Difference of longitude per range		Difference of latitude for—	
	On the parallel	Angle	In arc	In time	1 mi.	1 Tp.
°	*Lks.*	′ ″	′ ″	*Seconds*	′	′
25	33.9	2 25	5 44.34	22.96		
26	35.4	2 32	5 47.20	23.15		
27	37.0	2 39	5 50 22	23.35	0.871	5.229
28	38.6	2 46	5 53.40	23.56		
29	40.2	2 53	5 56.74	23.78		
30	41.9	3 0	6 0.26	24.02		
31	43.6	3 7	6 3.97	24.26		
32	45.4	3 15	6 7.87	24.52	0.871	5.225
33	47.2	3 23	6 11.96	24.80		
34	49.1	3 30	6 16.26	25.08		
35	50.9	3 38	6 20.78	25.39		
36	52.7	3 46	6 25.53	25.70		
37	54.7	3 55	6 30.52	26.03	0.870	5.221
38	56.8	4 4	6 35.76	26.38		
39	58.8	4 13	6 41.27	26.75		
40	60.9	4 22	6 47.06	27.14		
41	63.1	4 31	6 53.15	27.54		
42	65.4	4 41	6 59.56	27.97	0.869	5.216
43	67.7	4 51	7 6.29	28.42		
44	70.1	5 1	7 13.39	28.89		
45	72.6	5 12	7 20.86	29.39		
46	75.2	5 23	7 28.74	29.92		
47	77.8	5 34	7 37.04	30.47	0.869	5.211
48	80.6	5 46	7 45.80	31.05		
49	83.5	5 59	7 55.05	31.67		
50	86.4	6 12	8 4.83	32.32		
51	89.6	6 25	8 15.17	33.03		
52	92.8	6 39	8 26.13	33.74	0.868	5.207
53	96.2	6 54	8 37.75	34.52		
54	99.8	7 9	8 50.07	35.34		
55	103.5	7 25	9 3.18	36.22		
56	107.5	7 42	9 17.12	37.14		
57	111.6	8 0	9 31.97	38.13	0.867	5.202
58	116.0	8 19	9 47.83	39.19		
59	120.6	8 38	10 4.78	40.32		
60	125.5	8 59	10 22.94	41.52		
61	130.8	9 22	10 42.42	42.83		
62	136.3	9 46	11 3.38	44.22	0.866	5.198
63	142.2	10 11	11 25.97	45.73		
64	148.6	10 38	11 50.37	47.36		
65	155.0	11 8	12 16.82	49.12		
66	162.8	11 39	12 45.55	51.04		
67	170.7	12 13	13 16.88	53.12	0.866	5.195
68	179.3	12 51	13 51.15	55.41		
69	188.7	13 31	14 28.77	57.92		
70	199.1	14 15	15 10.26	60.68	0.866	5.193

TABLE XII.—AZIMUTHS OF THE SECANT

Lat.	0 mi.	1 mi.	2 mi.	3 mi.	Deflection angle 6 mi.
°	° ′	° ′	° ′		′ ″
25	89 58.8	89 59.2	89 59.6	90°	2 25
26	58.7	59.2	59.6	E or W.	2 32
27	58.7	59.1	59.6	" " "	2 39
28	58 6	59.1	59.5	" " "	2 46
29	58.6	59.0	59.5	" " "	2 53
30	58.5	59.0	59.5	" " "	3 0
31	58.4	59.0	59.5	" " "	3 7
32	58.4	58.9	59.5	" " "	3 15
33	58.3	58.9	59.4	" " "	3 23
34	58.2	58.8	59.4	" " "	3 30
35	58.2	58.8	59.4	" " "	3 38
36	58.1	58.7	59.4	" " "	3 46
37	58.0	58.7	59.3	" " "	3 55
38	58.0	58.6	59.3	" " "	4 4
39	57.9	58.6	59.3	" " "	4 13
40	57.8	58.5	59.3	" " "	4 22
41	57.7	58.5	59.2	" " "	4 31
42	57.7	58.4	59.2	" " "	4 41
43	57.6	58.4	59 2	" " "	4 51
44	57.5	58.3	59.2	" " "	5 1
45	57.4	58.3	59.1	" " "	5 12
46	57.3	58.2	59.1	" " "	5 23
47	57.2	58.1	59.1	" " "	5 34
48	57.1	58.1	59.0	" " "	5 46
49	57.0	58.0	59.0	" " "	5 59
50	56.9	57.9	59.0	" " "	6 12
51	56.8	57.9	58.9	" " "	6 25
52	56.7	57.8	58.9	" " "	6 39
53	56.6	57.7	58.8	" " "	6 54
54	56.4	57.6	58.8	" " "	7 9
55	56.3	57.5	58.8	" " "	7 25
56	56.2	57.4	58.7	" " "	7 42
57	56.0	57.3	58.7	" " "	8 0
58	55.8	57.2	58.6	" " "	8 19
59	55.7	57.1	58.6	" " "	8 38
60	55.5	57.0	58.5	" " "	8 59
61	55.3	56.9	58.4	" " "	9 22
62	55.1	56.7	58.4	" " "	9 46
63	54.9	56.6	58.3	" " "	10 11
64	54.7	56.5	58.2	" " "	10 38
65	54.4	56.3	58.1	" " "	11 8
66	54.2	56.1	58.1	" " "	11 39
67	53.9	55.9	58.0	" " "	12 13
68	53.6	55.7	57.9	" " "	12 51
69	53.2	55.5	57.8	" " "	13 31
70	89° 52′.9	89° 55′.3	89° 57′.6	" " "	14′ 15″
	6 mi.	5 mi.	4 mi.	3 mi.	

TABLE XIII.—OFFSETS, IN LINKS, FROM THE SECANT TO THE PARALLEL

Lat.	0 mi.	½ mi.	1 mi.	1½ mi.	2 mi.	2½ mi.	3 mi.
°							
25	2 N.	1 N.	0	1 S.	1 S.	2 S.	2 S.
26	2	1	0	1	1	2	2
27	3	1	0	1	2	2	2
28	3	1	0	1	2	2	2
29	3	1	0	1	2	2	2
30	3	1	0	1	2	2	2
31	3	1	0	1	2	2	2
32	3	1	0	1	2	2	3
33	3	1	0	1	2	2	3
34	3	2	0	1	2	3	3
35	4	2	0	1	2	3	3
36	4	2	0	1	2	3	3
37	4	2	0	1	2	3	3
38	4	2	0	1	2	3	3
39	4	2	0	1	2	3	3
40	4	2	0	1	3	3	3
41	4	2	0	2	3	3	4
42	5	2	0	2	3	3	4
43	5	2	0	2	3	4	4
44	5	2	0	2	3	4	4
45	5	2	0	2	3	4	4
46	5	2	0	2	3	4	4
47	5	2	0	2	3	4	4
48	6	3	0	2	3	4	4
49	6	3	0	2	3	4	5
50	6	3	0	2	4	4	5
51	6	3	0	2	4	5	5
52	6	3	0	2	4	5	5
53	7	3	0	2	4	5	5
54	7	3	0	2	4	5	6
55	7	3	0	3	4	5	6
56	7	3	0	3	4	6	6
57	8	3	0	3	5	6	6
58	8	4	0	3	5	6	6
59	8	4	0	3	5	6	7
60	9	4	0	3	5	7	7
61	9	4	0	3	5	7	7
62	9	4	0	3	6	7	8
63	10	4	0	3	6	7	8
64	10	5	0	4	6	8	8
65	11	5	0	4	6	8	9
66	11	5	0	4	7	8	9
67	12	5	0	4	7	9	9
68	12	6	0	4	7	9	10
69	13	6	0	5	8	10	10
70	14 N.	6 N.	0	5 S.	8 S.	10 S.	11 S.
	6 mi.	5½ mi.	5 mi.	4½ mi.	4 mi.	3½ mi.	3 mi.

TABLE XIV.—COEFFICIENTS FOR SHARP-CRESTED RECTANGULAR WEIRS WITH TWO COMPLETE END CONTRACTIONS (SMITH)

Effective head, H, in ft.	Length of weir, B, in feet						
	0.66	1	2	3	5	10	19
0.1	0.632	0.639	0.646	0.652	0.653	0.655	0.656
0.15	.619	.625	.634	.638	.640	.641	.642
0.2	.611	.618	.626	.630	.631	.633	.634
0.25	.605	.612	.621	.624	.626	.628	.629
0.3	.601	.608	.616	.619	.621	.624	.625
0.4	.595	.601	.609	.613	.615	.618	.620
0.5	.590	.596	.605	.608	.611	.615	.617
0.6	.587	.593	.601	.605	.608	.613	.615
0.7	.....	.590	.598	.603	.606	.612	.614
0.8	.....	.....	.595	.600	.604	.611	.613
0.9	.....	.....	.592	.598	.603	.609	.612
1.0	.....	.....	.590	.595	.601	.608	.611
1.2	.....	.....	.585	.591	.597	.605	.610
1.4	.....	.....	.580	.587	.594	.602	.609
1.6	.....	.....	.....	.582	.591	.600	.607

For use in the formula $Q = c_d \frac{2}{3} B \sqrt{2g}(H + nh_0)^{3/2}$.

TABLE XV.—COEFFICIENTS FOR SHARP-CRESTED RECTANGULAR WEIRS WITH BOTH END CONTRACTIONS SUPPRESSED (SMITH)

Effective head, H, in ft.	Length of weir, B, in feet						
	2	3	4	5	7	10	19
0.1	.....	.....	.....	0.659	0.658	0.658	0.657
0.15	0.652	0.649	0.647	.645	.645	.644	.643
0.2	.645	.642	.641	.638	.637	.637	.635
0.25	.641	.638	.636	.634	.633	.632	.630
0.3	.639	.636	.633	.631	.629	.628	.626
0.4	.636	.633	.630	.628	.625	.623	.621
0.5	.637	.633	.630	.627	.624	.621	.619
0.6	.638	.634	.630	.627	.623	.620	.618
0.7	.640	.635	.631	.628	.624	.620	.618
0.8	.643	.637	.633	.629	.625	.621	.618
0.9	.645	.639	.635	.631	.627	.622	.619
1.0	.648	.641	.637	.633	.628	.624	.619
1.2	.....	.646	.641	.636	.632	.626	.620
1.4	.....	.....	.644	.640	.634	.629	.622
1.6	.....	.....	.647	.642	.637	.631	.623

For use in the formula $Q = c_d \frac{2}{3} B \sqrt{2g}(H + nh_0)^{3/2}$.

TABLE XVI.—COEFFICIENTS FOR SHARP-CRESTED RECTANGULAR WEIRS WITH TWO COMPLETE END CONTRACTIONS (SMITH)

Head, H, in ft.	Length of weir, B, in feet										
	0.66	1*	2	2.6	3	4	5	7	10	15	19
0.1	3.381	3.419	3.456	3.478	3.488	3.494	3.494	3.499	3.504	3.504	3.510
.15	3.312	3.344	3.392	3.408	3.413	3.419	3.424	3.424	3.429	3.435	3.435
.2	3.269	3.306	3.349	3.365	3.371	3.376	3.376	3.381	3.386	3.392	3.392
.25	3.237	3.274	3.322	3.333	3.338	3.344	3.349	3.354	3.360	3.360	3.365
.3	3.215	3.253	3.296	3.306	3.312	3.322	3.322	3.333	3.338	3.338	3.344
.4	3.183	3.215	3.258	3.274	3.280	3.285	3.290	3.301	3.306	3.312	3.317
.5	3.156	3.189	3.237	3.247	3.253	3.264	3.269	3.280	3.290	3.295	3.301
.6	3.140	3.172	3.215	3.231	3.237	3.247	3.253	3.269	3.280	3.285	3.290
.7	3.130	3.156	3.199	3.210	3.226	3.231	3.242	3.258	3.274	3.280	3.285
.8	.....	.....	3.183	3.199	3.215	3.221	3.231	3.247	3.269	3.274	3.280
.9	.....	.....	3.167	3.189	3.199	3.210	3.226	3.242	3.258	3.269	3.274
1.0	.....	.....	3.156	3.172	3.183	3.199	3.215	3.231	3.253	3.264	3.269
1.1	.....	.....	3.140	3.162	3.172	3.189	3.205	3.226	3.242	3.258	3.264
1.2	.....	.....	3.130	3.151	3.162	3.178	3.194	3.215	3.237	3.253	3.264
1.3	.....	.....	3.114	3.135	3.151	3.167	3.199	3.205	3 231	3.247	3.258
1.4	.....	.....	3.103	3.124	3.140	3.156	3.178	3.199	3.221	3.242	3.258
1.5	.....	.....	.....	3.114	3.130	3.151	3.167	3.189	3.215	3.237	3.253
1.6	.....	.....	.....	3.103	3.114	3.140	3.162	3.183	3.210	3.231	3.247
1.7	.....	.....	.....	.....	.....	.....	.....	3.178	3.205	3.226	3.247

For use in the formula $Q = c_d BH^{3/2}$.
* Approximate.

TABLE XVII.—COEFFICIENTS FOR SHARP-CRESTED RECTANGULAR WEIRS WITH BOTH END CONTRACTIONS SUPPRESSED (SMITH)

Head, H, in ft.	Length of weir, B, in feet								
	0.66*	2*	3*	4	5	7	10	15	19
0.1	3.611	.....	.....	.....	3.526	3.520	3.520	3.515	3.515
.15	3.542	3.488	3.472	3.461	3.451	3.451	3.445	3.445	3.440
.2	3.510	3.450	3.435	3.429	3.413	3.408	3.408	3.403	3.397
.25	3.494	3.429	3.413	3.403	3.392	3.386	3.381	3.376	3.371
.3	3.483	3.418	3.403	3.386	3.376	3.365	3.360	3.354	3.349
.4	3.478	3.403	3.386	3.371	3.360	3.344	3.333	3.328	3.322
.5	3.478	3.408	3.386	3.371	3.354	3.338	3.322	3.317	3.312
.6	3.483	3.413	3.392	3.371	3.354	3.333	3.317	3.312	3.306
.7	3.494	3.424	3.397	3.376	3.360	3.338	3.317	3.312	3.306
.8	3.510	3.441	3.408	3.386	3.365	3.344	3.322	3.317	3.306
.9	.....	3.451	3.418	3.397	3.375	3.354	3.328	3.317	3.312
1.0	.....	3.467	3.429	3.408	3.386	3.360	3.338	3.322	3.312
1.1	.....	.....	3.445	3.419	3.397	3.371	3.344	3.328	3.317
1.2	.....	.....	3.456	3.429	3.403	3.381	3.349	3.333	3.317
1.3	.....	.....	3.467	3.440	3.413	3.386	3.360	3.338	3.322
1.4	.....	.....	.....	3.445	3.424	3.392	3.365	3.344	3.328
1.5	.....	.....	.....	3.456	3.429	3.403	3.371	3.344	3.328
1.6	.....	.....	.....	3.461	3.435	3.408	3.376	3.349	3.333
1.7	.....	.....	.....	.....	.....	3.413	3.381	3.349	3.333

For use in the formula $Q = c_d BH^{3/2}$.
* Approximate.

No. 100 Log. 000 TABLE XVIII.—LOGARITHMS OF NUMBERS No. 109 Log. 040

N.	0	1	2	3	4	5	6	7	8	9	Diff.
100	00 0000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
2	8600	9026	9451	9876	0300	0724	1147	1570	1993	2415	424
3	01 2837	3259	3680	4100	4521	4940	5360	5779	6197	6616	420
4	7033	7451	7868	8284	8700	9116	9532	9947	0361	0775	416
105	02 1189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
7	9384	9789	0195	0600	1004	1408	1812	2216	2619	3021	404
8	03 3424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
9	7426	7825	8223	8620	9017	9414	9811	0207	0602	0998	397
	04										

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
434	43.4	86.8	130.2	173.6	217.0	260.4	303.8	347.2	390.6
433	43.3	86.6	129.9	173.2	216.5	259.8	303.1	346.4	389.7
432	43.2	86.4	129.6	172.8	216.0	259.2	302.4	345.6	388.8
431	43.1	86.2	129.3	172.4	215.5	258.6	301.7	344.8	387.9
430	43.0	86.0	129.0	172.0	215.0	258.0	301.0	344.0	387.0
429	42.9	85.8	128.7	171.6	214.5	257.4	300.3	343.2	386.1
428	42.8	85.6	128.4	171.2	214.0	256.8	299.6	342.4	385.2
427	42.7	85.4	128.1	170.8	213.5	256.2	298.9	341.6	384.3
426	42.6	85.2	127.8	170.4	213.0	255.6	298.2	340.8	383.4
425	42.5	85.0	127.5	170.0	212.5	255.0	297.5	340.0	382.5
424	42.4	84.8	127.2	169.6	212.0	254.4	296.8	339.2	381.6
423	42.3	84.6	126.9	169.2	211.5	253.8	296.1	338.4	380.7
422	42.2	84.4	126.6	168.8	211.0	253.2	295.4	337.6	379.8
421	42.1	84.2	126.3	168.4	210.5	252.6	294.7	336.8	378.9
420	42.0	84.0	126.0	168.0	210.0	252.0	294.0	336.0	378.0
419	41.9	83.8	125.7	167.6	209.5	251.4	293.3	335.2	377.1
418	41.8	83.6	125.4	167.2	209.0	250.8	292.6	334.4	376.2
417	41.7	83.4	125.1	166.8	208.5	250.2	291.9	333.6	375.3
416	41.6	83.2	124.8	166.4	208.0	249.6	291.2	332.8	374.4
415	41.5	83.0	124.5	166.0	207.5	249.0	290.5	332.0	373.5
414	41.4	82.8	124.2	165.6	207.0	248.4	289.8	331.2	372.6
413	41.3	82.6	123.9	165.2	206.5	247.8	289.1	330.4	371.7
412	41.2	82.4	123.6	164.8	206.0	247.2	288.4	329.6	370.8
411	41.1	82.2	123.3	164.4	205.5	246.6	287.7	328.8	369.9
410	41.0	82.0	123.0	164.0	205.0	246.0	287.0	328.0	369.0
409	40.9	81.8	122.7	163.6	204.5	245.4	286.3	327.2	368.1
408	40.8	81.6	122.4	163.2	204.0	244.8	285.6	326.4	367.2
407	40.7	81.4	122.1	162.8	203.5	244.2	284.9	325.6	366.3
406	40.6	81.2	121.8	162.4	203.0	243.6	284.2	324.8	365.4
405	40.5	81.0	121.5	162.0	202.5	243.0	283.5	324.0	364.5
404	40.4	80.8	121.2	161.6	202.0	242.4	282.8	323.2	363.6
403	40.3	80.6	120.9	161.2	201.5	241.8	282.1	322.4	362.7
402	40.2	80.4	120.6	160.8	201.0	241.2	281.4	321.6	361.8
401	40.1	80.2	120.3	160.4	200.5	240.6	280.7	320.8	360.9
400	40.0	80.0	120.0	160.0	200.0	240.0	280.0	320.0	360.0
399	39.9	79.8	119.7	159.6	199.5	239.4	279.3	319.2	359.1
398	39.8	79.6	119.4	159.2	199.0	238.8	278.6	318.4	358.2
397	39.7	79.4	119.1	158.8	198.5	238.2	277.9	317.6	357.3
396	39.6	79.2	118.8	158.4	198.0	237.6	277.2	316.8	356.4
395	39.5	79.0	118.5	158.0	197.5	237.0	276.5	316.0	351.5

No. 110 Log. 041 — TABLE XVIII.—*Continued.* — No. 119 Log. 078

N.	0	1	2	3	4	5	6	7	8	9	Diff.
110	04 1393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
2	9218	9606	9993	0380	0766	1153	1538	1924	2309	2694	386
3	05 3078	3463	3846	4230	4613	4996	5378	5760	6142	6524	383
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	0320	379
115	06 0698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
7	8186	8557	8928	9298	9668	0038	0407	0776	1145	1514	370
8	07 1882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
395	39.5	79.0	118.5	158.0	197.5	237.0	276.5	316.0	355.5
394	39.4	78.8	118.2	157.6	197.0	236.4	275.8	315.2	354.6
393	39.3	78.6	117.9	157.2	196.5	235.8	275.1	314.4	353.7
392	39.2	78.4	117.6	156.8	196.0	235.2	274.4	313.6	352.8
391	39.1	78.2	117.3	156.4	195.5	234.6	273.7	312.8	351.9
390	39.0	78.0	117.0	156.0	195.0	234.0	273.0	312.0	351.0
389	38.9	77.8	116.7	155.6	194.5	233.4	272.3	311.2	350.1
388	38.8	77.6	116.4	155.2	194.0	232.8	271.6	310.4	349.2
387	38.7	77.4	116.1	154.8	193.5	232.2	270.9	309.6	348.3
386	38.6	77.2	115.8	154.4	193.0	231.6	270.2	308.8	347.4
385	38.5	77.0	115.5	154.0	192.5	231.0	269.5	308.0	346.5
384	38.4	76.8	115.2	153.6	192.0	230.4	268.8	307.2	345.6
383	38.3	76.6	114.9	153.2	191.5	229.8	268.1	306.4	344.7
382	38.2	76.4	114.6	152.8	191.0	229.2	267.4	305.6	343.8
381	38.1	76.2	114.3	152.4	190.5	228.6	266.7	304.8	342.9
380	38.0	76.0	114.0	152.0	190.0	228.0	266.0	304.0	342.0
379	37.9	75.8	113.7	151.6	189.5	227.4	265.3	303.2	341.1
378	37.8	75.6	113.4	151.2	189.0	226.8	264.6	302.4	340.2
377	37.7	75.4	113.1	150.8	188.5	226.2	263.9	301.6	339.3
376	37.6	75.2	112.8	150.4	188.0	225.6	263.2	300.8	338.4
375	37.5	75.0	112.5	150.0	187.5	225.0	262.5	300.0	337.5
374	37.4	74.8	112.2	149.6	187.0	224.4	261.8	299.2	336.6
373	37.3	74.6	111.9	149.2	186.5	223.8	261.1	298.4	335.7
372	37.2	74.4	111.6	148.8	186.0	223.2	260.4	297.6	334.8
371	37.1	74.2	111.3	148.4	185.5	222.6	259.7	296.8	333.9
370	37.0	74.0	111.0	148.0	185.0	222.0	259.0	296.0	333.0
369	36.9	73.8	110.7	147.6	184.5	221.4	258.3	295.2	332.1
368	36.8	73.6	110.4	147.2	184.0	220.8	257.6	294.4	331.2
367	36.7	73.4	110.1	146.8	183.5	220.2	256.9	293.6	330.3
366	36.6	73.2	109.8	146.4	183.0	219.6	256.2	292.8	329.4
365	36.5	73.0	109.5	146.0	182.5	219.0	255.7	292.0	328.5
364	36.4	72.8	109.2	145.6	182.0	218.4	254.8	291.2	327.6
363	36.3	72.6	108.9	145.2	181.5	217.8	254.1	290.4	326.7
362	36.2	72.4	108.6	144.8	181.0	217.2	253.4	289.6	325.8
361	36.1	72.2	108.3	144.4	180.5	216.6	252.7	288.8	324.9
360	36.0	72.0	108.0	144.0	180.0	216.0	252.0	288.0	324.0
359	35.9	71.8	107.7	143.6	179.5	215.4	251.3	287.2	323.1
358	35.8	71.6	107.4	143.2	179.0	214.8	250.6	286.4	322.2
357	35.7	71.4	107.1	142.8	178.5	214.2	249.9	285.6	321.3
356	35.6	71.2	106.8	142.4	178.0	213.6	249.2	284.8	320.4

No. 120 Log. 079 — Table XVIII.—*Continued.* — No. 134 Log. 130

N.	0	1	2	3	4	5	6	7	8	9	Diff.
120	07 9181	9543	9904	0266	0626	0987	1347	1707	2067	2426	360
1	08 2785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
3	9905	0258	0611	0963	1315	1667	2018	2370	2721	3071	352
4	09 3422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	0026	346
6	10 0371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
8	7210	7549	7888	8227	8565	8903	9241	9579	9916	0253	338
9	11 0590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	0245	330
2	12 0574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	0012	323
	13										

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
355	35.5	71.0	106.5	142.0	177.5	213.0	248.5	284.0	319.5
354	35.4	70.8	106.2	141.6	177.0	212.4	247.8	283.2	318.6
353	35.3	70.6	105.9	141.2	176.5	211.8	247.1	282.4	317.7
352	35.2	70.4	105.6	140.8	176.0	211.2	246.4	281.6	316.8
351	35.1	70.2	105.3	140.4	175.5	210.6	245.7	280.8	315.9
350	35.0	70.0	105.0	140.0	175.0	210.0	245.0	280.0	315.0
349	34.9	69.8	104.7	139.6	174.5	209.4	244.3	279.2	314.1
348	34.8	69.6	104.4	139.2	174.0	208.8	243.6	278.4	313.2
347	34.7	69.4	104.1	138.8	173.5	208.2	242.9	277.6	312.3
346	34.6	69.2	103.8	138.4	173.0	207.6	242.2	276.8	311.4
345	34.5	69.0	103.5	138.0	172.5	207.0	241.5	276.0	310.5
344	34.4	68.8	103.2	137.6	172.0	206.4	240.8	275.2	309.6
343	34.3	68.6	102.9	137.2	171.5	205.8	240.1	274.4	308.7
342	34.2	68.4	102.6	136.8	171.0	205.2	239.4	273.6	307.8
341	34.1	68.2	102.3	136.4	170.5	204.6	238.7	272.8	306.9
340	34.0	68.0	102.0	136.0	170.0	204.0	238.0	272.0	306.0
339	33.9	67.8	101.7	135.6	169.5	203.4	237.3	271.2	305.1
338	33.8	67.6	101.4	135.2	169.0	202.8	236.6	270.4	304.2
337	33.7	67.4	101.1	134.8	168.5	202.2	235.9	269.6	303.3
336	33.6	67.2	100.8	134.4	168.0	201.6	235.2	268.8	302.4
335	33.5	67.0	100.5	134.0	167.5	201.0	234.5	268.0	301.5
334	33.4	66.8	100.2	133.6	167.0	200.4	233.8	267.2	300.6
333	33.3	66.6	99.9	133.2	166.5	199.8	233.1	266.4	299.7
332	33.2	66.4	99.6	132.8	166.0	199.2	232.4	265.6	298.8
331	33.1	66.2	99.3	132.4	165.5	198.6	231.7	264.8	297.9
330	33.0	66.0	99.0	132.0	165.0	198.0	231.0	264.0	297.0
329	32.9	65.8	98.7	131.6	164.5	197.4	230.3	263.2	296.1
328	32.8	65.6	98.4	131.2	164.0	196.8	229.6	262.4	295.2
327	32.7	65.4	98.1	130.8	163.5	196.2	228.9	261.6	294.3
326	32.6	65.2	97.8	130.4	163.0	195.6	228.2	260.8	293.4
325	32.5	65.0	97.5	130.0	162.5	195.0	227.5	260.0	292.5
324	32.4	64.8	97.2	129.6	162.0	194.4	226.8	259.2	291.6
323	32.3	64.6	96.9	129.2	161.5	193.8	226.1	258.4	290.7
322	32.2	64.4	96.6	128.8	161.0	193.2	225.4	257.6	289.8

No. 135 Log. 130 TABLE XVIII.—*Continued.* No. 149 Log. 175

N.	0	1	2	3	4	5	6	7	8	9	Diff.
135	13 0334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
6	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
7	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
8	9879	0194	0508	0822	1136	1450	1763	2076	2389	2702	314
9	14 3015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
1	9219	9527	9835	0142	0449	0756	1063	1370	1676	1982	307
2	15 2288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
3	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
4	8362	8664	8965	9266	9567	9868	0168	0469	0769	1068	301
145	16 1368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
6	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
7	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
8	17 0262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
321	32.1	64.2	96.3	128.4	160.5	192.6	224.7	256.8	288.9
320	32.0	64.0	96.0	128.0	160.0	192.0	224.0	256.0	288.0
319	31.9	63.8	95.7	127.6	159.5	191.4	223.3	255.2	287.1
318	31.8	63.6	95.4	127.2	159.0	190.8	222.6	254.4	286.2
317	31.7	63.4	95.1	126.8	158.5	190.2	221.9	253.6	285.3
316	31.6	63.2	94.8	126.4	158.0	189.6	221.2	252.8	284.4
315	31.5	63.0	94.5	126.0	157.5	189.0	220.5	252.0	283.5
314	31.4	62.8	94.2	125.6	157.0	188.4	219.8	251.2	282.6
313	31.3	62.6	93.9	125.2	156.5	187.8	219.1	250.4	281.7
312	31.2	62.4	93.6	124.8	156.0	187.2	218.4	249.6	280.8
311	31.1	62.2	93.3	124.4	155.5	186.6	217.7	248.8	279.9
310	31.0	62.0	93.0	124.0	155.0	186.0	217.0	248.0	279.0
309	30.9	61.8	92.7	123.6	154.5	185.4	216.3	247.2	278.1
308	30.8	61.6	92.4	123.2	154.0	184.8	215.6	246.4	277.2
307	30.7	61.4	92.1	122.8	153.5	184.2	214.9	245.6	276.3
306	30.6	61.2	91.8	122.4	153.0	183.6	214.2	244.8	275.4
305	30.5	61.0	91.5	122.0	152.5	183.0	213.5	244.0	274.5
304	30.4	60.8	91.2	121.6	152.0	182.4	212.8	243.2	273.6
303	30.3	60.6	90.9	121.2	151.5	181.8	212.1	242.4	272.7
302	30.2	60.4	90.6	120.8	151.0	181.2	211.4	241.6	271.8
301	30.1	60.2	90.3	120.4	150.5	180.6	210.7	240.8	270.9
300	30.0	60.0	90.0	120.0	150.0	180.0	210.0	240.0	270.0
299	29.9	59.8	89.7	119.6	149.5	179.4	209.3	239.2	269.1
298	29.8	59.6	89.4	119.2	149.0	178.8	208.6	238.4	268.2
297	29.7	59.4	89.1	118.8	148.5	178.2	207.9	237.6	267.3
296	29.6	59.2	88.8	118.4	148.0	177.6	207.2	236.8	266.4
295	29.5	59.0	88.5	118.0	147.5	177.0	206.5	236.0	265.5
294	29.4	58.8	88.2	117.6	147.0	176.4	205.8	235.2	264.6
293	29.3	58.6	87.9	117.2	146.5	175.8	205.1	234.4	263.7
292	29.2	58.4	87.6	116.8	146.0	175.2	204.4	233.6	262.8
291	29.1	58.2	87.3	116.4	145.5	174.6	203.7	232.8	261.9
290	29.0	58.0	87.0	116.0	145.0	174.0	203.0	232.0	261.0
289	28.9	57.8	86.7	115.6	144.5	173.4	202.3	231.2	260.1
288	28.8	57.6	86.4	115.2	144.0	172.8	201.6	230.4	259.2
287	28.7	57.4	86.1	114.8	143.5	172.2	200.9	229.6	258.3
286	28.6	57.2	85.8	114.4	143.0	171.6	200.2	228.8	257.4

No. 150 Log. 176 TABLE XVIII.—*Continued.* No. 169 Log. 230

N.	0	1	2	3	4	5	6	7	8	9	Diff.
150	17 6091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
1	8977	9264	9552	9839	0126	0413	0699	0986	1272	1558	287
2	18 1844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
3	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
4	7521	7803	8084	8366	8647	8928	9209	9490	9771	0051	281
155	19 0332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
8	8657	8932	9206	9481	9755	0029	0303	0577	0850	1124	274
9	20 1397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
2	9515	9783	0051	0319	0586	0853	1121	1388	1654	1921	267
3	21 2188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
6	22 0108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
9	7887 23	8144	8400	8657	8913	9170	9426	9682	9938	0193	256

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
285	28.5	57.0	85.5	114.0	142.5	171.0	199.5	228.0	256.5
284	28.4	56.8	85.2	113.6	142.0	170.4	198.8	227.2	255.6
283	28.3	56.6	84.9	113.2	141.5	169.8	198.1	226.4	254.7
282	28.2	56.4	84.6	112.8	141.0	169.2	197.4	225.6	253.8
281	28.1	56.2	84.3	112.4	140.5	168.6	196.7	224.8	252.9
280	28.0	56.0	84.0	112.0	140.0	168.0	196.0	224.0	252.0
279	27.9	55.8	83.7	111.6	139.5	167.4	195.3	223.2	251.1
278	27.8	55.6	83.4	111.2	139.0	166.8	194.6	222.4	250.2
277	27.7	55.4	83.1	110.8	138.5	166.2	193.9	221.6	249.3
276	27.6	55.2	82.8	110.4	138.0	165.6	193.2	220.8	248.4
275	27.5	55.0	82.5	110.0	137.5	165.0	192.5	220.0	247.5
274	27.4	54.8	82.2	109.6	137.0	164.4	191.8	219.2	246.6
273	27.3	54.6	81.9	109.2	136.5	163.8	191.1	218.4	245.7
272	27.2	54.4	81.6	108.8	136.0	163.2	190.4	217.6	244.8
271	27.1	54.2	81.3	108.4	135.5	162.6	189.7	216.8	243.9
270	27.0	54.0	81.0	108.0	135.0	162.0	189.0	216.0	243.0
269	26.9	53.8	80.7	107.6	134.5	161.4	188.3	215.2	242.1
268	26.8	53.6	80.4	107.2	134.0	160.8	187.6	214.4	241.2
267	26.7	53.4	80.1	106.8	133.5	160.2	186.9	213.6	240.3
266	26.6	53.2	79.8	106.4	133.0	159.6	186.2	212.8	239.4
265	26.5	53.0	79.5	106.0	132.5	159.0	185.5	212.0	238.5
264	26.4	52.8	79.2	105.6	132.0	158.4	184.8	211.2	237.6
263	26.3	52.6	78.9	105.2	131.5	157.8	184.1	210.4	236.7
262	26.2	52.4	78.6	104.8	131.0	157.2	183.4	209.6	235.8
261	26.1	52.2	78.3	104.4	130.5	156.6	182.7	208.8	234.9
260	26.0	52.0	78.0	104.0	130.0	156.0	182.0	208.0	234.0
259	25.9	51.8	77.7	103.6	129.5	155.4	181.3	207.2	233.1
258	25.8	51.6	77.4	103.2	129.0	154.8	180.6	206.4	232.2
257	25.7	51.4	77.1	102.8	128.5	154.2	179.9	205.6	231.3
256	25.6	51.2	76.8	102.4	128.0	153.6	179.2	204.8	230.4
255	25.5	51.0	76.5	102.0	127.5	153.0	178.5	204.0	229.5

No. 170 — Log. 230 — TABLE XVIII.—*Continued.* — No. 189 — Log. 278

N.	0	1	2	3	4	5	6	7	8	9	Diff.
170	23 0449	0704	0960	1215	1470	1724	1979	2234	2488	2742	255
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
3	8046	8297	8548	8799	9049	9299	9550	9800	0050	0300	250
4	24 0549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	0176	245
8	25 0420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
2	26 0071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
6	9513	9746	9980	0213	0446	0679	0912	1144	1377	1609	233
7	27 1842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
255	25.5	51.0	76.5	102.0	127.5	153.0	178.5	204.0	229.5
254	25.4	50.8	76.2	101.6	127.0	152.4	177.8	203.2	228.6
253	25.3	50.6	75.9	101.2	126.5	151.8	177.1	202.4	227.7
252	25.2	50.4	75.6	100.8	126.0	151.2	176.4	201.6	226.8
251	25.1	50.2	75.3	100.4	125.5	150.6	175.7	200.8	225.9
250	25.0	50.0	75.0	100.0	125.0	150.0	175.0	200.0	225.0
249	24.9	49.8	74.7	99.6	124.5	149.4	174.3	199.2	224.1
248	24.8	49.6	74.4	99.2	124.0	148.8	173.6	198.4	223.2
247	24.7	49.4	74.1	98.8	123.5	148.2	172.9	197.6	222.3
246	24.6	49.2	73.8	98.4	123.0	147.6	172.2	196.8	221.4
245	24.5	49.0	73.5	98.0	122.5	147.0	171.5	196.0	220.5
244	24.4	48.8	73.2	97.6	122.0	146.4	170.8	195.2	219.6
243	24.3	48.6	72.9	97.2	121.5	145.8	170.1	194.4	218.7
242	24.2	48.4	72.6	96.8	121.0	145.2	169.4	193.6	217.8
241	24.1	48.2	72.3	96.4	120.5	144.6	168.7	192.8	216.9
240	24.0	48.0	72.0	96.0	120.0	144.0	168.0	192.0	216.0
239	23.9	47.8	71.7	95.6	119.5	143.4	167.3	191.2	215.1
238	23.8	47.6	71.4	95.2	119.0	142.8	166.6	190.4	214.2
237	23.7	47.4	71.1	94.8	118.5	142.2	165.9	189.6	213.3
236	23.6	47.2	70.8	94.4	118.0	141.6	165.2	188.8	212.4
235	23.5	47.0	70.5	94.0	117.5	141.0	164.5	188.0	211.5
234	23.4	46.8	70.2	93.6	117.0	140.4	163.8	187.2	210.6
233	23.3	46.6	69.9	93.2	116.5	139.8	163.1	186.4	209.7
232	23.2	46.4	69.6	92.8	116.0	139.2	162.4	185.6	208.8
231	23.1	46.2	69.3	92.4	115.5	138.6	161.7	184.8	207.9
230	23.0	46.0	69.0	92.0	115.0	138.0	161.0	184.0	207.0
229	22.9	45.8	68.7	91.6	114.5	137.4	160.3	183.2	206.1
228	22.8	45.6	68.4	91.2	114.0	136.8	159.6	182.4	205.2
227	22.7	45.4	68.1	90.8	113.5	136.2	158.9	181.6	204.3
226	22.6	45.2	67.8	90.4	113.0	135.6	158.2	180.8	203.4

No. 190 Log. 278 — Table XVIII.—*Continued.* — No. 214 Log. 332

N.	0	1	2	3	4	5	6	7	8	9	Diff.
190	27 8754	8982	9211	9439	9667	9895	0123	0351	0578	0806	228
1	28 1033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	29 0035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
9	8853	9071	9289	9507	9725	9943	0161	0378	0595	0813	218
200	30 1030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
1	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
2	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
3	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
4	9630	9843	0056	0268	0481	0693	0906	1118	1330	1542	212
205	31 1754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
6	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
7	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
8	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
9	32 0146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
1	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
2	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
3	8380	8583	8787	8991	9194	9398	9601	9805	0008	0211	203
4	33 0414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
225	22.5	45.0	67.5	90.0	112.5	135.0	157.5	180.0	202.5
224	22.4	44.8	67.2	89.6	112.0	134.4	156.8	179.2	201.6
223	22.3	44.6	66.9	89.2	111.5	133.8	156.1	178.4	200.7
222	22.2	44.4	66.6	88.8	111.0	133.2	155.4	177.6	199.8
221	22.1	44.2	66.3	88.4	110.5	132.6	154.7	176.8	198.9
220	22.0	44.0	66.0	88.0	110.0	132.0	154.0	176.0	198.0
219	21.9	43.8	65.7	87.6	109.5	131.4	153.3	175.2	197.1
218	21.8	43.6	65.4	87.2	109.0	130.8	152.6	174.4	196.2
217	21.7	43.4	65.1	86.8	108.5	130.2	151.9	173.6	195.3
216	21.6	43.2	64.8	86.4	108.0	129.6	151.2	172.8	194.4
215	21.5	43.0	64.5	86.0	107.5	129.0	150.5	172.0	193.5
214	21.4	42.8	64.2	85.6	107.0	128.4	149.8	171.2	192.6
213	21.3	42.6	63.9	85.2	106.5	127.8	149.1	170.4	191.7
212	21.2	42.4	63.6	84.8	106.0	127.2	148.4	169.6	190.8
211	21.1	42.2	63.3	84.4	105.5	126.6	147.7	168.8	189.9
210	21.0	42.0	63.0	84.0	105.0	126.0	147.0	168.0	189.0
209	20.9	41.8	62.7	83.6	104.5	125.4	146.3	167.2	188.1
208	20.8	41.6	62.4	83.2	104.0	124.8	145.6	166.4	187.2
207	20.7	41.4	62.1	82.8	103.5	124.2	144.9	165.6	186.3
206	20.6	41.2	61.8	82.4	103.0	123.6	144.2	164.8	185.4
205	20.5	41.0	61.5	82.0	102.5	123.0	143.5	164.0	184.5
204	20.4	40.8	61.2	81.6	102.0	122.4	142.8	163.2	183.6
203	20.3	40.6	60.9	81.2	101.5	121.8	142.1	162.4	182.7
202	20.2	40.4	60.6	80.8	101.0	121.2	141.4	161.6	181.8

No. 215 Log. 332 TABLE XVIII.—*Continued.* No. 239 Log. 380

N.	0	1	2	3	4	5	6	7	8	9	Diff.
215	33 2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
6	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
7	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
8	8456	8656	8855	9054	9253	9451	9650	9849	0047	0246	199
9	34 0444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
1	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
2	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
3	8305	8500	8694	8889	9083	9278	9472	9666	9860	0054	194
4	35 0248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
6	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
7	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
8	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
9	9835	0025	0215	0404	0593	0783	0972	1161	1350	1539	189
230	36 1728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
1	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
2	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
3	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
4	9216	9401	9587	9772	9958	0143	0328	0513	0698	0883	185
235	37 1068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
6	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
9	8398	8580	8761	8943	9124	9306	9487	9668	9849	0030	181
	38										

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
202	20.2	40.4	60.6	80.8	101.0	121.2	141.4	161.6	181.8
201	20.1	40.2	60.3	80.4	100.5	120.6	140.7	160.8	180.9
200	20.0	40.0	60.0	80.0	100.0	120.0	140.0	160.0	180.0
199	19.9	39.8	59.7	79.6	99.5	119.4	139.3	159.2	179.1
198	19.8	39.6	59.4	79.2	99.0	118.8	138.6	158.4	178.2
197	19.7	39.4	59.1	78.8	98.5	118.2	137.9	157.6	177.3
196	19.6	39.2	58.8	78.4	98.0	117.6	137.2	156.8	176.4
195	19.5	39.0	58.5	78.0	97.5	117.0	136.5	156.0	175.5
194	19.4	38.8	58.2	77.6	97.0	116.4	135.8	155.2	174.6
193	19.3	38.6	57.9	77.2	96.5	115.8	135.1	154.4	173.7
192	19.2	38.4	57.6	76.8	96.0	115.2	134.4	153.6	172.8
191	19.1	38.2	57.3	76.4	95.5	114.6	133.7	152.8	171.9
190	19.0	38.0	57.0	76.0	95.0	114.0	133.0	152.0	171.0
189	18.9	37.8	56.7	75.6	94.5	113.4	132.3	151.2	170.1
188	18.8	37.6	56.4	75.2	94.0	112.8	131.6	150.4	169.2
187	18.7	37.4	56.1	74.8	93.5	112.2	130.9	149.6	168.3
186	18.6	37.2	55.8	74.4	93.0	111.6	130.2	148.8	167.4
185	18.5	37.0	55.5	74.0	92.5	111.0	129.5	148.0	166.5
184	18.4	36.8	55.2	73.6	92.0	110.4	128.8	147.2	165.6
183	18.3	36.6	54.9	73.2	91.5	109.8	128.1	146.4	164.7
182	18.2	36.4	54.6	72.8	91.0	109.2	127.4	145.6	163.8
181	18.1	36.2	54.3	72.4	90.5	108.6	126.7	144.8	162.9
180	18.0	36.0	54.0	72.0	90.0	108.0	126.0	144.0	162.0
179	17.9	35.8	53.7	71.6	89.5	107.4	125.3	143.2	161.1

No. 240 No. 269
Log. 380 Table XVIII.—*Continued.* Log. 431

N.	0	1	2	3	4	5	6	7	8	9	Diff.
240	38 0211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
1	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
2	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
3	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
4	7390	7568	7746	7924	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	0051	0228	0405	0582	0759	177
6	39 0935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
8	4452	4627	4802	4077	5152	5326	5501	5676	5850	6025	175
9	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	7940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
1	9674	9847	0020	0192	0365	0538	0711	0883	1056	1228	173
2	40 1401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
3	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
4	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
6	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
7	9933	0102	0271	0440	0609	0777	0946	1114	1283	1451	169
8	41 1620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
9	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
1	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
2	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
3	9956	0121	0286	0451	0616	0781	0945	1110	1275	1439	165
4	42 1604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
6	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
7	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
8	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
9	9752	9914	0075	0236	0398	0559	0720	0881	1042	1203	161
	43										

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
178	17.8	35.6	53.4	71.2	89.0	106.8	124.6	142.4	160.2
177	17.7	35.4	53.1	70.8	88.5	106.2	123.9	141.6	159.3
176	17.6	35.2	52.8	70.4	88.0	105.6	123.2	140.8	158.4
175	17.5	35.0	52.5	70.0	87.5	105.0	122.5	140.0	157.5
174	17.4	34.8	52.2	69.6	87.0	104.4	121.8	139.2	156.6
173	17.3	34.6	51.9	69.2	86.5	103.8	121.1	138.4	155.7
172	17.2	34.4	51.6	68.8	86.0	103.2	120.4	137.6	154.8
171	17.1	34.2	51.3	68.4	85.5	102.6	119.7	136.8	153.9
170	17.0	34.0	51.0	68.0	85.0	102.0	119.0	136.0	153.0
169	16.9	33.8	50.7	67.6	84.5	101.4	118.3	135.2	152.1
168	16.8	33.6	50.4	67.2	84.0	100.8	117.6	134.4	151.2
167	16.7	33.4	50.1	66.8	83.5	100.2	116.9	133.6	150.3
166	16.6	33.2	49.8	66.4	83.0	99.6	116.2	132.8	149.4
165	16.5	33.0	49.5	66.0	82.5	99.0	115.5	132.0	148.5
164	16.4	32.8	49.2	65.6	82.0	98.4	114.8	131.2	147.6
163	16.3	32.6	48.9	65.2	81.5	97.8	114.1	130.4	146.7
162	16.2	32.4	48.5	64.8	81.0	97.2	113.4	129.6	145.8
161	16.1	32.2	48.3	64.4	80.5	96.6	112.7	128.8	144.9

No. 270 Log. 431 TABLE XVIII.—*Continued.* No. 299 Log. 476

N.	0	1	2	3	4	5	6	7	8	9	Diff.
270	43 1364	1525	1685	1846	2007	2167	2328	2488	2649	2809	161
1	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
2	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
3	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
4	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	9333	9491	9648	9806	9964	0122	0279	0437	0594	0752	158
6	44 0909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
7	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
8	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
9	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	7158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
1	8706	8861	9015	9170	9324	9478	9633	9787	9941	0095	154
2	45 0249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
3	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
4	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
6	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
7	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
8	9392	9543	9694	9845	9995	0146	0296	0447	0597	0748	151
9	46 0898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
1	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
2	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
3	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
4	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	9822	9969	0116	0263	0410	0557	0704	0851	0998	1145	147
6	47 1292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
7	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
8	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
9	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
161	16.1	32.2	48.3	64.4	80.5	96.6	112.7	128.8	144.9
160	16.0	32.0	48.0	64.0	80.0	96.0	112.0	128.0	144.0
159	15.9	31.8	47.7	63.6	79.5	95.4	111.3	127.2	143.1
158	15.8	31.6	47.4	63.2	79.0	94.8	110.6	126.4	142.2
157	15.7	31.4	47.1	62.8	78.5	94.2	109.9	125.6	141.3
156	15.6	31.2	46.8	62.4	78.0	93.6	109.2	124.8	140.4
155	15.5	31.0	46.5	62.0	77.5	93.0	108.5	124.0	139.5
154	15.4	30.8	46.2	61.6	77.0	92.4	107.8	123.2	138.6
153	15.3	30.6	45.9	61.2	76.5	91.8	107.1	122.4	137.7
152	15.2	30.4	45.6	60.8	76.0	91.2	106.4	121.6	136.8
151	15.1	30.2	45.3	60.4	75.5	90.6	105.7	120.8	135.9
150	15.0	30.0	45.0	60.0	75.0	90.0	105.0	120.0	135.0
149	14.9	29.8	44.7	59.6	74.5	89.4	104.3	119.2	134.1
148	14.8	29.6	44.4	59.2	74.0	88.8	103.6	118.4	133.2
147	14.7	29.4	44.1	58.8	73.5	88.2	102.9	117.6	132.3
146	14.6	29.2	43.8	58.4	73.0	87.6	102.2	116.8	131.4
145	14.5	29.0	43.5	58.0	72.5	87.0	101.5	116.0	130.5
144	14.4	28.8	43.2	57.6	72.0	86.4	100.8	115.2	129.6
143	14.3	28.6	42.9	57.2	71.5	85.8	100.1	114.4	128.7
142	14.2	28.4	42.6	56.8	71.0	85.2	99.4	113.6	127.8
141	14.1	28.2	42.3	56.4	70.5	84.6	98.7	112.8	126.9
140	14.0	28.0	42.0	56.0	70.0	84.0	98.0	112.0	126.0

No. 300 — Log. 477 — Table XVIII.—*Continued.* — No. 339 — Log. 531

N.	0	1	2	3	4	5	6	7	8	9	Diff.
300	47 7121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145
1	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
2	48 0007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
3	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
4	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
6	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
7	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
8	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
9	9958	0099	0239	0380	0520	0661	0801	0941	1081	1222	140
310	49 1362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
1	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
2	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
3	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
4	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
6	9687	9824	9962	0099	0236	0374	0511	0648	0785	0922	137
7	50 1059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
8	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
9	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
1	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
2	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
3	9203	9337	9471	9606	9740	9874	0009	0143	0277	0411	134
4	51 0545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
6	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	133
7	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
8	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
9	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
1	9828	9959	0090	0221	0353	0484	0615	0745	0876	1007	131
2	52 1138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
3	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
4	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
6	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
7	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
8	8917	9045	9174	9302	9430	9559	9687	9815	9943	0072	128
9	53 0200	0328	0456	0584	0712	0840	0968	1096	1223	1351	128

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
139	13.9	27.8	41.7	55.6	69.5	83.4	97.3	111.2	125.1
138	13.8	27.6	41.4	55.2	69.0	82.8	96.6	110.4	124.2
137	13.7	27.4	41.1	54.8	68.5	82.2	95.9	109.6	123.3
136	13.6	27.2	40.8	54.4	68.0	81.6	95.2	108.8	122.4
135	13.5	27.0	40.5	54.0	67.5	81.0	94.5	108.0	121.5
134	13.4	26.8	40.2	53.6	67.0	80.4	93.8	107.2	120.6
133	13.3	26.6	39.9	53.2	66.5	79.8	93.1	106.4	119.7
132	13.2	26.4	39.6	52.8	66.0	79.2	92.4	105.6	118.8
131	13.1	26.2	39.3	52.4	65.5	78.6	91.7	104.8	117.9
130	13.0	26.0	39.0	52.0	65.0	78.0	91.0	104.0	117.0
129	12.9	25.8	38.7	51.6	64.5	77.4	90.3	103.2	116.1
128	12.8	25.6	38.4	51.2	64.0	76.8	89.6	102.4	115.2
127	12.7	25.4	38.1	50.8	63.5	76.2	88.9	101.6	114.3

No. 340 Log. 531 — Table XVIII.—*Continued.* — No. 379 Log. 579

N.	0	1	2	3	4	5	6	7	8	9	Diff.
340	53 1479	1607	1734	1862	1990	2117	2245	2372	2500	2627	128
1	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
2	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
3	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
4	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
6	9076	9202	9327	9452	9578	9703	9829	9954	0079	0204	125
7	54 0329	0455	0580	0705	0830	0955	1080	1205	1330	1454	125
8	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
9	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
350	4068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124
1	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
2	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
3	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
4	9003	9126	9249	9371	9494	9616	9739	9861	9984	0106	123
355	55 0228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
6	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
7	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
8	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
9	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	6303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
1	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
2	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
3	9907	0026	0146	0265	0385	0504	0624	0743	0863	0982	119
4	56 1101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
6	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
7	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
8	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
9	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
1	9374	9491	9608	9725	9842	9959	0076	0193	0309	0426	117
2	57 0543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
3	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
4	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
375	4031	4147	4263	4379	4494	4610	4726	4841	4057	5072	116
6	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
7	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
8	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
9	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
128	12.8	25.6	38.4	51.2	64.0	76.8	80.6	102.4	115.2
127	12.7	25.4	38.1	50.8	63.5	76.2	88.9	101.6	114.3
126	12.6	25.2	37.8	50.4	63.0	75.6	88.2	100.8	113.4
125	12.5	25.0	37.5	50.0	62.5	75.0	87.5	100.0	112.5
124	12.4	24.8	37.2	49.6	62.0	74.4	86.8	99.2	111.6
123	12.3	24.6	36.9	49.2	61.5	73.8	86.1	98.4	110.7
122	12.2	24.4	36.6	48.8	61.0	73.2	85.4	97.6	109.8
121	12.1	24.2	36.3	48.4	60.5	72.6	84.7	96.8	108.9
120	12.0	24.0	36.0	48.0	60.0	72.0	84.0	96.0	108.0
110	11.9	23.8	35.7	47.6	59.5	71.4	83.3	95.2	107.1

No. 380 — Log. 579 | TABLE XVIII.—*Continued.* | No. 414 — Log. 617

N.	0	1	2	3	4	5	6	7	8	9	Diff.
380	57 9784	9898	0012	0126	0241	0355	0469	0583	0697	0811	114
1	58 0925	1039	1153	1267	1381	1495	1608	1722	1836	1950	
2	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	
3	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	
4	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	
6	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	
7	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
8	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	
9	9950	0061	0173	0284	0396	0507	0619	0730	0842	0953	
390	59 1065	1176	1287	1399	1510	1621	1732	1843	1955	2066	
1	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
2	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	
3	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	
4	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
6	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	
7	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	
8	9883	9992	0101	0210	0319	0428	0537	0646	0755	0864	109
9	60 0973	1082	1191	1299	1408	1517	1625	1734	1843	1951	
400	2060	2169	2277	2386	2494	2603	2711	2819	2928	3036	
1	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
2	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	
3	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	
4	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
6	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	
7	9594	9701	9808	9914	0021	0128	0234	0341	0447	0554	
8	61 0660	0767	0873	0979	1086	1192	1298	1405	1511	1617	
9	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736	
1	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	
2	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	
3	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
4	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
118	11.8	23.6	35.4	47.2	59.0	70.8	82.6	94.4	106.2
117	11.7	23.4	35.1	46.8	58.5	70.2	81.9	93.6	105.3
116	11.6	23.2	34.8	46.4	58.0	69.6	81.2	92.8	104.4
115	11.5	23.0	34.5	46.0	57.5	69.0	80.5	92.0	103.5
114	11.4	22.8	34.2	45.6	57.0	68.4	79.8	91.2	102.6
113	11.3	22.6	33.9	45.2	56.5	67.8	79.1	90.4	101.7
112	11.2	22.4	33.6	44.8	56.0	67.2	78.4	89.6	100.8
111	11.1	22.2	33.3	44.4	55.5	66.6	77.7	88.8	99.9
110	11.0	22.0	33.0	44.0	55.0	66.0	77.0	88.0	99.0
109	10.9	21.8	32.7	43.6	54.5	65.4	76.3	87.2	98.1
108	10.8	21.6	32.4	43.2	54.0	64.8	75.6	86.4	97.2
107	10.7	21.4	32.1	42.8	53.5	64.2	74.9	85.6	96.3
106	10.6	21.2	31.8	42.4	53.0	63.6	74.2	84.8	95.4
105	10.5	21.0	31.5	42.0	52.5	63.0	73.5	84.0	94.5
104	10.4	20.8	31.2	41.6	52.0	62.4	72.8	83.2	93.6

No. 415 — Log. 618 — TABLE XVIII.—*Continued.* — No. 459 — Log. 662

N.	0	1	2	3	4	5	6	7	8	9	Diff.
415	61 8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
6	9093	9198	9302	9406	9511	9615	9719	9824	9928	0032	
7	62 0136	0240	0344	0448	0552	0656	0760	0864	0968	1072	104
8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	
9	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179	
1	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
2	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	
3	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	
4	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
6	9410	9512	9613	9715	9817	9919	0021	0123	0224	0326	
7	63 0428	0530	0631	0733	0835	0936	1038	1139	1241	1342	
8	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	
9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376	101
1	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	
2	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	
3	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	
4	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	100
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	
6	9486	9586	9686	9785	9885	9984	0084	0183	0283	0382	
7	64 0481	0581	0680	0779	0879	0978	1077	1177	1276	1375	
8	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	
9	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340	
1	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	
2	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	
3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
4	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	
6	9335	9432	9530	9627	9724	9821	9919	0016	0113	0210	
7	65 0308	0405	0502	0599	0696	0793	0890	0987	1084	1181	
8	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	
450	3213	3309	3405	3502	3598	3695	3791	3888	3984	4080	
1	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	
2	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
3	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	
4	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	
6	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	
7	9916	0011	0106	0201	0296	0391	0486	0581	0676	0771	95
8	66 0865	0960	1055	1150	1245	1339	1434	1529	1623	1718	
9	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
105	10.5	21.0	31.5	42.0	52.5	63.0	73.5	84.0	94.5
104	10.4	20.8	31.2	41.6	52.0	62.4	72.8	83.2	93.6
103	10.3	20.6	30.9	41.2	51.5	61.8	72.1	82.4	92.7
102	10.2	20.4	30.6	40.8	51.0	61.2	71.4	81.6	91.8
101	10.1	20.2	30.3	40.4	50.5	60.6	70.7	80.8	90.9
100	10.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0
99	9.9	19.8	29.7	39.6	49.5	59.4	69.3	79.2	89.1

No. 460 | Log. 662 — Table XVIII.—*Continued.* — No. 499 | Log. 698

N.	0	1	2	3	4	5	6	7	8	9	Diff.
460	66 2758	2852	2947	3041	3135	3230	3324	3418	3512	3607	
1	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	
2	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
3	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	
4	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	
6	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	
7	9317	9410	9503	9596	9689	9782	9875	9967	0060	0153	93
8	67 0246	0339	0431	0524	0617	0710	0802	0895	0988	1080	
9	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	
470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929	
1	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	
2	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
3	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	
4	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	
6	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	
7	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
8	9428	9519	9610	9700	9791	9882	9973	0063	0154	0245	
9	68 0336	0426	0517	0607	0698	0789	0879	0970	1060	1151	
480	1241	1332	1422	1513	1603	1693	1784	1874	1964	2055	
1	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	
2	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
3	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	
4	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	
6	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	
9	9309	9398	9486	9575	9664	9753	9841	9930	0019	0107	
490	69 0196	0285	0373	0462	0550	0639	0728	0816	0905	0993	
1	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	
2	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	
3	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	
6	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	
7	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	
8	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	
9	8100	8188	8275	8362	8449	8535	8622	8709	8796	8883	87

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
98	9.8	19.6	29.4	39.2	49.0	58.8	68.6	78.4	88.2
97	9.7	19.4	29.1	38.8	48.5	58.2	67.9	77.6	87.3
96	9.6	19.2	28.8	38.4	48.0	57.6	67.2	76.8	86.4
95	9.5	19.0	28.5	38.0	47.5	57.0	66.5	76.0	85.5
94	9.4	18.8	28.2	37.6	47.0	56.4	65.8	75.2	84.6
93	9.3	18.6	27.9	37.2	46.5	55.8	65.1	74.4	83.7
92	9.2	18.4	27.6	36.8	46.0	55.2	64.4	73.6	82.8
91	9.1	18.2	27.3	36.4	45.5	54.6	63.7	72.8	81.9
90	9.0	18.0	27.0	36.0	45.0	54.0	63.0	72.0	81.0
89	8.9	17.8	26.7	35.6	44.5	53.4	62.3	71.2	80.1
88	8.8	17.6	26.4	35.2	44.0	52.8	61.6	70.4	79.2
87	8.7	17.4	26.1	34.8	43.5	52.2	60.9	69.6	78.3
86	8.6	17.2	25.8	34.4	43.0	51.6	60.2	68.8	77.4

No. 500 Log. 698 — TABLE XVIII.—*Continued.* — No. 544 Log. 736

N.	0	1	2	3	4	5	6	7	8	9	Diff.
500	69 8970	9057	9144	9231	9317	9404	9491	9578	9664	9751	
1	9838	9924	0011	0098	0184	0271	0358	0444	0531	0617	
2	70 0704	0790	0877	0963	1050	1136	1222	1309	1395	1482	
3	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	
4	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
6	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	
7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	
8	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	
9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336	85
1	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	
2	9270	9355	9440	9524	9609	9694	9779	9863	9948	0033	
3	71 0117	0202	0287	0371	0456	0540	0625	0710	0794	0879	
4	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	
6	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
7	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	
8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	
9	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754	
1	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	
2	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	
3	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
4	9331	9414	9497	9580	9663	9745	9828	9911	9994	0077	
525	72 0159	0242	0325	0407	0490	0573	0655	0738	0821	0903	
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013	
1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	
2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	
3	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	
4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	
7	9974	0055	0136	0217	0298	0378	0459	0540	0621	0702	
8	73 0782	0863	0944	1024	1105	1186	1266	1347	1428	1508	
9	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117	
1	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	
2	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
3	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	
4	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
87	8.7	17.4	26.1	34.8	43.5	52.2	60.9	69.6	78.3
86	8.6	17.2	25.8	34.4	43.0	51.6	60.2	68.8	77.4
85	8.5	17.0	25.5	34.0	42.5	51.0	59.5	68.0	76.5
84	8.4	16.8	25.2	33.6	42.0	50.4	58.8	67.2	75.6

No. 545 — Log. 736 TABLE XVIII.—*Continued.* No. 584 — Log. 767

N.	0	1	2	3	4	5	6	7	8	9	Diff.
545	73 6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	
6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	
9	9572	9651	9731	9810	9889	9968	0047	0126	0205	0284	79
550	74 0363	0442	0521	0600	0678	0757	0836	0915	0994	1073	
1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	
555	4203	4371	4449	4528	4606	4684	4762	4840	4919	4997	
6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885	
1	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	
2	9736	9814	9891	9968	0045	0123	0200	0277	0354	0431	
3	75 0508	0586	0663	0740	0817	0894	0971	1048	1125	1202	
4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560	
1	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	
3	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	
4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	
575	9668	9743	9819	9894	9970	0045	0121	0196	0272	0347	
6	76 0422	0498	0573	0649	0724	0799	0875	0950	1025	1101	
7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	
8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	
9	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	75
580	3428	3503	3578	3653	3727	3802	3877	3952	4027	4101	
1	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	
2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	
3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
83	8.3	16.6	24.9	33.2	41.5	49.8	58.1	66.4	74.7
82	8.2	16.4	24.6	32.8	41.0	49.2	57.4	65.6	73.8
81	8.1	16.2	24.3	32.4	40.5	48.6	56.7	64.8	72.9
80	8.0	16.0	24.0	32.0	40.0	48.0	56.0	64.0	72.0
79	7.9	15.8	23.7	31.6	39.5	47.4	55.3	63.2	71.1
78	7.8	15.6	23.4	31.2	39.0	46.8	54.6	62.4	70.2
77	7.7	15.4	23.1	30.8	38.5	46.2	53.9	61.6	69.3
76	7.6	15.2	22.8	30.4	38.0	45.6	53.2	60.8	68.4
75	7.5	15.0	22.5	30.0	37.5	45.0	52.5	60.0	67.5
74	7.4	14.8	22.2	29.6	37.0	44.4	51.8	59.2	66.6

No. 585 — No. 629
Log. 767 — Log. 799

TABLE XVIII.—*Continued.*

N.	0	1	2	3	4	5	6	7	8	9	Diff.
585	76 7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	0042	
9	77 0115	0189	0263	0336	0410	0484	0557	0631	0705	0778	
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514	
1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	
600	8151	8224	8296	8368	8441	8513	8585	8658	8730	8802	
1	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	
2	9596	9669	9741	9813	9885	9957	0029	0101	0173	0245	72
3	78 0317	0389	0461	0533	0605	0677	0749	0821	0893	0965	
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970	
1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	
6	9581	9651	9722	9792	9863	9933	0004	0074	0144	0215	
7	79 0285	0356	0426	0496	0567	0637	0707	0778	0848	0918	
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022	70
1	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	
2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	
3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
75	7.5	15.0	22.5	30.0	37.5	45.0	52.5	60.0	67.5
74	7.4	14.8	22.2	29.6	37.0	44.4	51.8	59.2	66.6
73	7.3	14.6	21.9	29.2	36.5	43.8	51.1	58.4	65.7
72	7.2	14.4	21.6	28.8	36.0	43.2	50.4	57.6	64.8
71	7.1	14.2	21.3	28.4	35.5	42.6	49.7	56.8	63.9
70	7.0	14.0	21.0	28.0	35.0	42.0	49.0	56.0	63.0
69	6.9	13.8	20.7	27.6	34.5	41.4	48.3	55.2	62.1

No. 630 Log. 799 — Table XVIII.—*Continued.* — No. 674 Log. 829

N.	0	1	2	3	4	5	6	7	8	9	Diff.
630	79 9341	9409	9478	9547	9616	9685	9754	9823	9892	9961	
1	80 0029	0098	0167	0236	0305	0373	0442	0511	0580	0648	
2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	
640	80 6180	6248	6316	6384	6451	6519	6587	6655	6723	6790	
1	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	
2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	
4	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	
645	9560	9627	9694	9762	9829	9806	9964	0031	0098	0165	
6	81 0233	0300	0367	0434	0501	0569	0636	0703	0770	0837	
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	
650	2913	2980	3047	3114	3181	3247	3314	3381	3448	3514	
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
660	9544	9610	9676	9741	9807	9873	9939	0004	0070	0136	
1	82 0201	0267	0333	0399	0464	0530	0595	0661	0727	0792	
2	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	
3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	
4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	
670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658	
1	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	
2	7369	7434	7499	7563	7628	7602	7757	7821	7886	7951	
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
68	6.8	13.6	20.4	27.2	34.0	40.8	47.6	54.4	61.2
67	6.7	13.4	20.1	26.8	33.5	40.2	46.9	53.6	60.3
66	6.6	13.2	19.8	26.4	33.0	39.6	46.2	52.8	59.4
65	6.5	13.0	19.5	26.0	32.5	39.0	45.5	52.0	58.5
64	6.4	12.8	19.2	25.6	32.0	38.4	44.8	51.2	57.6

No. 675 — Log. 829 | Table XVIII.—*Continued.* | No. 719 — Log. 857

N.	0	1	2	3	4	5	6	7	8	9	Diff.
675	82 9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	
6	9947	0011	0075	0139	0204	0268	0332	0396	0460	0525	
7	83 0589	0653	0717	0781	0845	0909	0973	1037	1102	1166	
8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	
680	2509	2573	2637	2700	2764	2828	2892	2956	3020	3083	
1	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	
2	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	
3	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	
4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	
7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	
8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	8849	8912	8975	9038	9101	9164	9227	9289	9352	9415	
1	9478	9541	9604	9667	9729	9792	9855	9918	9981	0043	
2	84 0106	0169	0232	0294	0357	0420	0482	0545	0608	0671	
3	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	
4	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	
9	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	
700	5098	5160	5222	5284	5346	5408	5470	5532	5594	5656	62
1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	
2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	
8	85 0033	0095	0156	0217	0279	0340	0401	0462	0524	0585	
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809	
1	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	
2	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
3	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	
4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	
8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	
9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
65	6.5	13.0	19.5	26.0	32.5	39.0	45.5	52.0	58.5
64	6.4	12.8	19.2	25.6	32.0	38.4	44.8	51.2	57.6
63	6.3	12.6	18.9	25.2	31.5	37.8	44.1	50.4	56.7
62	6.2	12.4	18.6	24.8	31.0	37.2	43.4	49.6	55.8
61	6.1	12.2	18.3	24.4	30.5	36.6	42.7	48.8	54.9
60	6.0	12.0	18.0	24.0	30.0	36.0	42.0	48.0	54.0

No. 720 — Log. 857 | Table XVIII.—*Continued.* | No. 764 — Log. 883

N.	0	1	2	3	4	5	6	7	8	9	Diff.
720	85 7332	7393	7453	7513	7574	7634	7694	7755	7815	7875	
1	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	
2	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	
3	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
4	9739	9799	9859	9918	9978	0038	0098	0158	0218	0278	
725	86 0338	0398	0458	0518	0578	0637	0697	0757	0817	0877	
6	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	
7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858	
1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	
2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	
3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	
8	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	
9	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	
740	9232	9290	9349	9408	9466	9525	9584	9642	9701	9760	
1	9818	9877	9935	9994	0053	0111	0170	0228	0287	0345	
2	87 0404	0462	0521	0579	0638	0696	0755	0813	0872	0930	
3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	
8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	
750	5061	5119	5177	5235	5293	5351	5409	5466	5524	5582	
1	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	
2	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	
3	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	
4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	
6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	
8	9669	9726	9784	9841	9898	9956	0013	0070	0127	0185	
9	88 0242	0299	0356	0413	0471	0528	0585	0642	0699	0756	
760	0814	0871	0928	0985	1042	1099	1156	1213	1271	1328	
1	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	
2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
3	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	
4	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
59	5.9	11.8	17.7	23.6	29.5	35.4	41.3	47.2	53.1
58	5.8	11.6	17.4	23.2	29.0	34.8	40.6	46.4	52.2
57	5.7	11.4	17.1	22.8	28.5	34.2	39.9	45.6	51.3
56	5.6	11.2	16.8	22.4	28.0	33.6	39.2	44.8	50.4

No. 765
Log. 883

TABLE XVIII.—*Continued.*

No. 809
Log. 908

N.	0	1	2	3	4	5	6	7	8	9	Diff.
765	88 3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	
6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	
7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	
8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998	
1	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	
2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	
3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	
4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
6	9862	9918	9974	0030	0086	0141	0197	0253	0309	0365	
7	89 0421	0477	0533	0589	0645	0700	0756	0812	0868	0924	
8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	
9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	
730	2095	2150	2206	2262	2317	2373	2429	2484	2540	2595	
1	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	
2	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	
3	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	
4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	
6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	
790	7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	55
1	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	
2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	
3	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	
4	9821	9875	9930	9985	0039	0094	0149	0203	0258	0312	
795	90 0367	0422	0476	0531	0586	0640	0695	0749	0804	0859	
6	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	
7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	
8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	
9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	
800	3090	3144	3199	3253	3307	3361	3416	3470	3524	3578	
1	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	
2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	
3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	
4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	
6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	
9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
57	5.7	11.4	17.1	22.8	28.5	34.2	39.9	45.6	51.3
56	5.6	11.2	16.8	22.4	28.0	33.6	39.2	44.8	50.4
55	5.5	11.0	16.5	22.0	27.5	33.0	38.5	44.0	49.5
54	5.4	10.8	16.2	21.6	27.0	32.4	37.8	43.2	48.6

No. 810 Log. 908 — TABLE XVIII.—*Continued.* — No. 854 Log. 931

N.	0	1	2	3	4	5	6	7	8	9	Diff.
810	90 8485	8539	8592	8646	8699	8753	8807	8860	8914	8967	
1	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	
2	9556	9610	9663	9716	9770	9823	9877	9930	9984	0037	
3	91 0091	0144	0197	0251	0304	0358	0411	0464	0518	0571	
4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	
9	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290	
1	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	
2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	
3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	
4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	
6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	
7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549	
1	9601	9653	9706	9758	9810	9862	9914	9967	0019	0071	
2	92 0123	0176	0228	0280	0332	0384	0436	0489	0541	0593	
3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	
4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	
6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	
840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744	
1	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	
2	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	
3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	
4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	
6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	
7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	
850	9419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
1	9930	9981	0032	0083	0134	0185	0236	0287	0338	0389	
2	93 0440	0491	0542	0592	0643	0694	0745	0796	0847	0898	
3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	
4	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
53	5.3	10.6	15.9	21.2	26.5	31.8	37.1	42.4	47.7
52	5.2	10.4	15.6	20.8	26.0	31.2	36.4	41.6	46.8
51	5.1	10.2	15.3	20.4	25.5	30.6	35.7	40.8	45.9
50	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0

No. 855 Log. 931 — Table XVIII.—*Continued.* — No. 899 Log. 954

N.	0	1	2	3	4	5	6	7	8	9	Diff.
855	93 1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	
8	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	
9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953	
1	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	
2	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	
3	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	
4	6514	6564	6614	6665	6715	6765	6315	6865	6916	6966	
865	7016	7066	7116	7167	7217	7267	7317	7367	7418	7468	
6	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	
7	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
8	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	
9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	
870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968	
1	94 0018	0068	0118	0168	0218	0267	0317	0367	0417	0467	
2	0516	0566	0616	0666	0716	0765	0315	0865	0915	0964	
3	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	
4	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	
6	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	
7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	
8	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	
9	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927	
1	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	
2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	
3	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	
4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	
885	6943	6992	7041	7090	7139	7189	7238	7287	7336	7385	
6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	
8	8413	8462	8511	8560	8608	8657	8706	8755	8804	8853	
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	
890	9390	9439	9488	9536	9585	9634	9083	9731	9780	9829	
1	9878	9926	9975	0024	0073	0121	0170	0219	0267	0316	
2	95 0365	0414	0462	0511	0560	0608	0657	0706	0754	0803	
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	
4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
51	5.1	10.2	15.3	20.4	25.5	30.6	35.7	40.8	45.9
50	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0
49	4.9	9.8	14.7	19.6	24.5	29.4	34.3	39.2	44.1
48	4.8	9.6	14.4	19.2	24.0	28.8	33.6	38.4	43.2

No. 900 Log. 954 — TABLE XVIII.—*Continued.* — No. 944 Log. 975

N.	0	1	2	3	4	5	6	7	8	9	Diff.
900	95 4243	4291	4339	4387	4435	4484	4532	4580	4628	4677	
1	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	
2	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	
3	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	
4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	
6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	
1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	
2	9995	0042	0090	0138	0185	0233	0280	0328	0376	0423	
3	96 0471	0518	0566	0613	0661	0709	0756	0804	0851	0899	
4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	
7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	
8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	
1	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	
2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903	
1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	
2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	
3	9882	9928	9975	0021	0068	0114	0161	0207	0254	0300	
4	97 0347	0393	0440	0486	0533	0579	0626	0672	0719	0765	
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	
8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	
9	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543	
1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	
2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
47	4.7	9.4	14.1	18.8	23.5	28.2	32.9	37.6	42.3
46	4.6	9.2	13.8	18.4	23.0	27.6	32.2	36.8	41.4

No. 945 Log. 975 TABLE XVIII.—*Continued.* No. 989 Log. 995

N.	0	1	2	3	4	5	6	7	8	9	Diff.
945	97 5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	
950	7724	7769	7815	7861	7906	7952	7998	8043	8089	8135	
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	
955	98 0003	0049	0094	0140	0185	0231	0276	0322	0367	0412	
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	
8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678	
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175	
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	
7	9895	9939	9983	0028	0072	0117	0161	0206	0250	0294	
8	99 0339	0383	0428	0472	0516	0561	0605	0650	0694	0738	
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625	
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
46	4.6	9.2	13.8	18.4	23.0	27.6	32.2	36.8	41.4
45	4.5	9.0	13.5	18.0	22.5	27.0	31.5	36.0	40.5
44	4.4	8.8	13.2	17.6	22.0	26.4	30.8	35.2	39.6
43	4.3	8.6	12.9	17.2	21.5	25.8	30.1	34.4	38.7

No. 990 No. 999
Log. 995 Table XVIII.—*Concluded.* Log. 999

N.	0	1	2	3	4	5	6	7	8	9	Diff.
990	99 5635	5679	5723	5767	5811	5854	5898	5942	5986	6030	
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43

Table XIXa.—Values of *S*, *T*, and *C* in Table XIX, Pages 935 and 936

If we were to plot the values of the logarithmic functions given in Table XIX as ordinates and corresponding minutes as abscissas it would be found that the points for each function were on a curve with variable radius. It would further be noted that the curves for sines, tangents, and cotangents were of comparatively small radii when the angles were small; that the curves for cosines, cotangents, and tangents of angles near 90° respectively had the same shape as the curves for sines, tangents, and cotangents of the complements of the angles; and that other than the portions of the curves just mentioned were nearly straight lines for short distances.

When seconds are involved it will be sufficiently accurate to interpolate in the ordinary manner between adjacent values in the tables—or in other words, to assume that the curve joining two adjacent points is a straight line—for all functions between 2° and 88°, and also for sines of angles between 88° and 90° and for cosines of angles between 0° and 2°. The values in the columns headed *S*, *T*, and *C* provide a means (1) of accurately determining for any given angle between 0° and 2° the logarithmic sine, tangent, or cotangent, and for any given angle between 88° and 90°, the logarithmic cosine, cotangent, or tangent; or (2) for a given value of the logarithmic sine, tangent, or cotangent of accurately determining the angle when it lies between 0° and 2°, and for any given value of the cosine, cotangent, or tangent, the angle when it lies between 88° and 90°.

TABLE XIXa.—*Continued.*

Given*: *angle.* *Required*: *logarithmic function.

$$\left.\begin{aligned} \log\sin\alpha &= \log\alpha \text{ (in seconds)} + S \\ \log\tan\alpha &= \log\alpha \text{ (in seconds)} + T \\ \log\cot\alpha &= C - \log\alpha \text{ (in seconds)} \end{aligned}\right\} \text{In which } \alpha \text{ is less than } 2^\circ.$$

$$\left.\begin{aligned} \log\cos\beta &= \log(90^\circ-\beta) \text{ (in seconds)} + S \\ \log\tan\beta &= C - \log(90^\circ-\beta) \text{ (in seconds)} \\ \log\cot\beta &= \log(90^\circ-\beta) \text{ (in seconds)} + T \end{aligned}\right\} \text{In which } \beta \text{ lies between } 88^\circ \text{ and } 90^\circ$$

Given*: *logarithmic function.* *Required*: *angle.

$$\left.\begin{aligned} \log\alpha \text{ (in seconds)} &= \log\sin\alpha - S \\ &= \log\tan\alpha - T \\ &= C - \log\cot\alpha \end{aligned}\right\} \text{In which } \alpha \text{ is less than } 2^\circ.$$

$$\left.\begin{aligned} \log(90^\circ-\beta) \text{ (in seconds)} &= \log\cos\beta - S \\ &= C - \log\tan\beta \\ &= \log\cot\beta - T \end{aligned}\right\} \text{In which } \beta \text{ lies between } 88^\circ \text{ and } 90^\circ.$$

EXAMPLES

Given*: *angle.

Required*: *logarithmic function.

When $\left\{\begin{matrix}\alpha\\90^\circ-\beta\end{matrix}\right\} = 19'\,22'' = 1162''$

$\log 1162'' = 3.065206$

S (for $19'$) $= 4.685573$

$\left.\begin{matrix}\log\sin 19'\,22''\\ \log\cos 89^\circ\,40'\,38''\end{matrix}\right\} = 7.750779$

When $\left\{\begin{matrix}\alpha\\90^\circ-\beta\end{matrix}\right\} = 23'\,21'' = 1401''$

$\log 1401'' = 3.146438$

C (for $23'$) $= 15.314419$

$\left.\begin{matrix}\log\cot 23'\,21''\\ \log\tan 89^\circ 36' 39''\end{matrix}\right\} = 12.167981$

Given*: *logarithmic function.

Required*: *angle.

When $\left\{\begin{matrix}\log\sin\alpha\\ \log\cos\beta\end{matrix}\right\} = 7.750779$

S (for $19'$) $= 4.685573$

$\log\left\{\begin{matrix}\alpha\\90^\circ-\beta\end{matrix}\right\}$ (in seconds) $= 3.065206$

or $\begin{aligned}\alpha &= 1162'' = 19'\,22''\\ \beta &= 89^\circ\,40'\,38''\end{aligned}$

When $\left\{\begin{matrix}\log\cot\alpha\\ \log\tan\beta\end{matrix}\right\} = 12.167981$

C (for $23'$) $= 15.314419$

$\log\left\{\begin{matrix}\alpha\\90^\circ-\beta\end{matrix}\right\}$ (in seconds) $= 3.146438$

or $\begin{aligned}\alpha &= 1401'' = 23'\,21''\\ \beta &= 89^\circ\,36'\,39''\end{aligned}$

TABLE XIX.—LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS

0° 179°

″	′	Sine.	S.*	T.*	Tang.	Cotang.	C.*	D. 1″.	Cosine.	′
			4.685				15.314			
0	**0**	Inf. neg.	575	575	Inf. neg.	Inf. pos.	425		**10.00 0000**	**60**
60	1	6.46 3726	575	575	6.46 3726	13.53 6274	425		0000	59
120	2	.76 4756	575	575	.76 4756	.23 5244	425		0000	58
180	3	6.94 0847	575	575	6.94 0847	13.05 9153	425		0000	57
240	4	7.06 5786	575	575	7.06 5786	12.93 4214	425		0000	56
300	**5**	**7.16 2696**	575	575	**7.16 2696**	**12.83 7304**	425	.02	**10.00 0000**	**55**
360	6	.24 1877	575	575	.24 1878	.75 8122	425	.00	9.99 9999	54
420	7	.30 8824	575	575	.30 8825	.69 1175	425	.00	9999	53
480	8	.36 6816	574	576	.36 6817	.63 3183	424	.00	9999	52
540	9	.41 7968	574	576	.41 7970	.58 2030	424	.02	9999	51
600	**10**	**7.46 3726**	574	576	**7.46 3727**	**12.53 6273**	424	.00	**9.99 9998**	**50**
660	11	.50 5118	574	576	.50 5120	.49 4880	424	.02	9998	49
720	12	.54 2906	574	577	.54 2909	.45 7091	423	.00	9997	48
780	13	.57 7668	574	577	.57 7672	.42 2328	423	.02	9997	47
840	14	.60 9853	574	577	.60 9857	.39 0143	423	.00	9996	46
900	**15**	**7.63 9816**	573	578	**7.63 9820**	**12.36 0180**	422	.02	**9.99 9996**	**45**
960	16	.66 7845	573	578	.66 7849	.33 2151	422	.00	9995	44
1020	17	.69 4173	573	578	.69 4179	.30 5821	422	.02	9995	43
1080	18	.71 8997	573	579	.71 9003	.28 0997	421	.02	9994	42
1140	19	.74 2478	573	579	.74 2484	.25 7516	421	.00	9993	41
1200	**20**	**7.76 4754**	572	580	**7.76 4761**	**12.23 5239**	420	.02	**9.99 9993**	**40**
1260	21	.78 5943	572	580	.78 5951	.21 4049	420	.02	9992	39
1320	22	.80 6146	572	581	.80 6155	.19 3845	419	.02	9991	38
1380	23	.82 5451	572	581	.82 5460	.17 4540	419	.02	9990	37
1440	24	.84 3934	571	582	.84 3944	.15 6056	418	.00	9989	36
1500	**25**	**7.86 1662**	571	583	**7.86 1674**	**12.13 8326**	417	.02	**9.99 9989**	**35**
1560	26	.87 8695	571	583	.87 8708	.12 1202	417	.02	9988	34
1620	27	.89 5085	570	584	.89 5099	.10 4901	416	.02	9987	33
1680	28	.01 0879	570	584	.91 0894	.08 9106	416	.02	9986	32
1740	29	.92 6119	570	585	.92 6134	.07 3866	415	.03	9985	31
1800	**30**	**7.94 0842**	569	586	**7.94 0858**	**12.05 9142**	414	.02	**9.99 9983**	**30**
1860	31	.95 5082	569	587	.95 5100	.04 4900	413	.02	9982	29
1920	32	.96 8870	569	587	.96 8889	.03 1111	413	.02	9981	28
1980	33	.98 2233	568	588	.98 2253	.01 7747	412	.02	9980	27
2040	34	7.99 5198	568	589	7.99 5219	12.00 4781	411	.03	9979	26
2100	**35**	**8.00 7787**	567	590	**8.00 7809**	**11.99 2191**	410	.02	**9.99 9977**	**25**
2160	36	.02 0021	567	591	.02 0044	.97 9956	409	.02	9976	24
2220	37	.03 1919	566	592	.03 1945	.96 8055	408	.03	9975	23
2280	38	.04 3501	566	593	.04 3527	.95 6473	407	.02	9973	22
2340	39	.05 4781	566	593	.05 4809	.94 5191	407	.02	9972	21
2400	**40**	**8.06 5576**	565	594	**8.06 5806**	**11.93 4194**	406	.03	**9.99 9971**	**20**
2460	41	.07 6500	565	595	.07 6531	.92 3460	405	.02	9969	19
2520	42	.08 6965	564	596	.08 6997	.91 3003	404	.03	9968	18
2580	43	.00 7183	564	598	.09 7217	.90 2783	402	.03	9966	17
2640	44	.10 7167	563	599	.10 7203	.89 2797	401	.02	9964	16
2700	**45**	**8.11 6926**	562	600	**8.11 6963**	**11.88 3037**	400	.03	**9.99 9963**	**15**
2760	46	.12 6471	562	601	.12 6510	.87 3490	399	.03	9961	14
2820	47	.13 5810	561	602	.13 5851	.86 4149	398	.02	9959	13
2880	48	.14 4953	561	603	.14 4996	.85 5004	397	.03	9958	12
2940	49	.15 3907	560	604	.15 3952	.84 6048	396	.03	9956	11
3000	**50**	**8.16 2681**	560	605	**8.16 2727**	**11.83 7273**	395	.03	**9.99 9954**	**10**
3060	51	.17 1280	559	607	.17 1328	.82 8672	393	.03	9952	9
3120	52	.17 9713	558	608	.17 9763	.82 0237	392	.03	9950	8
3180	53	.18 7985	558	609	.18 8036	.81 1964	391	.03	9948	7
3240	54	.19 6102	557	611	.19 6156	.80 3844	389	.03	9946	6
3300	**55**	**8.20 4070**	556	612	**8.20 4126**	**11.79 5874**	388	.03	**9.99 9944**	**5**
3360	56	.21 1805	556	613	.21 1953	.78 8047	387	.03	9942	4
3420	57	.21 9581	555	615	.21 9641	.78 0359	385	.03	9940	3
3480	58	.22 7134	554	616	.22 7195	.77 2805	384	.03	9938	2
3540	59	.23 4557	554	618	.23 4621	.76 5379	382	.03	9936	1
3600	**60**	**8.24 1855**	553	610	**8.24 1921**	**11.75 8079**	381		**9.99 9934**	**0**
			4.685				15.314			
″	′	Cosine.	S.*	T.*	Cotang.	Tang.	C.*	D. 1″.	Sine.	′

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* For use of S, T, and C see Table XIXa, page 933.

1° TABLE XIX.—*Continued* 178°

″	′	Sine.	S.*	T.*	Tang.	Cotang.	C.*	D. 1″.	Cosine.	′
			4.685				15.314			
3600	**0**	**8.24 1855**	553	619	**8.24 1921**	**11.75 8079**	381	.03	**9.99 9934**	**60**
3660	1	.24 9033	552	620	.24 9102	.75 0898	380	.05	9932	59
3720	2	.25 6094	551	622	.25 6165	.74 3835	378	.03	9929	58
3780	3	.26 3042	551	623	.26 3115	.73 6885	377	.03	9927	57
3840	4	.26 9881	550	625	.26 9956	.73 0044	375	.05	9925	56
3900	**5**	**8.27 6614**	549	627	**8.27 6691**	**11.72 3309**	373	.03	**9.99 9922**	**55**
3960	6	.28 3243	548	628	.28 3323	.71 6677	372	.03	9920	54
4020	7	.28 9773	547	630	.28 9856	.71 0144	370	.05	9918	53
4080	8	.29 6207	546	632	.29 6292	.70 3708	368	.03	9915	52
4140	9	.30 2546	546	633	.30 2634	.69 7366	367	.05	9913	51
4200	**10**	**8.30 8794**	545	635	**8.30 8884**	**11.69 1116**	365	.05	**9.99 9910**	**50**
4260	11	.31 4954	544	637	.31 5046	.68 4954	363	.03	9907	49
4320	12	.32 1027	543	638	.32 1122	.67 8878	362	.05	9905	48
4380	13	.32 7016	542	640	.32 7114	.67 2886	360	.05	9902	47
4440	14	.33 2924	541	642	.33 3025	.66 6975	358	.03	9899	46
4500	**15**	**8.33 8753**	540	644	**8.33 8856**	**11.66 1144**	356	.05	**9.99 9897**	**45**
4560	16	.34 4504	539	646	.34 4610	.65 5390	354	.05	9894	44
4620	17	.35 0181	539	648	.35 0289	.64 9711	352	.05	9891	43
4680	18	.35 5783	538	649	.35 5895	.64 4105	351	.05	9888	42
4740	19	.36 1315	537	651	.36 1430	.63 8570	349	.05	9885	41
4800	**20**	**8.36 6777**	536	653	**8.36 6895**	**11.63 3105**	347	.05	**9.99 9882**	**40**
4860	21	.37 2171	535	655	.37 2292	.62 7708	345	.05	9879	39
4920	22	.37 7499	534	657	.37 7622	.62 2378	343	.05	9876	38
4980	23	.38 2762	533	659	.38 2889	.61 7111	341	.05	9873	37
5040	24	.38 7962	532	661	.38 8092	.61 1908	339	.05	9870	36
5100	**25**	**8.39 3101**	531	663	**8.39 3234**	**11.60 6766**	337	.05	**9.99 9867**	**35**
5160	26	.39 8179	530	666	.39 8315	.60 1685	334	.05	9864	34
5220	27	.40 3199	529	668	.40 3338	.59 6662	332	.05	9861	33
5280	28	.40 8161	527	670	.40 8304	.59 1696	330	.07	9858	32
5340	29	.41 3068	526	672	.41 3213	.58 6787	328	.05	9854	31
5400	**30**	**8.41 7919**	525	674	**8.41 8068**	**11.58 1932**	326	.05	**9.99 9851**	**30**
5460	31	.42 2717	524	676	.42 2869	.57 7131	324	.07	9848	29
5520	32	.42 7462	523	679	.42 7618	.57 2382	321	.05	9844	28
5580	33	.43 2156	522	681	.43 2315	.56 7685	319	.05	9841	27
5640	34	.43 6800	521	683	.43 6962	.56 3038	317	.07	9838	26
5700	**35**	**8.44 1394**	520	685	**8.44 1560**	**11.55 8440**	315	.05	**9.99 9834**	**25**
5760	36	.44 5941	518	688	.44 6110	.55 3890	312	.07	9831	24
5820	37	.45 0440	517	690	.45 0613	.54 9387	310	.05	9827	23
5880	38	4893	516	693	5070	4930	307	.07	9824	22
5940	39	.45 9301	515	695	.45 9481	.54 0519	305	.07	9820	21
6000	**40**	**8.46 3665**	514	697	**8.46 3849**	**11.53 6151**	303	.05	**9.99 9816**	**20**
6060	41	.46 7985	512	700	.46 8172	.53 1828	300	.07	9813	19
6120	42	.47 2263	511	702	.47 2454	.52 7546	298	.07	9809	18
6180	43	.47 6498	510	705	.47 6693	.52 3307	295	.07	9805	17
6240	44	.48 0693	509	707	.48 0892	.51 9108	293	.07	9801	16
6300	**45**	**8.48 4848**	507	710	**8.48 5050**	**11.51 4950**	290	.05	**9.99 9797**	**15**
6360	46	.48 8963	506	713	.48 9170	.51 0830	287	.07	9794	14
6420	47	.49 3040	505	715	.49 3250	.50 6750	285	.07	9790	13
6480	48	.49 7078	503	718	.49 7293	.50 2707	282	.07	9786	12
6540	49	.50 1080	502	720	.50 1298	.49 8702	280	.07	9782	11
6600	**50**	**8.50 5045**	501	723	**8.50 5267**	**11.49 4733**	277	.07	**9.99 9778**	**10**
6660	51	.50 8974	499	726	.50 9200	.49 0800	274	.08	9774	9
6720	52	.51 2867	498	729	.51 3098	.48 6902	271	.07	9769	8
6780	53	.51 6726	497	731	.51 6961	.48 3039	269	.07	9765	7
6840	54	.52 0551	495	734	.52 0790	.47 9210	266	.07	9761	6
6900	**55**	**8.52 4343**	494	737	**8.52 4586**	**11.47 5414**	263	.07	**9.99 9757**	**5**
6960	56	.52 8102	492	740	.52 8349	.47 1651	260	.08	9753	4
7020	57	.53 1828	491	743	.53 2080	.46 7920	257	.07	9748	3
7080	58	5523	490	745	5779	4221	255	.07	9744	2
7140	59	.53 9186	488	748	.53 9447	.46 0553	252	.08	9740	1
7200	**60**	**8.54 2819**	487	751	**8.54 3084**	**11.45 6916**	249		**9.99 9735**	**0**
			4.685				15.314			
″	′	Cosine.	S.*	T.*	Cotang.	Tang.	C.*	D. 1″.	Sine.	′

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* For use of *S*, *T*, and *C* see Table XIXa, page 933.

2° TABLE XIX.—*Continued.* 177°

′	Sine.	D. 1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**8.54 2819**	60.05	**9.99 9735**	.07	**8.54 3084**	60.12	**11.45 6916**	**60**
1	6422	59.55	9731	.08	.54 6691	59.62	.45 3309	59
2	.54 9995	59.07	9726	.07	.55 0268	59.15	.44 9732	58
3	.55 3539	58.58	9722	.08	3817	58.65	6183	57
4	.55 7054	58.10	9717	.07	.55 7336	58.20	.44 2664	56
5	**8.56 0540**	57.65	**9.99 9713**	.08	**8.56 0828**	57.72	**11.43 9172**	**55**
6	3999	57.20	9708	.07	4291	57.27	5709	54
7	.56 7431	56.75	9704	.08	.56 7727	56.83	.43 2273	53
8	.57 0836	56.30	9699	.08	.57 1137	56.38	.42 8863	52
9	4214	55.87	9694	.08	4520	55.95	5480	51
10	**8.57 7566**	55.43	**9.99 9689**	.07	**8.57 7877**	55.52	**11.42 2123**	**50**
11	.58 0892	55.02	9685	.08	.58 1208	55.10	.41 8792	49
12	4193	54.60	9680	.08	4514	54.68	5486	48
13	.58 7469	54.20	9675	.08	.58 7795	54.27	.41 2205	47
14	.59 0721	53.78	9670	.08	.59 1051	53.87	.40 8949	46
15	**8.59 3948**	53.40	**9.99 9665**	.08	**8.59 4283**	53.48	**11.40 5717**	**45**
16	.59 7152	53.00	9660	.08	.59 7492	53.08	.40 2508	44
17	.60 0332	52.62	9655	.08	.60 0677	52.70	.39 9323	43
18	3489	52.23	9650	.08	3839	52.32	6161	42
19	6623	51.85	9645	.08	.60 6978	51.93	.39 3022	41
20	**8.60 9734**	51.48	**9.99 9640**	.08	**8.61 0094**	51.58	**11.38 9906**	**40**
21	.61 2823	51.13	9635	.10	3189	51.22	6811	39
22	5891	50.77	9629	.08	6262	50.85	3738	38
23	.61 8937	50.42	9624	.08	.61 9313	50.50	.38 0687	37
24	.62 1962	50.05	9619	.08	.62 2343	50.15	.37 7657	36
25	**8.62 4965**	49.72	**9.99 9614**	.10	**8.62 5352**	49.80	**11.37 4648**	**35**
26	.62 7948	49.38	9608	.08	.62 8340	49.47	.37 1660	34
27	.63 0911	49.05	9603	.10	.63 1308	49.13	.36 8692	33
28	3854	48.70	9597	.08	4256	48.80	5744	32
29	6776	48.40	9592	.10	.63 7184	48.48	.36 2816	31
30	**8.63 9680**	48.05	**9.99 9586**	.08	**8.64 0093**	48.15	**11.35 9907**	**30**
31	.64 2563	47.75	9581	.10	2982	47.85	7018	29
32	5428	47.43	9575	.08	5853	47.52	4147	28
33	.64 8274	47.13	9570	.10	.64 8704	47.22	.35 1296	27
34	.65 1102	46.82	9564	.10	.65 1537	46.92	.34 8463	26
35	**8.65 3911**	46.52	**9.99 9558**	.08	**8.65 4352**	46.62	**11.34 5648**	**25**
36	6702	46.22	9553	.10	7149	46.32	2851	24
37	.65 9475	45.92	9547	.10	.65 9928	46.02	.34 0072	23
38	.66 2230	45.63	9541	.10	.66 2689	45.73	.33 7311	22
39	4968	45.35	9535	.10	5433	45.45	4567	21
40	**8.66 7689**	45.07	**9.99 9529**	.08	**8.66 8160**	45.17	**11.33 1840**	**20**
41	.67 0393	44.78	9524	.10	.67 0870	44.88	.32 9130	19
42	3080	44.52	9518	.10	3563	44.60	6437	18
43	5751	44.23	9512	.10	6239	44.35	3761	17
44	.67 8405	43.97	9506	.10	.67 8900	44.07	.32 1100	16
45	**8.68 1043**	43.70	**9.99 9500**	.12	**8.68 1544**	43.80	**11.31 8456**	**15**
46	3665	43.45	9493	.10	4172	43.53	5828	14
47	6272	43.18	9487	.10	6784	43.28	3216	13
48	.68 8863	42.92	9481	.10	.68 9381	43.03	.31 0619	12
49	.69 1438	42.67	9475	.10	.69 1963	42.77	.30 8037	11
50	**8.69 3998**	42.42	**9.99 9469**	.10	**8.69 4529**	42.53	**11.30 5471**	**10**
51	6543	42.17	9463	.12	7081	42.27	2919	9
52	.69 9073	41.93	9456	.10	.69 9617	42.03	.30 0383	8
53	.70 1589	41.68	9450	.12	.70 2139	41.78	.29 7861	7
54	4090	41.45	9443	.10	4646	41.57	5354	6
55	**8.70 6577**	41.20	**9.99 9437**	.10	**8.70 7140**	41.30	**11.29 2860**	**5**
56	.70 9049	40.97	9431	.12	.70 9618	41.08	.29 0382	4
57	.71 1507	40.75	9424	.10	.71 2083	40.85	.28 7917	3
58	3952	40.52	9418	.12	4534	40.63	5466	2
59	6383	40.28	9411	.12	6972	40.40	3028	1
60	**8.71 8800**		**9.99 9404**		**8.71 9396**		**11.28 0604**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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3° TABLE XIX.—*Continued.* 176°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**8.71 8800**	40.07	**9.99 9404**	.10	**8.71 9396**	40.17	**11.28 0604**	**60**
1	.72 1204	39.85	9398	.12	.72 1806	39.97	.27 8194	59
2	3595	39.62	9391	.12	4204	39.73	5796	58
3	5972	39.42	9384	.10	6588	39.52	3412	57
4	.72 8337	39.18	9378	.12	.72 8959	39.30	.27 1041	56
5	**8.73 0688**	38.98	**9.99 9371**	.12	**8.73 1317**	39.10	**11.26 8683**	**55**
6	3027	38.78	9364	.12	3663	38.88	6337	54
7	5354	38.55	9357	.12	5996	38.68	4004	53
8	7667	38.37	9350	.12	.73 8317	38.48	.26 1683	52
9	.73 9969	38.17	9343	.12	.74 0626	38.27	.25 9374	51
10	**8.74 2259**	37.95	**9.99 9336**	.12	**8.74 2922**	38.08	**11.25 7078**	**50**
11	4536	37.77	9329	.12	5207	37.87	4793	49
12	6802	37.55	9322	.12	7479	37.68	2521	48
13	.74 9055	37.37	9315	.12	.74 9740	37.48	.25 0260	47
14	.75 1297	37.18	9308	.12	.75 1989	37.30	.24 8011	46
15	**8.75 3528**	36.98	**9.99 9301**	.12	**8.75 4227**	37.10	**11.24 5773**	**45**
16	5747	36.80	9294	.12	6453	36.92	3547	44
17	.75 7955	36.60	9287	.13	.75 8668	36.73	.24 1332	43
18	.76 0151	36.43	9279	.12	.76 0872	36.55	.23 9128	42
19	2337	36.23	9272	.12	3065	36.35	6935	41
20	**8.76 4511**	36.07	**9.99 9265**	.13	**8.76 5246**	36.18	**11.23 4754**	**40**
21	6675	35.88	9257	.12	7417	36.02	2583	39
22	.76 8828	35.70	9250	.13	.76 9578	35.82	.23 0422	38
23	.77 0970	35.52	9242	.12	.77 1727	35.65	.22 8273	37
24	3101	35.37	9235	.13	3866	35.48	6134	36
25	**8.77 5223**	35.17	**9.99 9227**	.12	**8.77 5995**	35.32	**11.22 4005**	**35**
26	7333	35.02	9220	.13	.77 8114	35.13	.22 1886	34
27	.77 9434	34.83	9212	.12	.78 0222	34.97	.21 9778	33
28	.78 1524	34.68	9205	.13	2320	34.80	7680	32
29	3605	34.50	9197	.13	4408	34.63	5592	31
30	**8.78 5675**	34.35	**9.99 9189**	.13	**8.78 6486**	34.47	**11.21 3514**	**30**
31	7736	34.18	9181	.12	.78 8554	34.32	.21 1446	29
32	.78 9787	34.02	9174	.13	.79 0613	34.15	.20 9387	28
33	.79 1828	33.85	9166	.13	2662	33.98	7338	27
34	3859	33.70	9158	.13	4701	33.83	5299	26
35	**8.79 5881**	33.55	**9.99 9150**	.13	**8.79 6731**	33.68	**11.20 3269**	**25**
36	7894	33.38	9142	.13	.79 8752	33.52	.20 1248	24
37	.79 9897	33.25	9134	.13	.80 0763	33.37	.19 9237	23
38	.80 1892	33.07	9126	.13	2765	33.22	7235	22
39	3876	32.93	9118	.13	4758	33.07	5242	21
40	**8.80 5852**	32.78	**9.99 9110**	.13	**8.80 6742**	32.92	**11.19 3258**	**20**
41	7819	32.63	9102	.13	.80 8717	32.77	.19 1283	19
42	.80 9777	32.48	9094	.13	.81 0683	32.63	.18 9317	18
43	.81 1726	32.35	9086	.15	2641	32.47	7359	17
44	3667	32.20	9077	.13	4589	32.33	5411	16
45	**8.81 5599**	32.05	**9.99 9069**	.13	**8.81 6529**	32.20	**11.18 3471**	**15**
46	7522	31.90	9061	.13	.81 8461	32.05	.18 1539	14
47	.81 9436	31.78	9053	.15	.82 0384	31.90	.17 9616	13
48	.82 1343	31.62	9044	.13	2298	31.78	7702	12
49	3240	31.50	9036	.15	4205	31.63	5795	11
50	**8.82 5130**	31.35	**9.99 9027**	.13	**8.82 6103**	31.48	**11.17 3897**	**10**
51	7011	31.22	9019	.15	7992	31.37	2008	9
52	.82 8884	31.08	9010	.13	.82 9874	31.23	.17 0126	8
53	.83 0749	30.97	9002	.15	.83 1748	31.08	.16 8252	7
54	2607	30.82	8993	.15	3613	30.97	6387	6
55	**8.83 4456**	30.68	**9.99 8984**	.13	**8.83 5471**	30.83	**11.16 4529**	**5**
56	6297	30.55	8976	.15	7321	30.70	2679	4
57	8130	30.43	8967	.15	.83 9163	30.58	.16 0837	3
58	.83 9956	30.30	8958	.13	.84 0998	30.45	.15 9002	2
59	.84 1774	30.18	8950	.15	2825	30.32	7175	1
60	**8.84 3585**		**9.99 8941**		**8.84 4644**		**11.15 5356**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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4° TABLE XIX.—*Continued.* 175°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**8.84 3585**	30.03	**9.99 8941**	.15	**8.84 4644**	30.18	**11.15 5356**	**60**
1	5387	29.93	8932	.15	6455	30.08	3545	59
2	7183	29.80	8923	.15	.84 8260	29.95	.15 1740	58
3	.84 8971	29.67	8914	.15	.85 0057	29.82	.14 9943	57
4	.85 0751	29.57	8905	.15	1846	29.70	8154	56
5	**8.85 2525**	29.43	**9.99 8896**	.15	**8.85 3628**	29.58	**11.14 6372**	**55**
6	4291	29.30	8887	.15	5403	29.47	4597	54
7	6049	29.20	8878	.15	7171	29.35	2829	53
8	7801	29.08	8869	.15	.85 8932	29.23	.14 1068	52
9	.85 9546	28.95	8860	.15	.86 0686	29.12	.13 9314	51
10	**8.86 1283**	28.85	**9.99 8851**	.17	**8.86 2433**	29.00	**11.13 7567**	**50**
11	3014	28.73	8841	.15	4173	28.88	5827	49
12	4738	28.62	8832	.15	5906	28.77	4094	48
13	6455	28.50	8823	.17	7632	28.65	2368	47
14	8165	28.38	8813	.15	.86 9351	28.55	.13 0649	46
15	**8.86 9868**	28.28	**9.99 8804**	.15	**8.87 1064**	28.43	**11.12 8936**	**45**
16	.87 1565	28.17	8795	.17	2770	28.32	7230	44
17	3255	28.05	8785	.15	4469	28.22	5531	43
18	4938	27.95	8776	.17	6162	28.12	3838	42
19	6615	27.83	8766	.15	7849	28.00	2151	41
20	**8.87 8285**	27.73	**9.99 8757**	.17	**8.87 9529**	27.88	**11.12 0471**	**40**
21	.87 9949	27.63	8747	.15	.88 1202	27.78	.11 8798	39
22	.88 1607	27.52	8738	.17	2869	27.68	7131	38
23	3258	27.42	8728	.17	4530	27.58	5470	37
24	4903	27.32	8718	.17	6185	27.47	3815	36
25	**8.88 6542**	27.20	**9.99 8708**	.15	**8.88 7833**	27.88	**11.11 2167**	**35**
26	8174	27.12	8699	.17	.88 9476	27.27	.11 0524	34
27	.88 9801	27.00	8689	.17	.89 1112	27.17	.10 8888	33
28	.89 1421	26.90	8679	.17	2742	27.07	7258	32
29	3035	26.80	8669	.17	4366	26.97	5634	31
30	**8.89 4643**	26.72	**9.99 8659**	.17	**8.89 5984**	26.87	**11.10 4016**	**30**
31	6246	26.60	8649	.17	7596	26.78	2404	29
32	7842	26.50	8639	.17	.89 9203	26.67	.10 0797	28
33	.89 9432	26.42	8629	.17	.90 0803	26.58	.09 9197	27
34	.90 1017	26.32	8619	.17	2398	26.48	7602	26
35	**8.90 2596**	26.22	**9.99 8609**	.17	**8.90 3987**	26.38	**11.09 6013**	**25**
36	4169	26.12	8599	.17	5570	26.28	4430	24
37	5736	26.02	8589	.18	7147	26.20	2853	23
38	7297	25.93	8578	.17	.90 8719	26.10	.09 1281	22
39	.90 8853	25.85	8568	.17	.91 0285	26.02	.08 9715	21
40	**8.91 0404**	25.75	**9.99 8558**	.17	**8.91 1846**	25.92	**11.08 8154**	**20**
41	1949	25.65	8548	.18	3401	25.83	6599	19
42	3488	25.57	8537	.17	4951	25.73	5049	18
43	5022	25.47	8527	.18	6495	25.63	3505	17
44	6550	25.38	8516	.17	8034	25.57	1966	16
45	**8.91 8073**	25.30	**9.99 8506**	.18	**8.91 9568**	25.47	**11.08 0432**	**15**
46	.91 9591	25.20	8495	.17	.92 1096	25.38	.07 8904	14
47	.92 1103	25.12	8485	.18	2619	25.28	7381	13
48	2610	25.03	8474	.17	4136	25.22	5864	12
49	4112	24.95	8464	.18	5649	25.12	4351	11
50	**8.92 5609**	24.85	**9.99 8453**	.18	**8.92 7156**	25.03	**11.07 2844**	**10**
51	7100	24.78	8442	.18	.92 8658	24.95	.07 1342	9
52	.92 8587	24.68	8431	.17	.93 0155	24.87	.06 9845	8
53	.93 0068	24.60	8421	.18	1647	24.78	8353	7
54	1544	24.52	8410	.18	3134	24.70	6866	6
55	**8.93 3015**	24.43	**9.99 8399**	.18	**8.93 4616**	24.62	**11.06 5384**	**5**
56	4481	24.35	8388	.18	6093	24.53	3907	4
57	5942	24.27	8377	.18	7565	24.45	2435	3
58	7398	24.20	8366	.18	.93 9032	24.37	.06 0968	2
59	.93 8850	24.10	8355	.18	.94 0494	24.30	.05 9506	1
60	**8.94 0296**		**9.99 8344**		**8.94 1952**		**11.05 8048**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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5° TABLE XIX.—*Continued.* 174°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**8.94 0296**	24.03	**9.99 8344**	.18	**8.94 1952**	24.20	**11.05 8048**	**60**
1	1738	23.93	8333	.18	3404	24.13	6596	59
2	3174	23.87	8322	.18	4852	24.05	5148	58
3	4606	23.80	8311	.18	6295	23.98	3705	57
4	6034	23.70	8300	.18	7734	23.90	2266	56
5	**8.94 7456**	23.63	**9.99 8289**	.20	**8.94 9168**	23.82	**11.05 0832**	**55**
6	.94 8874	23.55	8277	.18	.95 0597	23.73	.04 9403	54
7	.95 0287	23.48	8266	.18	2021	23.67	7979	53
8	1696	23.40	8255	.20	3441	23.58	6559	52
9	3100	23.32	8243	.18	4856	23.52	5144	51
10	**8.95 4499**	23.25	**9.99 8232**	.20	**8.95 6267**	23.45	**11.04 3733**	**50**
11	5894	23.17	8220	.18	7674	23.35	2326	49
12	7284	23.10	8209	.20	.95 9075	23.30	.04 0925	48
13	.95 8670	23.03	8197	.18	.96 0473	23.22	.03 9527	47
14	.96 0052	22.95	8186	.20	1866	23.15	8134	46
15	**8.96 1429**	22.87	**9.99 8174**	.18	**8.96 3255**	23.07	**11.03 6745**	**45**
16	2801	22.82	8163	.20	4639	23.00	5361	44
17	4170	22.73	8151	.20	6019	22.92	3981	43
18	5534	22.65	8139	.18	7394	22.87	2606	42
19	6893	22.60	8128	.20	.96 8766	22.78	.03 1234	41
20	**8.96 8249**	22.52	**9.99 8116**	.20	**8.97 0133**	22.72	**11.02 9867**	**40**
21	.96 9600	22.45	8104	.20	1496	22.65	8504	39
22	.97 0947	22.37	8092	.20	2855	22.57	7145	38
23	2289	22.32	8080	.20	4209	22.52	5791	37
24	3628	22.23	8068	.20	5560	22.43	4440	36
25	**8.97 4962**	22.18	**9.99 8056**	.20	**8.97 6906**	22.37	**11.02 3094**	**35**
26	6293	22.10	8044	.20	8248	22.30	1752	34
27	7619	22.03	8032	.20	.97 9586	22.25	.02 0414	33
28	.97 8941	21.97	8020	.20	.98 0921	22.17	.01 9079	32
29	.98 0259	21.90	8008	.20	2251	22.10	7749	31
30	**8.98 1573**	21.83	**9.99 7996**	.20	**8.98 3577**	22.03	**11.01 6423**	**30**
31	2883	21.77	7984	.20	4899	21.97	5101	29
32	4189	21.72	7972	.22	6217	21.92	3783	28
33	5491	21.63	7959	.20	7532	21.83	2468	27
34	6789	21.57	7947	.20	.98 8842	21.78	.01 1158	26
35	**8.98 8083**	21.52	**9.99 7935**	.22	**8.99 0149**	21.70	**11.00 9851**	**25**
36	.98 9374	21.43	7922	.20	1451	21.65	8549	24
37	.99 0660	21.38	7910	.22	2750	21.58	7250	23
38	1943	21.32	7897	.20	4045	21.53	5955	22
39	3222	21.25	7885	.22	5337	21.45	4663	21
40	**8.99 4497**	21.18	**9.99 7872**	.20	**8.99 6624**	21.40	**11.00 3376**	**20**
41	5768	21.13	7860	.22	7908	21.33	2092	19
42	7036	21.05	7847	.20	8.99 9188	21.28	11.00 0812	18
43	8299	21.02	7835	.22	9.00 0465	21.22	10.99 9535	17
44	8.99 9560	20.93	7822	.22	1738	21.15	8262	16
45	**9.00 0816**	20.88	**9.99 7809**	.20	**9.00 3007**	21.08	**10.99 6993**	**15**
46	2069	20.82	7797	.22	4272	21.03	5728	14
47	3318	20.75	7784	.22	5534	20.97	4466	13
48	4563	20.70	7771	.22	6792	20.92	3208	12
49	5805	20.65	7758	.22	8047	20.85	1953	11
50	**9.00 7044**	20.57	**9.99 7745**	.22	**9.00 9298**	20.80	**10.99 0702**	**10**
51	8278	20.53	7732	.22	.01 0546	20.73	.98 9454	9
52	.00 9510	20.45	7719	.22	1790	20.68	8210	8
53	.01 0737	20.42	7706	.22	3031	20.62	6969	7
54	1962	20.33	7693	.22	4268	20.57	5732	6
55	**9.01 3182**	20.30	**9.99 7680**	.22	**9.01 5502**	20.50	**10.98 4498**	**5**
56	4400	20.22	7667	.22	6732	20.45	3268	4
57	5613	20.18	7654	.22	7959	20.40	2041	3
58	6824	20.12	7641	.22	.01 9183	20.33	.98 0817	2
59	8031	20.07	7628	.23	.02 0403	20.28	.97 9597	1
60	**9.01 9235**		**9.99 7614**		**9.02 1620**		**10.97 8380**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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6° TABLE XIX.—*Continued.* 173°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.01 9235**	20.00	**9.99 7614**	.22	**9.02 1620**	20.23	**10.97 8380**	60
1	.02 0435	19.95	7601	.22	2834	20.17	7166	59
2	1632	19.88	7588	.23	4044	20.12	5956	58
3	2825	19.85	7574	.22	5251	20.07	4749	57
4	4016	19.78	7561	.23	6455	20.00	3545	56
5	**9.02 5203**	19.72	**9.99 7547**	.22	**9.02 7655**	19.95	**10.97 2345**	55
6	6386	19.68	7534	.23	.02 8852	19.90	.97 1148	54
7	7567	19.62	7520	.22	.03 0046	19.85	.96 9954	53
8	8744	19.57	7507	.23	1237	19.80	8763	52
9	.02 9918	19.52	7493	.22	2425	19.73	7575	51
10	**9.03 1089**	19.47	**9.99 7480**	.23	**9.03 3609**	19.70	**10.96 6391**	50
11	2257	19.40	7466	.23	4791	19.63	5209	49
12	3421	19.35	7452	.22	5969	19.58	4031	48
13	4582	19.32	7439	.23	7144	19.53	2856	47
14	5741	19.25	7425	.23	8316	19.48	1684	46
15	**9.03 6896**	19.20	**9.99 7411**	.23	**9.03 9485**	19.43	**10.96 0515**	45
16	8048	19.15	7397	.23	.04 0651	19.37	.95 9349	44
17	.03 9197	19.08	7383	.23	1813	19.33	8187	43
18	.04 0342	19.05	7369	.23	2973	19.28	7027	42
19	1485	19.00	7355	.23	4130	19.23	5870	41
20	**9.04 2625**	18.95	**9.99 7341**	.23	**9.04 5284**	19.17	**10.95 4716**	40
21	3762	18.88	7327	.23	6434	19.13	3566	39
22	4895	18.85	7313	.23	7582	19.08	2418	38
23	6026	18.80	7299	.23	8727	19.03	1273	37
24	7154	18.75	7285	.23	.04 9869	18.98	.95 0131	36
25	**9.04 8279**	18.68	**9.99 7271**	.23	**9.05 1008**	18.93	**10.94 8992**	35
26	.04 9400	18.65	7257	.25	2144	18.88	7856	34
27	.05 0519	18.60	7242	.23	3277	18.83	6723	33
28	1635	18.57	7228	.23	4407	18.80	5593	32
29	2749	18.50	7214	.25	5535	18.73	4465	31
30	**9.05 3859**	18.45	**9.99 7199**	.23	**9.05 6659**	18.70	**10.94 3341**	30
31	4966	18.42	7185	.25	7781	18.65	2219	29
32	6071	18.35	7170	.23	.05 8900	18.60	.94 1100	28
33	7172	18.32	7156	.25	.06 0016	18.57	.93 9984	27
34	8271	18.27	7141	.23	1130	18.50	8870	26
35	**9.05 9367**	18.22	**9.99 7127**	.25	**9.06 2240**	18.47	**10.93 7760**	25
36	.06 0460	18.18	7112	.23	3348	18.42	6652	24
37	1551	18.13	7098	.25	4453	18.38	5547	23
38	2639	18.08	7083	.25	5556	18.32	4444	22
39	3724	18.03	7068	.25	6655	18.28	3345	21
40	**9.06 4806**	17.98	**9.99 7053**	.23	**9.06 7752**	18.25	**10.93 2248**	20
41	5885	17.95	7039	.25	8846	18.20	1154	19
42	6962	17.90	7024	.25	.06 9938	18.15	.93 0062	18
43	8036	17.85	7009	.25	.07 1027	18.10	.92 8973	17
44	.06 9107	17.82	6994	.25	2113	18.07	7887	16
45	**9.07 0176**	17.77	**9.99 6979**	.25	**9.07 3197**	18.02	**10.92 6803**	15
46	1242	17.73	6964	.25	4278	17.97	5722	14
47	2306	17.67	6949	.25	5356	17.93	4644	13
48	3366	17.63	6934	.25	6432	17.88	3568	12
49	4424	17.60	6919	.25	7505	17.85	2495	11
50	**9.07 5480**	17.55	**9.99 6904**	.25	**9.07 8576**	17.80	**10.92 1424**	10
51	6533	17.50	6889	.25	.07 9644	17.77	.92 0356	9
52	7583	17.47	6874	.27	.08 0710	17.72	.91 9290	8
53	8631	17.42	6858	.25	1773	17.67	8227	7
54	.07 9676	17.38	6843	.27	2833	17.63	7167	6
55	**9.08 0719**	17.33	**9.99 6828**	.27	**9.08 3891**	17.60	**10.91 6109**	5
56	1759	17.30	6812	.25	4947	17.55	5053	4
57	2797	17.25	6797	.25	6000	17.50	4000	3
58	3832	17.20	6782	.27	7050	17.47	2950	2
59	4864	17.17	6766	.25	8098	17.43	1902	1
60	**9.08 5894**		**9.99 6751**		**9.08 9144**		**10.91 0856**	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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7° TABLE XIX.—*Continued.* 172°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.08 5894**	17.13	**9.99 6751**	.27	**9.08 9144**	17.38	**10.91 0856**	**60**
1	6922	17.08	6735	.25	.09 0187	17.35	.90 9813	59
2	7947	17.05	6720	.27	1228	17.30	8772	58
3	8970	17.00	6704	.27	2266	17.27	7734	57
4	.08 9990	16.97	6688	.25	3302	17.23	6698	56
5	**9.09 1008**	16.93	**9.99 6673**	.27	**9.09 4336**	17.18	**10.90 5664**	**55**
6	2024	16.88	6657	.27	5367	17.13	4633	54
7	3037	16.83	6641	.27	6395	17.12	3605	53
8	4047	16.82	6625	.25	7422	17.07	2578	52
9	5056	16.77	6610	.27	8446	17.03	1554	51
10	**9.09 6062**	16.72	**9.99 6594**	.27	**9.09 9468**	16.98	**10.90 0532**	**50**
11	7065	16.68	6578	.27	.10 0487	16.95	.89 9513	49
12	8066	16.65	6562	.27	1504	16.92	8496	48
13	.09 9065	16.62	6546	.27	2519	16.88	7481	47
14	.10 0062	16.57	6530	.27	3532	16.83	6468	46
15	**9.10 1056**	16.53	**9.99 6514**	.27	**9.10 4542**	16.80	**10.89 5458**	**45**
16	2048	16.48	6498	.27	5550	16.77	4450	44
17	3037	16.47	6482	.28	6556	16.72	3444	43
18	4025	16.42	6465	.27	7559	16.68	2441	42
19	5010	16.37	6449	.27	8560	16.65	1440	41
20	**9.10 5992**	16.35	**9.99 6433**	.27	**9.10 9559**	16.62	**10.89 0441**	**40**
21	6973	16.30	6417	.28	.11 0556	16.58	.88 9444	39
22	7951	16.27	6400	.27	1551	16.53	8449	38
23	8927	16.23	6384	.27	2543	16.50	7457	37
24	.10 9901	16.20	6368	.28	3533	16.47	6467	36
25	**9.11 0873**	16.15	**9.99 6351**	.27	**9.11 4521**	16.43	**10.88 5479**	**35**
26	1842	16.12	6335	.28	5507	16.40	4493	34
27	2809	16.08	6318	.27	6491	16.35	3509	33
28	3774	16.05	6302	.28	7472	16.33	2528	32
29	4737	16.02	6285	.27	8452	16.28	1548	31
30	**9.11 5698**	15.97	**9.99 6269**	.28	**9.11 9429**	16.25	**10.88 0571**	**30**
31	6656	15.95	6252	.28	.12 0404	16.22	.87 9596	29
32	7613	15.90	6235	.27	1377	16.18	8623	28
33	8567	15.87	6219	.28	2348	16.15	7652	27
34	.11 9519	15.83	6202	.28	3317	16.12	6683	26
35	**9.12 0469**	15.80	**9.99 6185**	.28	**9.12 4284**	16.08	**10.87 5716**	**25**
36	1417	15.75	6168	.28	5249	16.03	4751	24
37	2362	15.73	6151	.28	6211	16.02	3789	23
38	3306	15.70	6134	.28	7172	15.97	2828	22
39	4248	15.65	6117	.28	8130	15.95	1870	21
40	**9.12 5187**	15.63	**9.99 6100**	.28	**9.12 9087**	15.90	**10.87 0913**	**20**
41	6125	15.58	6083	.28	.13 0041	15.88	.86 9959	19
42	7060	15.55	6066	.28	0994	15.83	9006	18
43	7993	15.53	6049	.28	1944	15.82	8056	17
44	8925	15.48	6032	.28	2893	15.77	7107	16
45	**9.12 9854**	15.45	**9.99 6015**	.28	**9.13 3839**	15.75	**10.86 6161**	**15**
46	.13 0781	15.42	5998	.30	4784	15.70	5216	14
47	1706	15.40	5980	.28	5726	15.68	4274	13
48	2630	15.35	5963	.28	6667	15.63	3333	12
49	3551	15.32	5946	.30	7605	15.62	2395	11
50	**9.13 4470**	15.28	**9.99 5928**	.28	**9.13 8542**	15.57	**10.86 1458**	**10**
51	5387	15.27	5911	.28	.13 9476	15.55	.86 0524	9
52	6303	15.22	5894	.30	.14 0409	15.52	.85 9591	8
53	7216	15.20	5876	.28	1340	15.48	8660	7
54	8128	15.15	5859	.30	2269	15.45	7731	6
55	**9.13 9037**	15.12	**9.99 5841**	.30	**9.14 3196**	15.42	**10.85 6804**	**5**
56	.13 9944	15.10	5823	.28	4121	15.38	5879	4
57	.14 0850	15.07	5806	.30	5044	15.37	4956	3
58	1754	15.02	5788	.28	5966	15.32	4034	2
59	2655	15.00	5771	.30	6885	15.30	3115	1
60	**9.14 3555**		**9.99 5753**		**9.14 7803**		**10.85 2197**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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8° TABLE XIX.—*Continued.* 171°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.14 3555**	14.97	**9.99 5753**	.30	**9.14 7803**	15.25	**10.85 2197**	**60**
1	4453	14.93	5735	.30	8718	15.23	1282	59
2	5349	14.90	5717	.30	.14 9632	15.20	.85 0368	58
3	6243	14.88	5699	.30	.15 0544	15.17	.84 9456	57
4	7136	14.83	5681	.28	1454	15.15	8546	56
5	**9.14 8026**	14.82	**9.99 5664**	.30	**9.15 2363**	15.10	**10.84 7637**	**55**
6	8915	14.78	5646	.30	3269	15.08	6731	54
7	.14 9802	14.73	5628	.30	4174	15.05	5826	53
8	.15 0686	14.72	5610	.32	5077	15.02	4923	52
9	1569	14.70	5591	.30	5978	14.98	4022	51
10	**9.15 2451**	14.65	**9.99 5573**	.30	**9.15 6877**	14.97	**10.84 3123**	**50**
11	3330	14.63	5555	.30	7775	14.93	2225	49
12	4208	14.58	5537	.30	8671	14.90	1329	48
13	5083	14.57	5519	.30	.15 9565	14.87	.84 0435	47
14	5957	14.55	5501	.32	.16 0457	14.83	.83 9543	46
15	**9.15 6830**	14.50	**9.99 5482**	.30	**9.16 1347**	14.82	**10.83 8653**	**45**
16	7700	14.48	5464	.30	2236	14.78	7764	44
17	8569	14.43	5446	.32	3123	14.75	6877	43
18	.15 9435	14.43	5427	.30	4008	14.73	5992	42
19	.16 0301	14.38	5409	.32	4892	14.70	5108	41
20	**9.16 1164**	14.35	**9.99 5390**	.30	**9.16 5774**	14.67	**10.83 4226**	**40**
21	2025	14.33	5372	.32	6654	14.63	3346	39
22	2885	14.30	5353	.32	7532	14.62	2468	38
23	3743	14.28	5334	.30	8409	14.58	1591	37
24	4600	14.23	5316	.32	.16 9284	14.55	.83 0716	36
25	**9.16 5454**	14.22	**9.99 5297**	.32	**9.17 0157**	14.53	**10.82 9843**	**35**
26	6307	14.20	5278	.30	1029	14.50	8971	34
27	7159	14.15	5260	.32	1899	14.47	8101	33
28	8008	14.13	5241	.32	2767	14.45	7233	32
29	8856	14.10	5222	.32	3634	14.42	6366	31
30	**9.16 9702**	14.08	**9.99 5203**	.32	**9.17 4499**	14.38	**10.82 5501**	**30**
31	.17 0547	14.03	5184	.32	5362	14.37	4638	29
32	1389	14.02	5165	.32	6224	14.33	3776	28
33	2230	14.00	5146	.32	7084	14.30	2916	27
34	3070	13.97	5127	.32	7942	14.28	2058	26
35	**9.17 3908**	13.93	**9.99 5108**	.32	**9.17 8799**	14.27	**10.82 1201**	**25**
36	4744	13.90	5089	.32	.17 9655	14.22	.82 0345	24
37	5578	13.88	5070	.32	.18 0508	14.20	.81 9492	23
38	6411	13.85	5051	.32	1360	14.18	8640	22
39	7242	13.83	5032	.32	2211	14.13	7789	21
40	**9.17 8072**	13.80	**9.99 5013**	.33	**9.18 3059**	14.13	**10.81 6941**	**20**
41	8900	13.77	4993	.32	3907	14.08	6093	19
42	.17 9726	13.75	4974	.32	4752	14.08	5248	18
43	.18 0551	13.72	4955	.33	5597	14.03	4403	17
44	1374	13.70	4935	.32	6439	14.02	3561	16
45	**9.18 2196**	13.67	**9.99 4916**	.33	**9.18 7280**	14.00	**10.81 2720**	**15**
46	3016	13.63	4896	.32	8120	13.97	1880	14
47	3834	13.62	4877	.33	8958	13.93	1042	13
48	4651	13.58	4857	.32	.18 9794	13.92	.81 0206	12
49	5466	13.57	4838	.33	.19 0629	13.88	.80 9371	11
50	**9.18 6280**	13.53	**9.99 4818**	.33	**9.19 1462**	13.87	**10.80 8538**	**10**
51	7092	13.52	4798	.32	2294	13.83	7706	9
52	7903	13.48	4779	.33	3124	13.82	6876	8
53	8712	13.45	4759	.33	3953	13.78	6047	7
54	.18 9519	13.43	4739	.32	4780	13.77	5220	6
55	**9.19 0325**	13.42	**9.99 4720**	.33	**9.19 5606**	13.73	**10.80 4394**	**5**
56	1130	13.38	4700	.33	6430	13.72	3570	4
57	1933	13.35	4680	.33	7253	13.68	2747	3
58	2734	13.33	4660	.33	8074	13.67	1926	2
59	3534	13.30	4640	.33	8894	13.65	1106	1
60	**9.19 4332**		**9.99 4620**		**9.19 9713**		**10.80 0287**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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9° TABLE XIX.—*Continued.* 170°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.19 4332	13.28	9.99 4620	.33	9.19 9713	13.60	10.80 0287	60
1	5129	13.27	4600	.33	.20 0529	13.60	.79 9471	59
2	5925	13.23	4580	.33	1345	13.57	8655	58
3	6719	13.20	4560	.33	2159	13.53	7841	57
4	7511	13.18	4540	.35	2971	13.52	7029	56
5	9.19 8302	13.15	9.99 4519	.33	9.20 3782	13.50	10.79 6218	55
6	9091	13.13	4499	.33	4592	13.47	5408	54
7	.19 9879	13.12	4479	.33	5400	13.45	4600	53
8	.20 0666	13.08	4459	.35	6207	13.43	3793	52
9	1451	13.05	4438	.33	7013	13.40	2987	51
10	9.20 2234	13.05	9.99 4418	.33	9.20 7817	13.37	10.79 2183	50
11	3017	13.00	4398	.35	8619	13.35	1381	49
12	3797	13.00	4377	.33	.20 9420	13.33	.79 0580	48
13	4577	12.95	4357	.35	.21 0220	13.30	.78 9780	47
14	5354	12.95	4336	.33	1018	13.28	8982	46
15	9.20 6131	12.92	9.99 4316	.35	9.21 1815	13.27	10.78 8185	45
16	6906	12.88	4295	.35	2611	13.23	7389	44
17	7679	12.88	4274	.33	3405	13.22	6595	43
18	8452	12.83	4254	.35	4198	13.18	5802	42
19	9222	12.83	4233	.35	4989	13.18	5011	41
20	9.20 9992	12.80	9.99 4212	.35	9.21 5780	13.13	10.78 4220	40
21	.21 0760	12.77	4191	.33	6568	13.13	3432	39
22	1526	12.75	4171	.35	7356	13.10	2644	38
23	2291	12.73	4150	.35	8142	13.07	1858	37
24	3055	12.72	4129	.35	8926	13.07	1074	36
25	9.21 3818	12.68	9.99 4108	.35	9.21 9710	13.03	10.78 0290	35
26	4579	12.65	4087	.35	.22 0492	13.00	.77 9508	34
27	5338	12.65	4066	.35	1272	13.00	8728	33
28	6097	12.62	4045	.35	2052	12.97	7948	32
29	6854	12.58	4024	.35	2830	12.95	7170	31
30	9.21 7609	12.57	9.99 4003	.35	9.22 3607	12.92	10.77 6393	30
31	8363	12.55	3982	.37	4382	12.90	5618	29
32	9116	12.53	3960	.35	5156	12.88	4844	28
33	.21 9868	12.50	3939	.35	5929	12.85	4071	27
34	.22 0618	12.48	3918	.35	6700	12.85	3300	26
35	9.22 1367	12.47	9.99 3897	.37	9.22 7471	12.80	10.77 2529	25
36	2115	12.43	3875	.35	8239	12.80	1761	24
37	2861	12.42	3854	.37	9007	12.77	0993	23
38	3606	12.38	3832	.35	.22 9773	12.77	.77 0227	22
39	4349	12.38	3811	.37	.23 0539	12.72	.76 9461	21
40	9.22 5092	12.35	9.99 3789	.35	9.23 1302	12.72	10.76 8698	20
41	5833	12.33	3768	.37	2065	12.68	7935	19
42	6573	12.30	3746	.35	2826	12.67	7174	18
43	7311	12.28	3725	.37	3586	12.65	6414	17
44	8048	12.27	3703	.37	4345	12.63	5655	16
45	9.22 8784	12.23	9.99 3681	.35	9.23 5103	12.60	10.76 4897	15
46	.22 9518	12.23	3660	.37	5859	12.58	4141	14
47	.23 0252	12.20	3638	.37	6614	12.57	3386	13
48	0984	12.18	3616	.37	7368	12.53	2632	12
49	1715	12.15	3594	.37	8120	12.53	1880	11
50	9.23 2444	12.13	9.99 3572	.37	9.23 8872	12.50	10.76 1128	10
51	3172	12.12	3550	.37	.23 9622	12.48	.76 0378	9
52	3899	12.10	3528	.37	.24 0371	12.45	.75 9629	8
53	4625	12.07	3506	.37	1118	12.45	8882	7
54	5349	12.07	3484	.37	1865	12.42	8135	6
55	9.23 6073	12.03	9.99 3462	.37	9.24 2610	12.40	10.75 7390	5
56	6795	12.00	3440	.37	3354	12.38	6646	4
57	7515	12.00	3418	.37	4097	12.37	5903	3
58	8235	11.97	3396	.37	4839	12.33	5161	2
59	8953	11.95	3374	.38	5579	12.33	4421	1
60	9.23 9670		9.99 3351		9.24 6319		10.75 3681	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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10° TABLE XIX.—*Continued.* 169°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.23 9670**	11.93	**9.99 3351**	.37	**9.24 6319**	12.30	**10.75 3681**	**60**
1	.24 0386	11.92	3329	.37	7057	12.28	2943	59
2	1101	11.88	3307	.38	7794	12.27	2206	58
3	1814	11.87	3284	.37	8530	12.23	1470	57
4	2526	11.85	3262	.37	9264	12.23	0736	56
5	**9.24 3237**	11.83	**9.99 3240**	.38	**9.24 9998**	12.20	**10.75 0002**	**55**
6	3947	11.82	3217	.37	.25 0730	12.18	.74 9270	54
7	4656	11.78	3195	.38	1461	12.17	8539	53
8	5363	11.77	3172	.38	2191	12.15	7809	52
9	6069	11.77	3149	.37	2920	12.13	7080	51
10	**9.24 6775**	11.72	**9.99 3127**	.38	**9.25 3648**	12.10	**10.74 6352**	**50**
11	7478	11.72	3104	.38	4374	12.10	5626	49
12	8181	11.70	3081	.37	5100	12.07	4900	48
13	8883	11.67	3059	.38	5824	12.05	4176	47
14	.24 9583	11.65	3036	.38	6547	12.03	3453	46
15	**9.25 0282**	11.63	**9.99 3013**	.38	**9.25 7269**	12.02	**10.74 2731**	**45**
16	0980	11.62	2990	.38	7990	12.00	2010	44
17	1677	11.60	2967	.38	8710	11.98	1290	43
18	2373	11.57	2944	.38	.25 9429	11.95	.74 0571	42
19	3067	11.57	2921	.38	.26 0146	11.95	.73 9854	41
20	**9.25 3761**	11.53	**9.99 2898**	.38	**9.26 0863**	11.92	**10.73 9137**	**40**
21	4453	11.52	2875	.38	1578	11.90	8422	39
22	5144	11.50	2852	.38	2292	11.88	7708	38
23	5834	11.48	2829	.38	3005	11.87	6995	37
24	6523	11.47	2806	.38	3717	11.85	6283	36
25	**9.25 7211**	11.45	**9.99 2783**	.40	**9.26 4428**	11.83	**10.73 5572**	**35**
26	7898	11.42	2759	.38	5138	11.82	4862	34
27	8583	11.42	2736	.38	5847	11.80	4153	33
28	9268	11.38	2713	.38	6555	11.77	3445	32
29	.25 9951	11.37	2690	.40	7261	11.77	2739	31
30	**9.26 0633**	11.35	**9.99 2666**	.38	**9.26 7967**	11.73	**10.73 2033**	**30**
31	1314	11.33	2643	.40	8671	11.73	1329	29
32	1994	11.32	2619	.38	.26 9375	11.70	.73 0625	28
33	2673	11.30	2596	.40	.27 0077	11.70	.72 9923	27
34	3351	11.27	2572	.38	0779	11.67	9221	26
35	**9.26 4027**	11.27	**9.99 2549**	.40	**9.27 1479**	11.65	**10.72 8521**	**25**
36	4703	11.23	2525	.40	2178	11.63	7822	24
37	5377	11.23	2501	.38	2876	11.62	7124	23
38	6051	11.20	2478	.40	3573	11.60	6427	22
39	6723	11.20	2454	.40	4269	11.58	5731	21
40	**9.26 7395**	11.17	**9.99 2430**	.40	**9.27 4964**	11.57	**10.72 5036**	**20**
41	8065	11.15	2406	.40	5658	11.55	4342	19
42	8734	11.13	2382	.38	6351	11.53	3649	18
43	.26 9402	11.12	2359	.40	7043	11.52	2957	17
44	.27 0069	11.10	2335	.40	7734	11.50	2266	16
45	**9.27 0735**	11.08	**9.99 2311**	.40	**9.27 8424**	11.48	**10.72 1576**	**15**
46	1400	11.07	2287	.40	9113	11.47	0887	14
47	2064	11.03	2263	.40	.27 9801	11.45	.72 0199	13
48	2726	11.03	2239	.42	.28 0488	11.43	.71 9512	12
49	3388	11.02	2214	.40	1174	11.40	8826	11
50	**9.27 4049**	10.98	**9.99 2190**	.40	**9.28 1858**	11.40	**10.71 8142**	**10**
51	4708	10.98	2166	.40	2542	11.38	7458	9
52	5367	10.97	2142	.40	3225	11.37	6775	8
53	6025	10.93	2118	.42	3907	11.35	6093	7
54	6681	10.93	2093	.40	4588	11.33	5412	6
55	**9.27 7337**	10.90	**9.99 2069**	.42	**9.28 5268**	11.32	**10.71 4732**	**5**
56	7991	10.90	2044	.40	5947	11.28	4053	4
57	8645	10.87	2020	.40	6624	11.28	3376	3
58	9297	10.85	1996	.42	7301	11.27	2699	2
59	.27 9948	10.85	1971	.40	7977	11.25	2023	1
60	**9.28 0599**		**9.99 1947**		**9.28 8652**		**10.71 1348**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

100° 79°

11° TABLE XIX.—*Continued.* 168°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.28 0599**	10.82	**9.99 1947**	.42	**9.28 8652**	11.23	**10.71 1348**	**60**
1	1248	10.82	1922	.42	9326	11.22	0674	59
2	1897	10.78	1897	.40	.28 9999	11.20	.71 0001	58
3	2544	10.77	1873	.42	.29 0671	11.18	.70 9329	57
4	3190	10.77	1848	.42	1342	11.18	8658	56
5	**9.28 3836**	10.73	**9.99 1823**	.40	**9.29 2013**	11.15	**10.70 7987**	**55**
6	4480	10.73	1799	.42	2682	11.13	7318	54
7	5124	10.70	1774	.42	3350	11.12	6650	53
8	5766	10.70	1749	.42	4017	11.12	5983	52
9	6408	10.67	1724	.42	4684	11.08	5316	51
10	**9.28 7048**	10.67	**9.99 1699**	.42	**9.29 5349**	11.07	**10.70 4651**	**50**
11	7688	10.63	1674	.42	6013	11.07	3987	49
12	8326	10.63	1649	.42	6677	11.03	3323	48
13	8964	10.60	1624	.42	7339	11.03	2661	47
14	.28 9600	10.60	1599	.42	8001	11.02	1999	46
15	**9.29 0236**	10.57	**9.99 1574**	.42	**9.29 8662**	11.00	**10.70 1338**	**45**
16	0870	10.57	1549	.42	9322	10.97	0678	44
17	1504	10.55	1524	.43	.29 9980	10.97	.70 0020	43
18	2137	10.52	1498	.42	.30 0638	10.95	.69 9362	42
19	2768	10.52	1473	.42	1295	10.93	8705	41
20	**9.29 3399**	10.50	**9.99 1448**	.43	**9.30 1951**	10.93	**10.69 8049**	**40**
21	4029	10.48	1422	.42	2607	10.90	7393	39
22	4658	10.47	1397	.42	3261	10.88	6739	38
23	5286	10.45	1372	.43	3914	10.88	6086	37
24	5913	10.43	1346	.42	4567	10.85	5433	36
25	**9.29 6539**	10.42	**9.99 1321**	.43	**9.30 5218**	10.85	**10.69 4782**	**35**
26	7164	10.40	1295	.42	5869	10.83	4131	34
27	7788	10.40	1270	.43	6519	10.82	3481	33
28	8412	10.37	1244	.43	7168	10.80	2832	32
29	9034	10.35	1218	.42	7816	10.78	2184	31
30	**9.29 9655**	10.35	**9.99 1193**	.43	**9.30 8463**	10.77	**10.69 1537**	**30**
31	.30 0276	10.32	1167	.43	9109	10.75	0891	29
32	0895	10.32	1141	.43	.30 9754	10.75	.69 0246	28
33	1514	10.30	1115	.42	.31 0399	10.72	.68 9601	27
34	2132	10.27	1090	.43	1042	10.72	8958	26
35	**9.30 2748**	10.27	**9.99 1064**	.43	**9.31 1685**	10.70	**10.68 8315**	**25**
36	3364	10.25	1038	.43	2327	10.68	7673	24
37	3979	10.23	1012	.43	2968	10.67	7032	23
38	4593	10.23	0986	.43	3608	10.65	6392	22
39	5207	10.20	0960	.43	4247	10.63	5753	21
40	**9.30 5819**	10.18	**9.99 0934**	.43	**9.31 4885**	10.63	**10.68 5115**	**20**
41	6430	10.18	0908	.43	5523	10.60	4477	19
42	7041	10.15	0882	.45	6159	10.60	3841	18
43	7650	10.15	0855	.43	6795	10.58	3205	17
44	8259	10.13	0829	.43	7430	10.57	2570	16
45	**9.30 8867**	10.12	**9.99 0803**	.43	**9.31 8064**	10.55	**10.68 1936**	**15**
46	.30 9474	10.10	0777	.45	8697	10.55	1303	14
47	.31 0080	10.08	0750	.43	9330	10.52	0670	13
48	0685	10.07	0724	.45	.31 9961	10.52	.68 0039	12
49	1289	10.07	0697	.43	.32 0592	10.50	.67 9408	11
50	**9.31 1893**	10.03	**9.99 0671**	.43	**9.32 1222**	10.48	**10.67 8778**	**10**
51	2495	10.03	0645	.45	1851	10.47	8149	9
52	3097	10.02	0618	.45	2479	10.45	7521	8
53	3698	9.98	0591	.43	3106	10.45	6894	7
54	4297	10.00	0565	.45	3733	10.42	6267	6
55	**9.31 4897**	9.97	**9.99 0538**	.45	**9.32 4358**	10.42	**10.67 5642**	**5**
56	5495	9.95	0511	.43	4983	10.40	5017	4
57	6092	9.95	0485	.45	5607	10.40	4393	3
58	6689	9.92	0458	.45	6231	10.37	3769	2
59	7284	9.92	0431	.45	6853	10.37	3147	1
60	**9.31 7879**		**9.99 0404**		**9.32 7475**		**10.67 2525**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

101° 78°

12° TABLE XIX.—*Continued.* 167°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.31 7879	9.90	9.99 0404	.43	9.32 7475	10.33	10.67 2525	60
1	8473	9.88	0378	.45	8095	10.33	1905	59
2	9066	9.87	0351	.45	8715	10.32	1285	58
3	.31 9658	9.85	0324	.45	9334	10.32	0666	57
4	.32 0249	9.85	0297	.45	.32 9953	10.28	.67 0047	56
5	9.32 0840	9.83	9.99 0270	.45	9.33 0570	10.28	10.66 9430	55
6	1430	9.82	0243	.47	1187	10.27	8813	54
7	2019	9.80	0215	.45	1803	10.25	8197	53
8	2607	9.78	0188	.45	2418	10.25	7582	52
9	3194	9.77	0161	.45	3033	10.22	6967	51
10	9.32 3780	9.77	9.99 0134	.45	9.33 3646	10.22	10.66 6354	50
11	4366	9.73	0107	.47	4259	10.20	5741	49
12	4950	9.73	0079	.45	4871	10.18	5129	48
13	5534	9.72	0052	.45	5482	10.18	4518	47
14	6117	9.72	.99 0025	.47	6093	10.15	3907	46
15	9.32 6700	9.68	9.98 9997	.45	9.33 6702	10.15	10.66 3298	45
16	7281	9.68	9970	.47	7311	10.13	2689	44
17	7862	9.67	9942	.45	7919	10.13	2081	43
18	8442	9.65	9915	.47	8527	10.10	1473	42
19	9021	9.63	9887	.45	9133	10.10	0867	41
20	9.32 9599	9.62	9.98 9860	.47	9.33 9739	10.08	10.66 0261	40
21	.33 0176	9.62	9832	.47	.34 0344	10.07	.65 9656	39
22	0753	9.60	9804	.45	0948	10.07	9052	38
23	1329	9.57	9777	.47	1552	10.05	8448	37
24	1903	9.58	9749	.47	2155	10.03	7845	36
25	9.33 2478	9.55	9.98 9721	.47	9.34 2757	10.02	10.65 7243	35
26	3051	9.55	9693	.47	3358	10.00	6642	34
27	3624	9.52	9665	.47	3958	10.00	6042	33
28	4195	9.53	9637	.45	4558	9.98	5442	32
29	4767	9.50	9610	.47	5157	9.97	4843	31
30	9.33 5337	9.48	9.98 9582	.48	9.34 5755	9.97	10.65 4245	30
31	5906	9.48	9553	.47	6353	9.93	3647	29
32	6475	9.47	9525	.47	6949	9.93	3051	28
33	7043	9.45	9497	.47	7545	9.93	2455	27
34	7610	9.43	9469	.47	8141	9.90	1859	26
35	9.33 8176	9.43	9.98 9441	.47	9.34 8735	9.90	10.65 1265	25
36	8742	9.42	9413	.47	9329	9.88	0671	24
37	9307	9.40	9385	.48	.34 9922	9.87	.65 0078	23
38	.33 9871	9.38	9356	.47	.35 0514	9.87	.64 9486	22
39	.34 0434	9.37	9328	.47	1106	9.85	8894	21
40	9.34 0996	9.37	9.98 9300	.48	9.35 1697	9.83	10.64 8303	20
41	1558	9.35	9271	.47	2287	9.82	7713	19
42	2119	9.33	9243	.48	2876	9.82	7124	18
43	2679	9.33	9214	.47	3465	9.80	6535	17
44	3239	9.30	9186	.48	4053	9.78	5947	16
45	9.34 3797	9.30	9.98 9157	.48	9.35 4640	9.78	10.64 5360	15
46	4355	9.28	9128	.47	5227	9.77	4773	14
47	4912	9.28	9100	.48	5813	9.75	4187	13
48	5469	9.25	9071	.48	6398	9.73	3602	12
49	6024	9.25	9042	.47	6982	9.73	3018	11
50	9.34 6579	9.25	9.98 9014	.48	9.35 7566	9.72	10.64 2434	10
51	7134	9.22	8985	.48	8149	9.70	1851	9
52	7687	9.22	8956	.48	8731	9.70	1269	8
53	8240	9.20	8927	.48	9313	9.67	0687	7
54	8792	9.18	8898	.48	.35 9893	9.68	.64 0107	6
55	9.34 9343	9.17	9.98 8869	.48	9.36 0474	9.65	10.63 9526	5
56	.34 9893	9.17	8840	.48	1053	9.65	8947	4
57	.35 0443	9.15	8811	.48	1632	9.63	8368	3
58	0992	9.13	8782	.48	2210	9.62	7790	2
59	1540	9.13	8753	.48	2787	9.62	7213	1
60	9.35 2088		9.98 8724		9.36 3364		10.63 6636	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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13° TABLE XIX.—*Continued.* 166°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.35 2088**	9.12	**9.98 8724**	.48	**9.36 3364**	9.60	**10.63 6636**	**60**
1	2635	9.10	8695	.48	3940	9.58	6060	59
2	3181	9.08	8666	.50	4515	9.58	5485	58
3	3726	9.08	8636	.48	5090	9.57	4910	57
4	4271	9.07	8607	.48	5664	9.55	4336	56
5	**9.35 4815**	9.05	**9.98 8578**	.50	**9.36 6237**	9.55	**10.63 3763**	**55**
6	5358	9.05	8548	.48	6810	9.53	3190	54
7	5901	9.03	8519	.50	7382	9.52	2618	53
8	6443	9.02	8489	.48	7953	9.52	2047	52
9	6984	9.00	8460	.50	8524	9.50	1476	51
10	**9.35 7524**	9.00	**9.98 8430**	.48	**9.36 9094**	9.48	**10.63 0906**	**50**
11	8064	8.98	8401	.50	.36 9663	9.48	.63 0337	49
12	8603	8.97	8371	.48	.37 0232	9.45	.62 9768	48
13	9141	8.95	8342	.50	0799	9.47	9201	47
14	.35 9678	8.95	8312	.50	1367	9.43	8633	46
15	**9.36 0215**	8.95	**9.98 8282**	.50	**9.37 1933**	9.43	**10.62 8067**	**45**
16	0752	8.92	8252	.48	2499	9.42	7501	44
17	1287	8.92	8223	.50	3064	9.42	6936	43
18	1822	8.90	8193	.50	3629	9.40	6371	42
19	2356	8.88	8163	.50	4193	9.38	5807	41
20	**9.36 2889**	8.88	**9.98 8133**	.50	**9.37 4756**	9.38	**10.62 5244**	**40**
21	3422	8.87	8103	.50	5319	9.37	4681	39
22	3954	8.85	8073	.50	5881	9.35	4119	38
23	4485	8.85	8043	.50	6442	9.35	3558	37
24	5016	8.83	8013	.50	7003	9.33	2997	36
25	**9.36 5546**	8.82	**9.98 7983**	.50	**9.37 7563**	9.32	**10.62 2437**	**35**
26	6075	8.82	7953	.52	8122	9.32	1878	34
27	6604	8.78	7922	.50	8681	9.30	1319	33
28	7131	8.80	7892	.50	9239	9.30	0761	32
29	7659	8.77	7862	.50	.37 9797	9.28	.62 0203	31
30	**9.36 8185**	8.77	**9.98 7832**	.52	**9.38 0354**	9.27	**10.61 9646**	**30**
31	8711	8.75	7801	.50	0910	9.27	9090	29
32	9236	8.75	7771	.52	1466	9.23	8534	28
33	.36 9761	8.72	7740	.50	2020	9.25	7980	27
34	.37 0285	8.72	7710	.52	2575	9.23	7425	26
35	**9.37 0808**	8.70	**9.98 7679**	.50	**9.38 3129**	9.22	**10.61 6871**	**25**
36	1330	8.70	7649	.52	3682	9.20	6318	24
37	1852	8.68	7618	.50	4234	9.20	5766	23
38	2373	8.68	7588	.52	4786	9.18	5214	22
39	2894	8.67	7557	.52	5337	9.18	4663	21
40	**9.37 3414**	8.65	**9.98 7526**	.50	**9.38 5888**	9.17	**10.61 4112**	**20**
41	3933	8.65	7496	.52	6438	9.15	3562	19
42	4452	8.63	7465	.52	6987	9.15	3013	18
43	4970	8.62	7434	.52	7536	9.13	2464	17
44	5487	8.60	7403	.52	8084	9.12	1916	16
45	**9.37 6003**	8.60	**9.98 7372**	.52	**9.38 8631**	9.12	**10.61 1369**	**15**
46	6519	8.60	7341	.52	9178	9.10	0822	14
47	7035	8.57	7310	.52	.38 9724	9.10	.61 0276	13
48	7549	8.57	7279	.52	.39 0270	9.08	.60 9730	12
49	8063	8.57	7248	.52	0815	9.08	9185	11
50	**9.37 8577**	8.53	**9.98 7217**	.52	**9.39 1360**	9.05	**10.60 8640**	**10**
51	9089	8.53	7186	.52	1903	9.07	8097	9
52	.37 9601	8.53	7155	.52	2447	9.03	7553	8
53	.38 0113	8.52	7124	.53	2989	9.03	7011	7
54	0624	8.50	7092	.52	3531	9.03	6469	6
55	**9.38 1134**	8.48	**9.98 7061**	.52	**9.39 4073**	9.02	**10.60 5927**	**5**
56	1643	8.48	7030	.53	4614	9.00	5386	4
57	2152	8.48	6998	.52	5154	9.00	4846	3
58	2661	8.45	6967	.52	5694	8.98	4306	2
59	3168	8.45	6936	.53	6233	8.97	3767	1
60	**9.38 3675**		**9.98 6904**		**9.39 6771**		**10.60 3229**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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14° TABLE XIX.—*Continued.* 165°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.38 3675	8.45	9.98 6904	.52	9.39 6771	8.97	10.60 3229	60
1	4182	8.42	6873	.53	7309	8.95	2691	59
2	4687	8.42	6841	.53	7846	8.95	2154	58
3	5192	8.42	6809	.52	8383	8.93	1617	57
4	5697	8.40	6778	.53	8919	8.93	1081	56
5	9.38 6201	8.38	9.98 6746	.53	9.39 9455	8.92	10.60 0545	55
6	6704	8.38	6714	.52	.39 9990	8.90	.60 0010	54
7	7207	8.37	6683	.53	.40 0524	8.90	.59 9476	53
8	7709	8.35	6651	.53	1058	8.88	8942	52
9	8210	8.35	6619	.53	1591	8.88	8409	51
10	9.38 8711	8.33	9.98 6587	.53	9.40 2124	8.87	10.59 7876	50
11	9211	8.33	6555	.53	2656	8.85	7344	49
12	.38 9711	8.32	6523	.53	3187	8.85	6813	48
13	.39 0210	8.30	6491	.53	3718	8.85	6282	47
14	0708	8.30	6459	.53	4249	8.82	5751	46
15	9.39 1206	8.28	9.98 6427	.53	9.40 4778	8.83	10.59 5222	45
16	1703	8.27	6395	.53	5308	8.80	4692	44
17	2199	8.27	6363	.53	5836	8.80	4164	43
18	2695	8.27	6331	.53	6364	8.80	3636	42
19	3191	8.23	6299	.55	6892	8.78	3108	41
20	9.39 3685	8.23	9.98 6266	.53	9.40 7419	8.77	10.59 2581	40
21	4179	8.23	6234	.53	7945	8.77	2055	39
22	4673	8.22	6202	.55	8471	8.75	1529	38
23	5166	8.20	6169	.53	8996	8.75	1004	37
24	5658	8.20	6137	.55	.40 9521	8.73	.59 0479	36
25	9.39 6150	8.18	9.98 6104	.53	9.41 0045	8.73	10.58 9955	35
26	6641	8.18	6072	.55	0569	8.72	9431	34
27	7132	8.15	6039	.53	1092	8.72	8908	33
28	7621	8.17	6007	.55	1615	8.70	8385	32
29	8111	8.15	5974	.53	2137	8.68	7863	31
30	9.39 8600	8.13	9.98 5942	.55	9.41 2658	8.68	10.58 7342	30
31	9088	8.12	5909	.55	3179	8.67	6821	29
32	.39 9575	8.12	5876	.55	3699	8.67	6301	28
33	.40 0062	8.12	5843	.53	4219	8.65	5781	27
34	0549	8.10	5811	.55	4738	8.65	5262	26
35	9.40 1035	8.08	9.98 5778	.55	9.41 5257	8.63	10.58 4743	25
36	1520	8.08	5745	.55	5775	8.63	4225	24
37	2005	8.07	5712	.55	6293	8.62	3707	23
38	2489	8.05	5679	.55	6810	8.60	3190	22
39	2972	8.05	5646	.55	7326	8.60	2674	21
40	9.40 3455	8.05	9.98 5613	.55	9.41 7842	8.60	10.58 2158	20
41	3938	8.03	5580	.55	8358	8.58	1642	19
42	4420	8.02	5547	.55	8873	8.57	1127	18
43	4901	8.02	5514	.57	9387	8.57	0613	17
44	5382	8.00	5480	.55	.41 9901	8.57	.58 0099	16
45	9.40 5862	7.98	9.98 5447	.55	9.42 0415	8.55	10.57 9585	15
46	6341	7.98	5414	.55	0927	8.55	9073	14
47	6820	7.98	5381	.57	1440	8.53	8560	13
48	7299	7.97	5347	.55	1952	8.52	8048	12
49	7777	7.95	5314	.57	2463	8.52	7537	11
50	9.40 8254	7.95	9.98 5280	.55	9.42 2974	8.50	10.57 7026	10
51	8731	7.93	5247	.57	3484	8.48	6516	9
52	9207	7.93	5213	.55	3993	8.50	6007	8
53	.40 9682	7.92	5180	.57	4503	8.47	5497	7
54	.41 0157	7.92	5146	.55	5011	8.47	4989	6
55	9.41 0632	7.90	9.98 5113	.57	9.42 5519	8.47	10.57 4481	5
56	1106	7.88	5079	.57	6027	8.45	3973	4
57	1579	7.88	5045	.57	6534	8.45	3466	3
58	2052	7.87	5011	.55	7041	8.43	2959	2
59	2524	7.87	4978	.57	7547	8.42	2453	1
60	9.41 2996		9.98 4944		9.42 8052		10.57 1948	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

104° 75°

15° TABLE XIX.—*Continued.* 164°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.41 2996	7.85	9.98 4944	.57	9.42 8052	8.43	10.57 1948	60
1	3467	7.85	4910	.57	8558	8.40	1442	59
2	3938	7.83	4876	.57	9062	8.40	0938	58
3	4408	7.83	4842	.57	.42 9566	8.40	.57 0434	57
4	4878	7.82	4808	.57	.43 0070	8.38	.56 9930	56
5	9.41 5347	7.80	9.98 4774	.57	9.43 0573	8.37	10.56 9427	55
6	5815	7.80	4740	.57	1075	8.37	8925	54
7	6283	7.80	4706	.57	1577	8.37	8423	53
8	6751	7.77	4672	.57	2079	8.35	7921	52
9	7217	7.78	4638	.58	2580	8.33	7420	51
10	9.41 7684	7.77	9.98 4603	.57	9.43 3080	8.33	10.56 6920	50
11	8150	7.75	4569	.57	3580	8.33	6420	49
12	8615	7.73	4535	.58	4080	8.32	5920	48
13	9079	7.75	4500	.57	4579	8.32	5421	47
14	.41 9544	7.72	4466	.57	5078	8.30	4922	46
15	9.42 0007	7.72	9.98 4432	.58	9.43 5576	8.28	10.56 4424	45
16	0470	7.72	4397	.57	6073	8.28	3927	44
17	0933	7.70	4363	.58	6570	8.28	3430	43
18	1395	7.70	4328	.57	7067	8.27	2933	42
19	1857	7.68	4294	.58	7563	8.27	2437	41
20	9.42 2318	7.67	9.98 4259	.58	9.43 8059	8.25	10.56 1941	40
21	2778	7.67	4224	.57	8554	8.23	1446	39
22	3238	7.65	4190	.58	9048	8.25	0952	38
23	3697	7.65	4155	.58	.43 9543	8.22	.56 0457	37
24	4156	7.65	4120	.58	.44 0036	8.22	.55 9964	36
25	9.42 4615	7.63	9.98 4085	.58	9.44 0529	8.22	10.55 9471	35
26	5073	7.62	4050	.58	1022	8.20	8978	34
27	5530	7.62	4015	.57	1514	8.20	8486	33
28	5987	7.60	3981	.58	2006	8.18	7994	32
29	6443	7.60	3946	.58	2497	8.18	7503	31
30	9.42 6899	7.58	9.98 3911	.60	9.44 2988	8.18	10.55 7012	30
31	7354	7.58	3875	.58	3479	8.15	6521	29
32	7809	7.57	3840	.58	3968	8.17	6032	28
33	8263	7.57	3805	.58	4458	8.15	5542	27
34	8717	7.55	3770	.58	4947	8.13	5053	26
35	9.42 9170	7.55	9.98 3735	.58	9.44 5435	8.13	10.55 4565	25
36	.42 9623	7.53	3700	.60	5923	8.13	4077	24
37	.43 0075	7.53	3664	.58	6411	8.12	3589	23
38	0527	7.52	3629	.58	6898	8.10	3102	22
39	0978	7.52	3594	.60	7384	8.10	2616	21
40	9.43 1429	7.50	9.98 3558	.58	9.44 7870	8.10	10.55 2130	20
41	1879	7.50	3523	.60	8356	8.08	1644	19
42	2329	7.48	3487	.58	8841	8.08	1159	18
43	2778	7.47	3452	.60	9326	8.07	0674	17
44	3226	7.48	3416	.58	.44 9810	8.07	.55 0190	16
45	9.43 3675	7.45	9.98 3381	.60	9.45 0294	8.05	10.54 9706	15
46	4122	7.45	3345	.60	0777	8.05	9223	14
47	4569	7.45	3309	.60	1260	8.05	8740	13
48	5016	7.43	3273	.58	1743	8.03	8257	12
49	5462	7.43	3238	.60	2225	8.02	7775	11
50	9.43 5908	7.42	9.98 3202	.60	9.45 2706	8.02	10.54 7294	10
51	6353	7.42	3166	.60	3187	8.02	6813	9
52	6798	7.40	3130	.60	3668	8.00	6332	8
53	7242	7.40	3094	.60	4148	8.00	5852	7
54	7686	7.38	3058	.60	4628	7.98	5372	6
55	9.43 8129	7.38	9.98 3022	.60	9.45 5107	7.98	10.54 4893	5
56	8572	7.37	2986	.60	5586	7.97	4414	4
57	9014	7.37	2950	.60	6064	7.97	3936	3
58	9456	7.35	2914	.60	6542	7.95	3458	2
59	.43 9897	7.35	2878	.60	7019	7.95	2981	1
60	9.44 0338		9.98 2842		9.45 7496		10.54 2404	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

105° 74°

16° TABLE XIX.—*Continued.* 163°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.44 0338**	7.33	**9.98 2842**	.62	**9.45 7496**	7.95	**10.54 2504**	**60**
1	0778	7.33	2805	.60	7973	7.93	2027	59
2	1218	7.33	2769	.60	8449	7.93	1551	58
3	1658	7.30	2733	.62	8925	7.92	1075	57
4	2096	7.32	2696	.60	9400	7.92	0600	56
5	**9.44 2535**	7.30	**9.98 2660**	.60	**9.45 9875**	7.90	**10.54 0125**	**55**
6	2973	7.28	2624	.62	.46 0349	7.90	.53 9651	54
7	3410	7.28	2587	.60	0823	7.90	9177	53
8	3847	7.28	2551	.62	1297	7.88	8703	52
9	4284	7.27	2514	.62	1770	7.87	8230	51
10	**9.44 4720**	7.25	**9.98 2477**	.60	**9.46 2242**	7.88	**10.53 7758**	**50**
11	5155	7.25	2441	.62	2715	7.85	7285	49
12	5590	7.25	2404	.62	3186	7.87	6814	48
13	6025	7.23	2367	.60	3658	7.83	6342	47
14	6459	7.23	2331	.62	4128	7.85	5872	46
15	**9.44 6893**	7.22	**9.98 2294**	.62	**9.46 4599**	7.83	**10.53 5401**	**45**
16	7326	7.22	2257	.62	5069	7.83	4931	44
17	7759	7.20	2220	.62	5539	7.82	4461	43
18	8191	7.20	2183	.62	6008	7.82	3992	42
19	8623	7.18	2146	.62	6477	7.80	3523	41
20	**9.44 9054**	7.18	**9.98 2109**	.62	**9.46 6945**	7.80	**10.53 3055**	**40**
21	9485	7.17	2072	.62	7413	7.78	2587	39
22	.44 9915	7.17	2035	.62	7880	7.78	2120	38
23	.45 0345	7.17	1998	.62	8347	7.78	1653	37
24	0775	7.15	1961	.62	8814	7.77	1186	36
25	**9.45 1204**	7.13	**9.98 1924**	.63	**9.46 9280**	7.77	**10.53 0720**	**35**
26	1632	7.13	1886	.62	.46 9746	7.75	.53 0254	34
27	2060	7.13	1849	.62	.47 0211	7.75	.52 9789	33
28	2488	7.12	1812	.63	0676	7.75	9324	32
29	2915	7.12	1774	.62	1141	7.73	8859	31
30	**9.45 3342**	7.10	**9.98 1737**	.62	**9.47 1605**	7.73	**10.52 8395**	**30**
31	3768	7.10	1700	.63	2069	7.72	7931	29
32	4194	7.08	1662	.62	2532	7.72	7468	28
33	4619	7.08	1625	.63	2995	7.70	7005	27
34	5044	7.08	1587	.63	3457	7.70	6543	26
35	**9.45 5469**	7.07	**9.98 1549**	.62	**9.47 3919**	7.70	**10.52 6081**	**25**
36	5893	7.05	1512	.63	4381	7.68	5619	24
37	6316	7.05	1474	.63	4842	7.68	5158	23
38	6739	7.05	1436	.62	5303	7.67	4697	22
39	7162	7.03	1399	.63	5763	7.67	4237	21
40	**9.45 7584**	7.03	**9.98 1361**	.63	**9.47 6223**	7.67	**10.52 3777**	**20**
41	8006	7.02	1323	.63	6683	7.65	3317	19
42	8427	7.02	1285	.63	7142	7.65	2858	18
43	8848	7.00	1247	.63	7601	7.63	2399	17
44	9268	7.00	1209	.63	8059	7.63	1941	16
45	**9.45 9688**	7.00	**9.98 1171**	.63	**9.47 8517**	7.63	**10.52 1483**	**15**
46	.46 0108	6.98	1133	.63	8975	7.62	1025	14
47	0527	6.98	1095	.63	9432	7.62	0568	13
48	0946	6.97	1057	.63	.47 9889	7.60	.52 0111	12
49	1364	6.97	1019	.63	.48 0345	7.60	.51 9655	11
50	**9.46 1782**	6.95	**9.98 0981**	.65	**9.48 0801**	7.60	**10.51 9199**	**10**
51	2199	6.95	0942	.63	1257	7.58	8743	9
52	2616	6.93	0904	.63	1712	7.58	8288	8
53	3032	6.93	0866	.65	2167	7.57	7833	7
54	3448	6.93	0827	.63	2621	7.57	7379	6
55	**9.46 3864**	6.92	**9.98 0789**	.65	**9.48 3075**	7.57	**10.51 6925**	**5**
56	4279	6.92	0750	.63	3529	7.55	6471	4
57	4694	6.90	0712	.65	3982	7.55	6018	3
58	5108	6.90	0673	.63	4435	7.53	5565	2
59	5522	6.88	0635	.65	4887	7.53	5113	1
60	**9.46 5935**		**9.98 0596**		**9.48 5339**		**10.51 4661**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

106° 73°

17° TABLE XIX.—*Continued.* 162°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.46 5935**	6.88	**9.98 0596**	.63	**9.48 5339**	7.53	**10.51 4661**	**60**
1	6348	6.88	0558	.65	5791	7.52	4209	59
2	6761	6.87	0519	.65	6242	7.52	3758	58
3	7173	6.87	0480	.63	6693	7.50	3307	57
4	7585	6.85	0442	.65	7143	7.50	2857	56
5	**9.46 7996**	6.85	**9.98 0403**	.65	**9.48 7593**	7.50	**10.51 2407**	**55**
6	8407	6.83	0364	.65	8043	7.48	1957	54
7	8817	6.83	0325	.65	8492	7.48	1508	53
8	9227	6.83	0286	.65	8941	7.48	1059	52
9	.46 9637	6.82	0247	.65	9390	7.47	0610	51
10	**9.47 0046**	6.82	**9.98 0208**	.65	**9.48 9838**	7.47	**10.51 0162**	**50**
11	0455	6.80	0169	.65	.49 0286	7.45	.50 9714	49
12	0863	6.80	0130	.65	0733	7.45	9267	48
13	1271	6.80	0091	.65	1180	7.45	8820	47
14	1679	6.78	0052	.67	1627	7.43	8373	46
15	**9.47 2086**	6.77	**9.98 0012**	.65	**9.49 2073**	7.43	**10.50 7927**	**45**
16	2492	6.77	.97 9973	.65	2519	7.43	7481	44
17	2898	6.77	9934	.65	2965	7.42	7035	43
18	3304	6.77	9895	.67	3410	7.40	6590	42
19	3710	6.75	9855	.65	3854	7.42	6146	41
20	**9.47 4115**	6.73	**9.97 9816**	.67	**9.49 4299**	7.40	**10.50 5701**	**40**
21	4519	6.73	9776	.65	4743	7.38	5257	39
22	4923	6.73	9737	.67	5186	7.40	4814	38
23	5327	6.72	9697	.65	5630	7.38	4370	37
24	5730	6.72	9658	.67	6073	7.37	3927	36
25	**9.47 6133**	6.72	**9.97 9618**	.65	**9.49 6515**	7.37	**10.50 3485**	**35**
26	6536	6.70	9579	.67	6957	7.37	3043	34
27	6938	6.70	9539	.67	7399	7.37	2601	33
28	7340	6.68	9499	.67	7841	7.35	2159	32
29	7741	6.68	9459	.65	8282	7.33	1718	31
30	**9.47 8142**	6.67	**9.97 9420**	.67	**9.49 8722**	7.35	**10.50 1278**	**30**
31	8542	6.67	9380	.67	9163	7.33	0837	29
32	8942	6.67	9340	.67	.49 9603	7.32	.50 0397	28
33	9342	6.65	9300	.67	.50 0042	7.32	.49 9958	27
34	.47 9741	6.65	9260	.67	0481	7.32	9519	26
35	**9.48 0140**	6.65	**9.97 9220**	.67	**9.50 0920**	7.32	**10.49 9080**	**25**
36	0539	6.63	9180	.67	1359	7.30	8641	24
37	0937	6.62	9140	.67	1797	7.30	8203	23
38	1334	6.62	9100	.68	2235	7.28	7765	22
39	1731	6.62	9059	.67	2672	7.28	7328	21
40	**9.48 2128**	6.62	**9.97 9019**	.67	**9.50 3109**	7.28	**10.49 6891**	**20**
41	2525	6.60	8979	.67	3546	7.27	6454	19
42	2921	6.58	8939	.68	3982	7.27	6018	18
43	3316	6.60	8898	.67	4418	7.27	5582	17
44	3712	6.58	8858	.68	4854	7.25	5146	16
45	**9.48 4107**	6.57	**9.97 8817**	.67	**9.50 5289**	7.25	**10.49 4711**	**15**
46	4501	6.57	8777	.67	5724	7.25	4276	14
47	4895	6.57	8737	.68	6159	7.23	3841	13
48	5280	6.55	8696	.68	6593	7.23	3407	12
49	5682	6.55	8655	.67	7027	7.23	2973	11
50	**9.48 6075**	6.53	**9.97 8615**	.68	**9.50 7460**	7.22	**10.49 2540**	**10**
51	6467	6.55	8574	.68	7893	7.22	2107	9
52	6860	6.52	8533	.67	8326	7.22	1674	8
53	7251	6.53	8493	.68	8759	7.20	1241	7
54	7643	6.52	8452	.68	9191	7.18	0809	6
55	**9.48 8034**	6.50	**9.97 8411**	.68	**9.50 9622**	7.20	**10.49 0378**	**5**
56	8424	6.50	8370	.68	.51 0054	7.18	.48 9946	4
57	8814	6.50	8329	.68	0485	7.18	9515	3
58	9204	6.48	8288	.68	0916	7.17	9084	2
59	9593	6.48	8247	.68	1346	7.17	8654	1
60	**9.48 9982**		**9.97 8206**		**9.51 1776**		**10.48 8224**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

107° 72°

18° TABLE XIX.—*Continued.* 161°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.48 9982**	6.48	**9.97 8206**	.68	**9.51 1776**	7.17	**10.48 8224**	**60**
1	.49 0371	6.47	8165	.68	2206	7.15	7794	59
2	0759	6.47	8124	.68	2635	7.15	7365	58
3	1147	6.47	8083	.68	3064	7.15	6936	57
4	1535	6.45	8042	.68	3493	7.13	6507	56
5	**9.49 1922**	6.43	**9.97 8001**	.70	**9.51 3921**	7.13	**10.48 6079**	**55**
6	2308	6.45	7959	.68	4349	7.13	5651	54
7	2695	6.43	7918	.68	4777	7.12	5223	53
8	3081	6.42	7877	.70	5204	7.12	4796	52
9	3466	6.42	7835	.68	5631	7.10	4369	51
10	**9.49 3851**	6.42	**9.97 7794**	.70	**9.51 6057**	7.12	**10.48 3934**	**50**
11	4236	6.42	7752	.68	6484	7.10	3516	49
12	4621	6.40	7711	.70	6910	7.08	3090	48
13	5005	6.38	7669	.68	7335	7.10	2665	47
14	5388	6.40	7628	.70	7761	7.08	2239	46
15	**9.49 5772**	6.37	**9.97 7586**	.70	**9.51 8186**	7.07	**10.48 1814**	**45**
16	6154	6.38	7544	.68	8610	7.07	1390	44
17	6537	6.37	7503	.70	9034	7.07	0966	43
18	6919	6.37	7461	.70	9458	7.07	0542	42
19	7301	6.35	7419	.70	.51 9882	7.05	.48 0118	41
20	**9.49 7682**	6.35	**9.97 7377**	.70	**9.52 0305**	7.05	**10.47 9695**	**40**
21	8064	6.33	7335	.70	0728	7.05	9272	39
22	8444	6.35	7293	.70	1151	7.03	8849	38
23	8825	6.32	7251	.70	1573	7.03	8427	37
24	9204	6.33	7209	.70	1995	7.03	8005	36
25	**9.49 9584**	6.32	**9.97 7167**	.70	**9.52 2417**	7.02	**10.47 7583**	**35**
26	.49 9963	6.32	7125	.70	2838	7.02	7162	34
27	.50 0342	6.32	7083	.70	3259	7.02	6741	33
28	0721	6.30	7041	.70	3680	7.00	6320	32
29	1099	6.28	6999	.70	4100	7.00	5900	31
30	**9.50 1476**	6.30	**9.97 6957**	.72	**9.52 4520**	7.00	**10.47 5480**	**30**
31	1854	6.28	6914	.70	4940	6.98	5060	29
32	2231	6.27	6872	.70	5359	6.98	4641	28
33	2607	6.28	6830	.72	5778	6.98	4222	27
34	2984	6.27	6787	.70	6197	6.97	3803	26
35	**9.50 3360**	6.25	**9.97 6745**	.72	**9.52 6615**	6.97	**10.47 3385**	**25**
36	3735	6.25	6702	.70	7033	6.97	2967	24
37	4110	6.25	6660	.72	7451	6.95	2549	23
38	4485	6.25	6617	.72	7868	6.95	2132	22
39	4860	6.23	6574	.70	8285	6.95	1715	21
40	**9.50 5234**	6.23	**9.97 6532**	.72	**9.52 8702**	6.95	**10.47 1298**	**20**
41	5608	6.22	6489	.72	9119	6.93	0881	19
42	5981	6.22	6446	.70	9535	6.93	0465	18
43	6354	6.22	6404	.72	.52 9951	6.92	.47 0049	17
44	6727	6.20	6361	.72	.53 0366	6.92	.46 9634	16
45	**9.50 7099**	6.20	**9.97 6318**	.72	**9.53 0781**	6.92	**10.46 9219**	**15**
46	7471	6.20	6275	.72	1196	6.92	8804	14
47	7843	6.18	6232	.72	1611	6.90	8389	13
48	8214	6.18	6189	.72	2025	6.90	7975	12
49	8585	6.18	6146	.72	2439	6.90	7561	11
50	**9.50 8956**	6.17	**9.97 6103**	.72	**9.53 2853**	6.88	**10.46 7147**	**10**
51	9326	6.17	6060	.72	3266	6.88	6734	9
52	.50 9696	6.15	6017	.72	3679	6.88	6321	8
53	.51 0065	6.15	5974	.73	4092	6.87	5908	7
54	0434	6.15	5930	.72	4504	6.87	5496	6
55	**9.51 0803**	6.15	**9.97 5887**	.72	**9.53 4916**	6.87	**10.46 5084**	**5**
56	1172	6.13	5844	.73	5328	6.85	4672	4
57	1540	6.12	5800	.72	5739	6.85	4261	3
58	1907	6.13	5757	.72	6150	6.85	3850	2
59	2275	6.12	5714	.73	6561	6.85	3439	1
60	**9.51 2642**		**9.97 5670**		**9.53 6972**		**10.46 3028**	**0**
′	**Cosine.**	D.1″.	**Sine.**	D.1″.	**Cotang.**	D.1″.	**Tang.**	′

108° 71°

19° TABLE XIX.—*Continued.* 160°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.51 2642**	6.12	**9.97 5670**	.72	**9.53 6972**	6.83	**10.46 3028**	**60**
1	3009	6.10	5627	.73	7382	6.83	2618	59
2	3375	6.10	5583	.73	7792	6.83	2208	58
3	3741	6.10	5539	.72	8202	6.82	1798	57
4	4107	6.08	5496	.73	8611	6.82	1389	56
5	**9.51 4472**	6.08	**9.97 5452**	.73	**9.53 9020**	6.82	**10.46 0980**	**55**
6	4837	6.08	5408	.72	9429	6.80	0571	54
7	5202	6.07	5365	.73	.53 9837	6.80	.46 0163	53
8	5566	6.07	5321	.73	.54 0245	6.80	.45 9755	52
9	5930	6.07	5277	.73	0653	6.80	9347	51
10	**9.51 6294**	6.05	**9.97 5233**	.73	**9.54 1061**	6.78	**10 45 8939**	**50**
11	6657	6.05	5189	.73	1468	6.78	8532	49
12	7020	6.03	5145	.73	1875	6.77	8125	48
13	7382	6.05	5101	.73	2281	6.78	7719	47
14	7745	6.03	5057	.73	2688	6.77	7312	46
15	**9.51 8107**	6.02	**9.97 5013**	.73	**9.54 3094**	6.75	**10.45 6906**	**45**
16	8468	6.02	4969	.73	3499	6.77	6501	44
17	8829	6.02	4925	.75	3905	6.75	6095	43
18	9190	6.02	4880	.73	4310	6.75	5690	42
19	9551	6.00	4836	.73	4715	6.73	5285	41
20	**9.51 9911**	6.00	**9.97 4792**	.73	**9.54 5119**	6.75	**10.45 4881**	**40**
21	.52 0271	6.00	4748	.75	5524	6.73	4476	39
22	0631	5.98	4703	.73	5928	6.72	4072	38
23	0990	5.98	4659	.75	6331	6.73	3669	37
24	1349	5.97	4614	.73	6735	6.72	3265	36
25	**9.52 1707**	5.98	**9.97 4570**	.75	**9.54 7138**	6.70	**10.45 2862**	**35**
26	2066	5.97	4525	.73	7540	6.72	2460	34
27	2424	5.95	4481	.75	7943	6.70	2057	33
28	2781	5.95	4436	.75	8345	6.70	1655	32
29	3138	5.95	4391	.73	8747	6.70	1253	31
30	**9.52 3495**	5.95	**9.97 4347**	.75	**9.54 9149**	6.68	**10.45 0851**	**30**
31	3852	5.93	4302	.75	9550	6.68	0450	29
32	4208	5.93	4257	.75	.54 9951	6.68	.45 0049	28
33	4564	5.93	4212	.75	.55 0352	6.67	.44 9648	27
34	4920	5.92	4167	.75	0752	6.68	9248	26
35	**9.52 5275**	5.92	**9.97 4122**	.75	**9.55 1153**	6.65	**10.44 8847**	**25**
36	5630	5.90	4077	.75	1552	6.67	8448	24
37	5984	5.92	4032	.75	1952	6.65	8048	23
38	6339	5.90	3987	.75	2351	6.65	7649	22
39	6693	5.88	3942	.75	2750	6.65	7250	21
40	**9.52 7046**	5.90	**9.97 3897**	.75	**9.55 3149**	6.65	**10.44 6851**	**20**
41	7400	5.88	3852	.75	3548	6.63	6452	19
42	7753	5.87	3807	.77	3946	6.63	6054	18
43	8105	5.88	3761	.75	4344	6.62	5656	17
44	8458	5.87	3716	.75	4741	6.63	5259	16
45	**9.52 8810**	5.85	**9.97 3671**	.77	**9.55 5139**	6.62	**10.44 4861**	**15**
46	9161	5.87	3625	.75	5536	6.62	4464	14
47	9513	5.85	3580	.75	5933	6.60	4067	13
48	.52 9864	5.85	3535	.77	6329	6.60	3671	12
49	.53 0215	5.83	3489	.75	6725	6.60	3275	11
50	**9.53 0565**	5.83	**9.97 3444**	.77	**9.55 7121**	6.60	**10.44 2879**	**10**
51	0915	5.83	3398	.77	7517	6.60	2483	9
52	1265	5.82	3352	.75	7913	6.58	2087	8
53	1614	5.82	3307	.77	8308	6.58	1692	7
54	1963	5.82	3261	.77	8703	6.57	1297	6
55	**9.53 2312**	5.82	**9.97 3215**	.77	**9.55 9097**	6.57	**10.44 0903**	**5**
56	2661	5.80	3169	.75	9491	6.57	0509	4
57	3009	5.80	3124	.77	.55 9885	6.57	.44 0115	3
58	3357	5.78	3078	.77	.56 0279	6.57	.43 9721	2
59	3704	5.80	3032	.77	0673	6.55	9327	1
60	**9.53 4052**		**9.97 2986**		**9.56 1066**		**10.43 8934**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

109° 70°

20° TABLE XIX.—*Continued.* 159°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.53 4052**	5.78	**9.97 2986**	.77	**9.56 1066**	6.55	**10.43 8934**	**60**
1	4399	5.77	2940	.77	1459	6.53	8541	59
2	4745	5.78	2894	.77	1851	6.55	8149	58
3	5092	5.77	2848	.77	2244	6.53	7756	57
4	5438	5.75	2802	.78	2636	6.53	7364	56
5	**9.53 5783**	5.77	**9.97 2755**	.77	**9.56 3028**	6.52	**10.43 6972**	**55**
6	6129	5.75	2709	.77	3419	6.53	6581	54
7	6474	5.73	2663	.77	3811	6.52	6189	53
8	6818	5.75	2617	.78	4202	6.52	5798	52
9	7163	5.73	2570	.77	4593	6.50	5407	51
10	**9.53 7507**	5.73	**9.97 2524**	.77	**9.56 4983**	6.50	**10.43 5017**	**50**
11	7851	5.72	2478	.78	5373	6.50	4627	49
12	8194	5.73	2431	.77	5763	6.50	4237	48
13	8538	5.70	2385	.78	6153	6.48	3847	47
14	8880	5.72	2338	.78	6542	6.50	3458	46
15	**9.53 9223**	5.70	**9.97 2291**	.77	**9.56 6932**	6.47	**10.43 3068**	**45**
16	9565	5.70	2245	.78	7320	6.48	2680	44
17	.53 9907	5.68	2198	.78	7709	6.48	2291	43
18	.54 0249	5.68	2151	.77	8098	6.47	1902	42
19	0590	5.68	2105	.78	8486	6.45	1514	41
20	**9.54 0931**	5.68	**9.97 2058**	.78	**9.56 8873**	6.47	**10.43 1127**	**40**
21	1272	5.68	2011	.78	9261	6.45	0739	39
22	1613	5.67	1964	.78	.56 9648	6.45	.43 0352	38
23	1953	5.67	1917	.78	.57 0035	6.45	.42 9965	37
24	2293	5.65	1870	.78	0422	6.45	9578	36
25	**9.54 2632**	5.65	**9.97 1823**	.78	**9.57 0809**	6.43	**10.42 9191**	**35**
26	2971	5.65	1776	.78	1195	6.43	8805	34
27	3310	5.65	1729	.78	1581	6.43	8419	33
28	3649	5.63	1682	.78	1967	6.42	8033	32
29	3987	5.63	1635	.78	2352	6.43	7648	31
30	**9.54 4325**	5.63	**9.97 1588**	.80	**9.57 2738**	6.42	**10.42 7262**	**30**
31	4663	5.62	1540	.78	3123	6.40	6877	29
32	5000	5.63	1493	.78	3507	6.42	6493	28
33	5338	5.60	1446	.80	3892	6.40	6108	27
34	5674	5.62	1398	.78	4276	6.40	5724	26
35	**9.54 6011**	5.60	**9.97 1351**	.80	**9.57 4660**	6.40	**10.42 5340**	**25**
36	6347	5.60	1303	.78	5044	6.38	4956	24
37	6683	5.60	1256	.80	5427	6.38	4573	23
38	7019	5.58	1208	.78	5810	6.38	4190	22
39	7354	5.58	1161	.80	6193	6.38	3807	21
40	**9.54 7689**	5.58	**9.97 1113**	.78	**9.57 6576**	6.38	**10.42 3424**	**20**
41	8024	5.58	1066	.80	6959	6.37	3041	19
42	8359	5.57	1018	.80	7341	6.37	2659	18
43	8693	5.57	0970	.80	7723	6.35	2277	17
44	9027	5.55	0922	.80	8104	6.37	1896	16
45	**9.54 9360**	5.55	**9.97 0874**	.78	**9.57 8486**	6.35	**10.42 1514**	**15**
46	.54 9693	5.55	0827	.80	8867	6.35	1133	14
47	.55 0026	5.55	0779	.80	9248	6.35	0752	13
48	0359	5.55	0731	.80	.57 9629	6.33	.42 0371	12
49	0692	5.53	0683	.80	.58 0009	6.33	.41 9991	11
50	**9.55 1024**	5.53	**9.97 0635**	.82	**9.58 0389**	6.33	**10.41 9611**	**10**
51	1356	5.52	0586	.80	0769	6.33	9231	9
52	1687	5.52	0538	.80	1149	6.32	8851	8
53	2018	5.52	0490	.80	1528	6.32	8472	7
54	2349	5.52	0442	.80	1907	6.32	8093	6
55	**9.55 2680**	5.50	**9.97 0394**	.82	**9.58 2286**	6.32	**10.41 7714**	**5**
56	3010	5.52	0345	.80	2665	6.32	7335	4
57	3341	5.48	0297	.80	3044	6.30	6956	3
58	3670	5.50	0249	.82	3422	6.30	6578	2
59	4000	5.48	0200	.80	3800	6.28	6200	1
60	**9.55 4329**		**9.97 0152**		**9.58 4177**		**10.41 5823**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

110° 69°

21° TABLE XIX.—*Continued.* 158°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.55 4329	5.48	9.97 0152	.82	9.58 4177	6.30	10.41 5823	60
1	4658	5.48	0103	.80	4555	6.28	5445	59
2	4987	5.47	0055	.82	4932	6.28	5068	58
3	5315	5.47	.97 0006	.82	5309	6.28	4691	57
4	5643	5.47	.96 9957	.80	5686	6.27	4314	56
5	9.55 5971	5.47	9.96 9909	.82	9.58 6062	6.28	10.41 3938	55
6	6299	5.45	9860	.82	6439	6.27	3561	54
7	6626	5.45	9811	.82	6815	6.25	3185	53
8	6953	5.45	9762	.80	7190	6.27	2810	52
9	7280	5.43	9714	.82	7566	6.25	2434	51
10	9.55 7606	5.43	9.96 9665	.82	9.58 7941	6.25	10.41 2059	50
11	7932	5.43	9616	.82	8316	6.25	1684	49
12	8258	5.42	9567	.82	8691	6.25	1309	48
13	8583	5.43	9518	.82	9066	6.23	0934	47
14	8909	5.42	9469	.82	9440	6.23	0560	46
15	9.55 9234	5.40	9.96 9420	.83	9.58 9814	6.23	10.41 0186	45
16	9558	5.42	9370	.82	.59 0188	6.23	.40 9812	44
17	.55 9883	5.40	9321	.82	0562	6.22	9438	43
18	.56 0207	5.40	9272	.82	0935	6.22	9065	42
19	0531	5.40	9223	.83	1308	6.22	8692	41
20	9.56 0855	5.38	9.96 9173	.82	9.59 1681	6.22	10.40 8319	40
21	1178	5.38	9124	.82	2054	6.20	7946	39
22	1501	5.38	9075	.83	2426	6.22	7574	38
23	1824	5.37	9025	.82	2799	6.20	7201	37
24	2146	5.37	8976	.83	3171	6.18	6829	36
25	9.56 2468	5.37	9.96 8926	.82	9.59 3542	6.20	10.40 6458	35
26	2790	5.37	8877	.83	3914	6.18	6086	34
27	3112	5.35	8827	.83	4285	6.18	5715	33
28	3433	5.37	8777	.82	4656	6.18	5344	32
29	3755	5.33	8728	.83	5027	6.18	4973	31
30	9.56 4075	5.35	9.96 8678	.83	9.59 5398	6.17	10.40 4602	30
31	4396	5.33	8628	.83	5768	6.17	4232	29
32	4716	5.33	8578	.83	6138	6.17	3862	28
33	5036	5.33	8528	.82	6508	6.17	3492	27
34	5356	5.33	8479	.83	6878	6.15	3122	26
35	9.56 5676	5.32	9.96 8429	.83	9.59 7247	6.15	10.40 2753	25
36	5995	5.32	8379	.83	7616	6.15	2384	24
37	6314	5.30	8329	.85	7985	6.15	2015	23
38	6632	5.32	8278	.83	8354	6.13	1646	22
39	6951	5.30	8228	.83	8722	6.15	1278	21
40	9.56 7269	5.30	9.96 8178	.83	9.59 9091	6.13	10.40 0909	20
41	7587	5.28	8128	.83	9459	6.13	0541	19
42	7904	5.30	8078	.85	.59 9827	6.12	.40 0173	18
43	8222	5.28	8027	.83	.60 0194	6.13	.39 9806	17
44	8539	5.28	7977	.83	0562	6.12	9438	16
45	9.56 8856	5.27	9.96 7927	.85	9.60 0929	6.12	10.39 9071	15
46	9172	5.27	7876	.83	1296	6.12	8704	14
47	9488	5.27	7826	.85	1663	6.10	8337	13
48	.56 9804	5.27	7775	.83	2029	6.10	7971	12
49	.57 0120	5.25	7725	.85	2395	6.10	7605	11
50	9.57 0435	5.27	9.96 7674	.83	9.60 2761	6.10	10.39 7239	10
51	0751	5.25	7624	.85	3127	6.10	6873	9
52	1066	5.23	7573	.85	3493	6.08	6507	8
53	1380	5.25	7522	.85	3858	6.08	6142	7
54	1695	5.23	7471	.83	4223	6.08	5777	6
55	9.57 2009	5.23	9.96 7421	.85	9.60 4588	6.08	10.39 5412	5
56	2323	5.22	7370	.85	4953	6.07	5047	4
57	2636	5.23	7319	.85	5317	6.08	4683	3
58	2950	5.22	7268	.85	5682	6.07	4318	2
59	3263	5.20	7217	.85	6046	6.07	3954	1
60	9.57 3575		9.96 7166		9.60 6410		10.39 3590	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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22° TABLE XIX.—*Continued.* 157°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.57 3575**	5.22	**9.96 7166**	.85	**9.60 6410**	6.05	**10.39 3590**	**60**
1	3888	5.20	7115	.85	6773	6.07	3227	59
2	4200	5.20	7064	.85	7137	6.05	2863	58
3	4512	5.20	7013	.87	7500	6.05	2500	57
4	4824	5.20	6961	.85	7863	6.03	2137	56
5	**9.57 5136**	5.18	**9.96 6910**	.85	**9.60 8225**	6.05	**10.39 1775**	**55**
6	5447	5.18	6859	.85	8588	6.03	1412	54
7	5758	5.18	6808	.87	8950	6.03	1050	53
8	6069	5.17	6756	.85	9312	6.03	0688	52
9	6379	5.17	6705	.87	.60 9674	6.03	.39 0326	51
10	**9.57 6689**	5.17	**9.96 6653**	.85	**9.61 0036**	6.02	**10.38 9964**	**50**
11	6999	5.17	6602	.87	0397	6.03	9603	49
12	7309	5.15	6550	.85	0759	6.02	9241	48
13	7618	5.15	6499	.87	1120	6.00	8880	47
14	7927	5.15	6447	.87	1480	6.02	8520	46
15	**9.57 8236**	5.15	**9.96 6395**	.85	**9.61 1841**	6.00	**10.38 8159**	**45**
16	8545	5.13	6344	.87	2201	6.00	7799	44
17	8853	5.15	6292	.87	2561	6.00	7439	43
18	9162	5.13	6240	.87	2921	6.00	7079	42
19	9470	5.12	6188	.87	3281	6.00	6719	41
20	**9.57 9777**	5.13	**9.96 6136**	.85	**9.61 3641**	5.98	**10.38 6359**	**40**
21	.58 0085	5.12	6085	.87	4000	5.98	6000	39
22	0392	5.12	6033	.87	4359	5.98	5641	38
23	0699	5.10	5981	.87	4718	5.98	5282	37
24	1005	5.12	5929	.88	5077	5.97	4923	36
25	**9.58 1312**	5.10	**9.96 5876**	.87	**9.61 5435**	5.97	**10.38 4565**	**35**
26	1618	5.10	5824	.87	5793	5.97	4207	34
27	1924	5.08	5772	.87	6151	5.97	3849	33
28	2229	5.10	5720	.87	6509	5.97	3491	32
29	2535	5.08	5668	.88	6867	5.95	3133	31
30	**9.58 2840**	5.08	**9.96 5615**	.87	**9.61 7224**	5.97	**10.38 2776**	**30**
31	3145	5.07	5563	.87	7582	5.95	2418	29
32	3449	5.08	5511	.88	7939	5.93	2061	28
33	3754	5.07	5458	.87	8295	5.95	1705	27
34	4058	5.05	5406	.88	8652	5.93	1348	26
35	**9.58 4361**	5.07	**9.96 5353**	.87	**9.61 9008**	5.93	**10.38 0992**	**25**
36	4665	5.05	5301	.88	9364	5.93	0636	24
37	4968	5.07	5248	.88	.61 9720	5.93	.38 0280	23
38	5272	5.03	5195	.87	.62 0076	5.93	.37 9924	22
39	5574	5.05	5143	.88	0432	5.92	9568	21
40	**9.58 5877**	5.03	**9.96 5090**	.88	**9.62 0787**	5.92	**10.37 9213**	**20**
41	6179	5.05	5037	.88	1142	5.92	8858	19
42	6482	5.02	4984	.88	1497	5.92	8503	18
43	6783	5.03	4931	.87	1852	5.92	8148	17
44	7085	5.02	4879	.88	2207	5.90	7793	16
45	**9.58 7386**	5.03	**9.96 4826**	.88	**9.62 2561**	5.90	**10.37 7439**	**15**
46	7688	5.02	4773	.88	2915	5.90	7085	14
47	7989	5.00	4720	.90	3269	5.90	6731	13
48	8280	5.02	4666	.88	3623	5.88	6377	12
49	8590	5.00	4613	.88	3976	5.90	6024	11
50	**9.58 8890**	5.00	**9.96 4560**	.88	**9.62 4330**	5.88	**10.37 5670**	**10**
51	9190	4.98	4507	.88	4683	5.88	5317	9
52	9480	5.00	4454	.90	5036	5.87	4964	8
53	.58 9780	4.98	4400	.88	5388	5.88	4612	7
54	.59 0088	4.98	4347	.88	5741	5.87	4259	6
55	**9.59 0387**	4.98	**9.96 4294**	.90	**9.62 6093**	5.87	**10.37 3907**	**5**
56	0686	4.97	4240	.88	6445	5.87	3555	4
57	0984	4.97	4187	.90	6797	5.87	3203	3
58	1282	4.97	4133	.88	7149	5.87	2851	2
59	1580	4.97	4080	.90	7501	5.85	2499	1
60	**9.59 1878**		**9.96 4026**		**9.62 7852**		**10.37 2148**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

112° 67°

23° TABLE XIX.—*Continued.* 156°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.59 1878	4.97	9.96 4026	.90	9.62 7825	5.85	10.37 2148	60
1	2176	4.95	3972	.88	8203	5.85	1797	59
2	2473	4.95	3919	.90	8554	5.85	1446	58
3	2770	4.95	3865	.90	8905	5.83	1095	57
4	3067	4.93	3811	.90	9255	5.85	0745	56
5	9.59 3363	4.93	9.96 3757	.88	9.62 9606	5.83	10.37 0394	55
6	3659	4.93	3704	.90	.62 9956	5.83	.37 0044	54
7	3955	4.93	3650	.90	.63 0306	5.83	.36 9694	53
8	4251	4.93	3596	.90	0656	5.82	9344	52
9	4547	4.92	3542	.90	1005	5.83	8995	51
10	9.59 4842	4.92	9.96 3488	.90	9.63 1355	5.82	10.36 8645	50
11	5137	4.92	3434	.92	1704	5.82	8296	49
12	5432	4.92	3379	.90	2053	5.82	7947	48
13	5727	4.90	3325	.90	2402	5.80	7598	47
14	6021	4.90	3271	.90	2750	5.82	7250	46
15	9.59 6315	4.90	9.96 3217	.90	9.63 3099	5.80	10.36 6901	45
16	6609	4.90	3163	.92	3447	5.80	6553	44
17	6903	4.88	3108	.90	3795	5.80	6205	43
18	7196	4.90	3054	.92	4143	5.78	5857	42
19	7490	4.88	2999	.90	4490	5.80	5510	41
20	9.59 7783	4.87	9.96 2945	.92	9.63 4838	5.78	10.36 5162	40
21	8075	4.88	2890	.90	5185	5.78	4815	39
22	8368	4.87	2836	.92	5532	5.78	4468	38
23	8660	4.87	2781	.90	5879	5.78	4121	37
24	8952	4.87	2727	.92	6226	5.77	3774	36
25	9.59 9244	4.87	9.96 2672	.92	9.63 6572	5.78	10.36 3428	35
26	9536	4.85	2617	.92	6919	5.77	3081	34
27	.59 9827	4.85	2562	.90	7265	5.77	2735	33
28	.60 0118	4.85	2508	.92	7611	5.75	2389	32
29	0409	4.85	2453	.92	7956	5.77	2044	31
30	9.60 0700	4.83	9.96 2398	.92	9.63 8302	5.75	10.36 1698	30
31	0990	4.83	2343	.92	8647	5.75	1353	29
32	1280	4.83	2288	.92	8992	5.75	1008	28
33	1570	4.83	2233	.92	9337	5.75	0663	27
34	1860	4.83	2178	.92	.63 9682	5.75	.36 0318	26
35	9.60 2150	4.82	9.96 2123	.93	9.64 0027	5.73	10.35 9973	25
36	2439	4.82	2067	.92	0371	5.75	9629	24
37	2728	4.82	2012	.92	0716	5.73	9284	23
38	3017	4.80	1957	.92	1060	5.73	8940	22
39	3305	4.82	1902	.93	1404	5.72	8596	21
40	9.60 3594	4.80	9.96 1846	.92	9.64 1747	5.73	10.35 8253	20
41	3882	4.80	1791	.93	2091	5.72	7909	19
42	4170	4.78	1735	.92	2434	5.72	7566	18
43	4457	4.80	1680	.93	2777	5.72	7223	17
44	4745	4.78	1624	.92	3120	5.72	6880	16
45	9.60 5032	4.78	9.96 1569	.93	9.64 3463	5.72	10.35 6537	15
46	5319	4.78	1513	.92	3806	5.70	6194	14
47	5606	4.77	1458	.93	4148	5.70	5852	13
48	5892	4.78	1402	.93	4490	5.70	5510	12
49	6179	4.77	1346	.93	4832	5.70	5168	11
50	9.60 6465	4.77	9.96 1290	.92	9.64 5174	5.70	10.35 4826	10
51	6751	4.75	1235	.93	5516	5.68	4484	9
52	7036	4.77	1179	.93	5857	5.70	4143	8
53	7322	4.75	1123	.93	6199	5.68	3801	7
54	7607	4.75	1067	.93	6540	5.68	3460	6
55	9.60 7892	4.75	9.96 1011	.93	9.64 6881	5.68	10.35 3119	5
56	8177	4.73	0955	.93	7222	5.67	2778	4
57	8461	4.73	0899	.93	7562	5.68	2438	3
58	8745	4.73	0843	.95	7903	5.67	2097	2
59	9029	4.73	0786	.93	8243	5.67	1757	1
60	9.60 9313		9.96 0730		9.64 8583		10.35 1417	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

113° 66°

24° TABLE XIX.—*Continued.* 155°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.60 9313**	4.73	**9.96 0730**	.93	**9.64 8583**	5.67	**10.35 1417**	**60**
1	9597	4.72	0674	.93	8923	5.67	1077	59
2	.60 9880	4.73	0618	.95	9263	5.65	0737	58
3	.61 0164	4.72	0561	.93	9602	5.67	0398	57
4	0447	4.70	0505	.95	.64 9942	5.65	.35 0058	56
5	**9.61 0729**	4.72	**9.96 0448**	.93	**9.65 0281**	5.65	**10.34 9719**	**55**
6	1012	4.70	0392	.95	0620	5.65	9380	54
7	1294	4.70	0335	.93	0959	5.63	9041	53
8	1576	4.70	0279	.95	1297	5.65	8703	52
9	1858	4.70	0222	.95	1636	5.63	8364	51
10	**9.61 2140**	4.68	**9.96 0165**	.93	**9.65 1974**	5.63	**10.34 8026**	**50**
11	2421	4.68	0109	.95	2312	5.63	7688	49
12	2702	4.68	.96 0052	.95	2650	5.63	7350	48
13	2983	4.68	.95 9995	.95	2988	5.63	7012	47
14	3264	4.68	9938	.93	3326	5.62	6674	46
15	**9.61 3545**	4.67	**9.95 9882**	.95	**9.65 3663**	5.62	**10.34 6337**	**45**
16	3825	4.67	9825	.95	4000	5.62	6000	44
17	4105	4.67	9768	.95	4337	5.62	5663	43
18	4385	4.67	9711	.95	4674	5.62	5326	42
19	4665	4.65	9654	.97	5011	5.62	4989	41
20	**9.61 4944**	4.65	**9.95 9596**	.95	**9.65 5348**	5.60	**10.34 4652**	**40**
21	5223	4.65	9539	.95	5684	5.60	4316	39
22	5502	4.65	9482	.95	6020	5.60	3980	38
23	5781	4.65	9425	.95	6356	5.60	3644	37
24	6060	4.63	9368	.97	6692	5.60	3308	36
25	**9.61 6338**	4.63	**9.95 9310**	.95	**9.65 7028**	5.60	**10.34 2972**	**35**
26	6616	4.63	9253	.97	7364	5.58	2636	34
27	6894	4.63	9195	.95	7699	5.58	2301	33
28	7172	4.63	9138	.97	8034	5.58	1966	32
29	7450	4.62	9080	.95	8369	5.58	1631	31
30	**9.61 7727**	4.62	**9.95 9023**	.97	**9.65 8704**	5.58	**10.34 1296**	**30**
31	8004	4.62	8965	.95	9039	5.57	0961	29
32	8281	4.62	8908	.97	9373	5.58	0627	28
33	8558	4.60	8850	.97	.65 9708	5.57	.34 0292	27
34	8834	4.60	8792	.97	.66 0042	5.57	.33 9958	26
35	**9.61 9110**	4.60	**9.95 8734**	.95	**9.66 0376**	5.57	**10.33 9624**	**25**
36	9386	4.60	8677	.97	0710	5.55	9290	24
37	9662	4.60	8619	.97	1043	5.57	8957	23
38	.61 9938	4.58	8561	.97	1377	5.55	8623	22
39	.62 0213	4.58	8503	.97	1710	5.55	8290	21
40	**9.62 0488**	4.58	**9.95 8445**	.97	**9.66 2043**	5.55	**10.33 7957**	**20**
41	0763	4.58	8387	.97	2376	5.55	7624	19
42	1038	4.58	8329	.97	2709	5.55	7291	18
43	1313	4.57	8271	.97	3042	5.55	6958	17
44	1587	4.57	8213	.98	3375	5.53	6625	16
45	**9.62 1861**	4.57	**9.95 8154**	.97	**9.66 3707**	5.53	**10.33 6293**	**15**
46	2135	4.57	8096	.97	4039	5.53	5961	14
47	2409	4.55	8038	.98	4371	5.53	5629	13
48	2682	4.57	7979	.97	4703	5.53	5297	12
49	2956	4.55	7921	.97	5035	5.52	4965	11
50	**9.62 3229**	4.55	**9.95 7863**	.98	**9.66 5366**	5.53	**10.33 4634**	**10**
51	3502	4.53	7804	.97	5698	5.52	4302	9
52	3774	4.55	7746	.98	6029	5.52	3971	8
53	4047	4.53	7687	.98	6360	5.52	3640	7
54	4319	4.53	7628	.97	6691	5.50	3309	6
55	**9.62 4591**	4.53	**9.95 7570**	.98	**9.66 7021**	5.52	**10.33 2979**	**5**
56	4863	4.53	7511	.98	7352	5.50	2648	4
57	5135	4.52	7452	.98	7682	5.52	2318	3
58	5406	4.52	7393	.97	8013	5.50	1987	2
59	5677	4.52	7335	.98	8343	5.50	1657	1
60	**9.62 5948**		**9.95 7276**		**9.66 8673**		**10.33 1327**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

114° 65°

25° TABLE XIX.—*Continued.* 154°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.62 5948**	4.52	**9.95 7276**	.98	**9.66 8673**	5.48	**10.33 1327**	60
1	6219	4.52	7217	.98	9002	5.50	0998	59
2	6490	4.50	7158	.98	9332	5.48	0668	58
3	6760	4.50	7099	.98	9661	5.50	0339	57
4	7030	4.50	7040	.98	.66 9991	5.48	.33 0009	56
5	**9.62 7300**	4.50	**9.95 6981**	1.00	**9.67 0320**	5.48	**10.32 9680**	55
6	7570	4.50	6921	.98	0649	5.47	9351	54
7	7840	4.48	6862	.98	0977	5.48	9023	53
8	8109	4.48	6803	.98	1306	5.48	8694	52
9	8378	4.48	6744	1.00	1635	5.47	8365	51
10	**9.62 8647**	4.48	**9.95 6684**	.98	**9.67 1963**	5.47	**10.32 8037**	50
11	8916	4.48	6625	.98	2291	5.47	7709	49
12	9185	4.47	6566	1.00	2619	5.47	7381	48
13	9453	4.47	6506	.98	2947	5.45	7053	47
14	9721	4.47	6447	1.00	3274	5.47	6726	46
15	**9.62 9989**	4.47	**9.95 6387**	1.00	**9.67 3602**	5.45	**10.32 6398**	45
16	.63 0257	4.45	6327	.98	3929	5.47	6071	44
17	0524	4.47	6268	1.00	4257	5.45	5743	43
18	0792	4.45	6208	1.00	4584	5.45	5416	42
19	1059	4.45	6148	.98	4911	5.43	5089	41
20	**9.63 1326**	4.45	**9.95 6089**	1.00	**9.67 5237**	5.45	**10.32 4763**	40
21	1593	4.43	6029	1.00	5564	5.43	4436	39
22	1859	4.43	5969	1.00	5890	5.45	4110	38
23	2125	4.45	5909	1.00	6217	5.43	3783	37
24	2392	4.43	5849	1.00	6543	5.43	3457	36
25	**9.63 2658**	4.42	**9.95 5789**	1.00	**9.67 6869**	5.42	**10.32 3131**	35
26	2923	4.43	5729	1.00	7194	5.43	2806	34
27	3189	4.42	5669	1.00	7520	5.43	2480	33
28	3454	4.42	5609	.98	7846	5.42	2154	32
29	3719	4.42	5548	1.00	8171	5.42	1829	31
30	**9.63 3984**	4.42	**9.95 5488**	1.00	**9.67 8496**	5.42	**10.32 1504**	30
31	4249	4.42	5428	1.00	8821	5.42	1179	29
32	4514	4.40	5368	1.02	9146	5.42	0854	28
33	4778	4.40	5307	1.00	9471	5.40	0529	27
34	5042	4.40	5247	1.02	.67 9795	5.42	.32 0205	26
35	**9.63 5306**	4.40	**9.95 5186**	1.00	**9.68 0120**	5.40	**10.31 9880**	25
36	5570	4.40	5126	1.02	0444	5.40	9556	24
37	5834	4.38	5065	1.00	0768	5.40	9232	23
38	6097	4.38	5005	1.02	1092	5.40	8908	22
39	6360	4.38	4944	1.02	1416	5.40	8584	21
40	**9.63 6623**	4.38	**9.95 4883**	1.00	**9.68 1740**	5.38	**10.31 8260**	20
41	6886	4.37	4823	1.02	2063	5.40	7937	19
42	7148	4.38	4762	1.02	2387	5.38	7613	18
43	7411	4.37	4701	1.02	2710	5.38	7290	17
44	7673	4.37	4640	1.02	3033	5.38	6967	16
45	**9.63 7935**	4.37	**9.95 4579**	1.02	**9.68 3356**	5.38	**10.31 6644**	15
46	8197	4.35	4518	1.02	3679	5.37	6321	14
47	8458	4.37	4457	1.02	4001	5.38	5999	13
48	8720	4.35	4396	1.02	4324	5.37	5676	12
49	8981	4.35	4335	1.02	4646	5.37	5354	11
50	**9.63 9242**	4.35	**9.95 4274**	1.02	**9.68 4968**	5.37	**10.31 5032**	10
51	9503	4.35	4213	1.02	5290	5.37	4710	9
52	.63 9764	4.33	4152	1.03	5612	5.37	4388	8
53	.64 0024	4.33	4090	1.02	5934	5.35	4066	7
54	0284	4.33	4029	1.02	6255	5.37	3745	6
55	**9.64 0544**	4.33	**9.95 3968**	1.03	**9.68 6577**	5.35	**10.31 3423**	5
56	0804	4.33	3906	1.02	6898	5.35	3102	4
57	1064	4.33	3845	1.03	7219	5.35	2781	3
58	1324	4.32	3783	1.02	7540	5.35	2460	2
59	1583	4.32	3722	1.03	7861	5.35	2139	1
60	**9.64 1842**		**9.95 3660**		**9.68 8182**		**10.31 1818**	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

115° 64°

26° TABLE XIX.—*Continued.* 153°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.64 1842**	4.32	**9.95 3660**	1.02	**9.68 8182**	5.33	**10.31 1818**	**60**
1	2101	4.32	3599	1.03	8502	5.32	1498	59
2	2360	4.30	3537	1.03	8823	5.33	1177	58
3	2618	4.32	3475	1.03	9143	5.33	0857	57
4	2877	4.30	3413	1.02	9463	5.33	0537	56
5	**9.64 3135**	4.30	**9.95 3352**	1.03	**9.68 9783**	5.33	**10.31 0217**	**55**
6	3393	4.28	3290	1.03	.69 0103	5.33	.30 9897	54
7	3650	4.30	3228	1.03	0423	5.32	9577	53
8	3908	4.28	3166	1.03	0742	5.33	9258	52
9	4165	4.30	3104	1.03	1062	5.32	8938	51
10	**9.64 4423**	4.28	**9.95 3042**	1.03	**9.69 1381**	5.32	**10.30 8619**	**50**
11	4680	4.27	2980	1.03	1700	5.32	8300	49
12	4936	4.28	2918	1.05	2019	5.32	7981	48
13	5193	4.28	2855	1.03	2338	5.30	7662	47
14	5450	4.27	2793	1.03	2656	5.32	7344	46
15	**9.64 5706**	4.27	**9.95 2731**	1.03	**9.69 2975**	5.30	**10.30 7025**	**45**
16	5962	4.27	2669	1.05	3293	5.32	6707	44
17	6218	4.27	2606	1.03	3612	5.30	6388	43
18	6474	4.25	2544	1.05	3930	5.30	6070	42
19	6729	4.25	2481	1.03	4228	5.30	5752	41
20	**9.64 6984**	4.27	**9.95 2419**	1.05	**9.69 4566**	5.28	**10.30 5434**	**40**
21	7240	4.23	2356	1.03	4883	5.30	5117	39
22	7494	4.25	2294	1.05	5201	5.28	4799	38
23	7749	4.25	2231	1.05	5518	5.30	4482	37
24	8004	4.23	2168	1.03	5836	5.28	4164	36
25	**9.64 8258**	4.23	**9.95 2106**	1.05	**9.69 6153**	5.28	**10.30 3847**	**35**
26	8512	4.23	2043	1.05	6470	5.28	3530	34
27	8766	4.23	1980	1.05	6787	5.27	3213	33
28	9020	4.23	1917	1.05	7103	5.28	2897	32
29	9274	4.22	1854	1.05	7420	5.27	2580	31
30	**9.64 9527**	4.23	**9.95 1791**	1.05	**9.69 7736**	5.28	**10.30 2264**	**30**
31	.64 9781	4.22	1728	1.05	8053	5.27	1947	29
32	.65 0034	4.22	1665	1.05	8369	5.27	1631	28
33	0287	4.20	1602	1.05	8685	5.27	1315	27
34	0539	4.22	1539	1.05	9001	5.25	0999	26
35	**9.65 0792**	4.20	**9.95 1476**	1.07	**9.69 9316**	5.27	**10.30 0684**	**25**
36	1044	4.22	1412	1.05	9632	5.25	0368	24
37	1297	4.20	1349	1.05	.69 9947	5.27	.30 0053	23
38	1549	4.18	1286	1.07	.70 0263	5.25	.29 9737	22
39	1800	4.20	1222	1.05	0578	5.25	9422	21
40	**9.65 2052**	4.20	**9.95 1159**	1.05	**9.70 0893**	5.25	**10.29 9107**	**20**
41	2304	4.18	1096	1.07	1208	5.25	8792	19
42	2555	4.18	1032	1.07	1523	5.23	8477	18
43	2806	4.18	0968	1.05	1837	5.25	8163	17
44	3057	4.18	0905	1.07	2152	5.23	7848	16
45	**9.65 3308**	4.17	**9.95 0841**	1.05	**9.70 2466**	5.25	**10.29 7534**	**15**
46	3558	4.17	0778	1.07	2781	5.23	7219	14
47	3808	4.18	0714	1.07	3095	5.23	6905	13
48	4059	4.17	0650	1.07	3409	5.22	6591	12
49	4309	4.15	0586	1.07	3722	5.23	6278	11
50	**9.65 4558**	4.17	**9.95 0522**	1.07	**9.70 4036**	5.23	**10.29 5964**	**10**
51	4808	4.17	0458	1.07	4350	5.22	5650	9
52	5058	4.15	0394	1.07	4663	5.22	5337	8
53	5307	4.15	0330	1.07	4976	5.23	5024	7
54	5556	4.15	0266	1.07	5290	5.22	4710	6
55	**9.65 5805**	4.15	**9.95 0202**	1.07	**9.70 5603**	5.22	**10.29 4397**	**5**
56	6054	4.13	0138	1.07	5916	5.20	4084	4
57	6302	4.15	0074	1.07	6228	5.22	3772	3
58	6551	4.13	.95 0010	1.08	6541	5.22	3459	2
59	6799	4.13	.94 9945	1.07	6854	5.20	3146	1
60	**9.65 7047**		**9.94 9881**		**9.70 7166**		**10.29 2834**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

116° 63°

27° TABLE XIX.—*Continued.* 152°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	9.65 7047	4.13	9.94 9881	1.08	9.70 7166	5.20	10.29 2834	60
1	7295	4.12	9816	1.07	7478	5.20	2522	59
2	7542	4.13	9752	1.07	7790	5.20	2210	58
3	7790	4.12	9688	1.08	8102	5.20	1898	57
4	8037	4.12	9623	1.08	8414	5.20	1586	56
5	9.65 8284	4.12	9.94 9558	1.07	9.70 8726	5.18	10.29 1274	55
6	8531	4.12	9494	1.08	9037	5.20	0963	54
7	8778	4.12	9429	1.08	9349	5.18	0651	53
8	9025	4.10	9364	1.07	9660	5.18	0340	52
9	9271	4.10	9300	1.08	9971	5.18	.29 0029	51
10	9.65 9517	4.10	9.94 9235	1.08	9.71 0282	5.18	10.28 9718	50
11	.65 9763	4.10	9170	1.08	0593	5.18	9407	49
12	.66 0009	4.10	9105	1.08	0904	5.18	9096	48
13	0255	4.10	9040	1.08	1215	5.17	8785	47
14	0501	4.08	8975	1.08	.70 1525	5.18	8475	46
15	9.66 0746	4.08	9.94 8910	1.08	9.71 1836	5.17	10.28 8164	45
16	0991	4.08	8845	1.08	2146	5.17	7854	44
17	1236	4.08	8780	1.08	2456	5.17	7544	43
18	1481	4.08	8715	1.08	2766	5.17	7234	42
19	1726	4.07	8650	1.10	3076	5.17	6924	41
20	9.66 1970	4.07	9.94 8584	1.08	9.71 3386	5.17	10.28 6614	40
21	2214	4.08	8519	1.08	3696	5.15	6304	39
22	2459	4.07	8454	1.10	4005	5.15	5995	38
23	2703	4.05	8388	1.08	4314	5.17	5686	37
24	2946	4.07	8323	1.10	4624	5.15	5376	36
25	9.66 3190	4.05	9.94 8257	1.08	9.71 4933	5.15	10.28 5067	35
26	3433	4.07	8192	1.10	5242	5.15	4758	34
27	3677	4.05	8126	1.10	5551	5.15	4449	33
28	3920	4.05	8060	1.08	5860	5.13	4140	32
29	4163	4.05	7995	1.10	6168	5.15	3832	31
30	9.66 4406	4.03	9.94 7929	1.10	9.71 6477	5.13	10.28 3523	30
31	4648	4.05	7863	1.10	6785	5.13	3215	29
32	4891	4.03	7797	1.10	7093	5.13	2907	28
33	5133	4.03	7731	1.10	7401	5.13	2599	27
34	5375	4.03	7665	1.08	7709	5.13	2291	26
35	9.66 5617	4.03	9.94 7600	1.12	9.71 8017	5.13	10.28 1983	25
36	5859	4.02	7533	1.10	8325	5.13	1675	24
37	6100	4.03	7467	1.10	8633	5.12	1367	23
38	6342	4.02	7401	1.10	8940	5.13	1060	22
39	6583	4.02	7335	1.10	9248	5.12	0752	21
40	9.66 6824	4.02	9.94 7269	1.10	9.71 9555	5.12	10.28 0445	20
41	7065	4.00	7203	1.12	.71 9862	5.12	.28 0138	19
42	7305	4.02	7136	1.10	.72 0169	5.12	.27 9831	18
43	7546	4.00	7070	1.10	0476	5.12	9524	17
44	7786	4.02	7004	1.12	0783	5.10	9217	16
45	9.66 8027	4.00	9.94 6937	1.10	9.72 1089	5.12	10.27 8911	15
46	8267	3.98	6871	1.12	1396	5.10	8604	14
47	8506	4.00	6804	1.10	1702	5.12	8298	13
48	8746	4.00	6738	1.12	2009	5.10	7991	12
49	8986	3.98	6671	1.12	2315	5.10	7685	11
50	9.66 9225	3.98	9.94 6604	1.10	9.72 2621	5.10	10.27 7379	10
51	9464	3.98	6538	1.12	2927	5.08	7073	9
52	9703	3.98	6471	1.12	3232	5.10	6768	8
53	.66 9942	3.98	6404	1.12	3538	5.10	6462	7
54	.67 0181	3.97	6337	1.12	3844	5.08	6156	6
55	9.67 0419	3.98	9.94 6270	1.12	9.72 4149	5.08	10.27 5851	5
56	0658	3.97	6203	1.12	4454	5.10	5546	4
57	0896	3.97	6136	1.12	4760	5.08	5240	3
58	1134	3.97	6069	1.12	5065	5.08	4935	2
59	1372	3.95	6002	1.12	5370	5.07	4630	1
60	9.67 1609		9.94 5935		9.72 5674		10.27 4326	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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28° TABLE XIX.—*Continued.* 151°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.67 1609**	3.97	**9.94 5935**	1.12	**9.72 5674**	5.08	**10.27 4326**	**60**
1	1847	3.95	5868	1.13	5979	5.08	4021	59
2	2084	3.95	5800	1.12	6284	5.07	3716	58
3	2321	3.95	5733	1.12	6588	5.07	3412	57
4	2558	3.95	5666	1.13	6892	5.05	3108	56
5	**9.67 2795**	3.95	**9.94 5598**	1.12	**9.72 7197**	5.07	**10.27 2803**	**55**
6	3032	3.93	5531	1.12	7501	5.07	2499	54
7	3268	3.95	5464	1.13	7805	5.07	2195	53
8	3505	3.93	5396	1.13	8109	5.05	1891	52
9	3741	3.93	5328	1.12	8412	5.07	1588	51
10	**9.67 3977**	3.93	**9.94 5261**	1.13	**9.72 8716**	5.07	**10.27 1284**	**50**
11	4213	3.92	5193	1.13	9020	5.05	0980	49
12	4448	3.93	5125	1.12	9323	5.05	0677	48
13	4684	3.92	5058	1.13	9626	5.05	0374	47
14	4919	3.93	4990	1.13	.72 9929	5.07	.27 0071	46
15	**9.67 5155**	3.92	**9.94 4922**	1.13	**9.73 0233**	5.03	**10.26 9767**	**45**
16	5390	3.90	4854	1.13	0535	5.05	9465	44
17	5624	3.92	4786	1.13	0838	5.05	9162	43
18	5859	3.92	4718	1.13	1141	5.05	8859	42
19	6094	3.90	4650	1.13	1444	5.03	8556	41
20	**9.67 6328**	3.90	**9.94 4582**	1.13	**9.73 1746**	5.03	**10.26 8254**	**40**
21	6562	3.90	4514	1.13	2048	5.05	7952	39
22	6796	3.90	4446	1.15	2351	5.03	7649	38
23	7030	3.90	4377	1.13	2653	5.03	7347	37
24	7264	3.90	4309	1.13	2955	5.03	7045	36
25	**9.67 7498**	3.88	**9.94 4241**	1.15	**9.73 3257**	5.02	**10.26 6743**	**35**
26	7731	3.88	4172	1.13	3558	5.03	6442	34
27	7964	3.88	4104	1.13	3860	5.03	6140	33
28	8197	3.88	4036	1.15	4162	5.02	5838	32
29	8430	3.88	3967	1.13	4463	5.02	5537	31
30	**9.67 8663**	3.87	**9.94 3899**	1.15	**9.73 4764**	5.03	**10.26 5236**	**30**
31	8895	3.88	3830	1.15	5066	5.02	4934	29
32	9128	3.87	3761	1.13	5367	5.02	4633	28
33	9360	3.87	3693	1.15	5668	5.02	4332	27
34	9592	3.87	3624	1.15	5969	5.00	4031	26
35	**9.67 9824**	3.87	**9.94 3555**	1.15	**9.73 6269**	5.02	**10.26 3731**	**25**
36	.68 0056	3.87	3486	1.15	6570	5.00	3430	24
37	0288	3.85	3417	1.15	6870	5.02	3130	23
38	0519	3.85	3348	1.15	7171	5.00	2829	22
39	0750	3.87	3279	1.15	7471	5.00	2529	21
40	**9.68 0982**	3.85	**9.94 3210**	1.15	**9.73 7771**	5.00	**10.26 2229**	**20**
41	1213	3.83	3141	1.15	8071	5.00	1929	19
42	1443	3.85	3072	1.15	8371	5.00	1629	18
43	1674	3.85	3003	1.15	8671	5.00	1329	17
44	1905	3.83	2934	1.17	8971	5.00	1029	16
45	**9.68 2135**	3.83	**9.94 2864**	1.15	**9.73 9271**	4.98	**10.26 0729**	**15**
46	2365	3.83	2795	1.15	9570	5.00	0430	14
47	2595	3.83	2726	1.17	.73 9870	4.98	.26 0130	13
48	2825	3.83	2656	1.15	.74 0169	4.98	.25 9831	12
49	3055	3.82	2587	1.17	0468	4.98	9532	11
50	**9.68 3284**	3.83	**9.94 2517**	1.15	**9.74 0767**	4.98	**10.25 9233**	**10**
51	3514	3.82	2448	1.17	1066	4.98	8934	9
52	3743	3.82	2378	1.17	1365	4.98	8635	8
53	3972	3.82	2308	1.15	1664	4.97	8336	7
54	4201	3.82	2239	1.17	1962	4.98	8038	6
55	**9.68 4430**	3.80	**9.94 2169**	1.17	**9.74 2261**	4.97	**10.25 7739**	**5**
56	4658	3.82	2099	1.17	2559	4.98	7441	4
57	4887	3.80	2029	1.17	2858	4.97	7142	3
58	5115	3.80	1959	1.17	3156	4.97	6844	2
59	5343	3.80	1889	1.17	3454	4.97	6546	1
60	**9.68 5571**		**9.94 1819**		**9.74 3752**		**10.25 6248**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

118° 61°

29° TABLE XIX.—*Continued.* 150°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.68 5571**	3.80	**9.94 1819**	1.17	**9.74 3752**	4.97	**10.25 6248**	**60**
1	5799	3.80	1749	1.17	4050	4.97	5950	59
2	6027	3.78	1679	1.17	4348	4.95	5652	58
3	6254	3.80	1609	1.17	4645	4.97	5355	57
4	6482	3.78	1539	1.17	4943	4.95	5057	56
5	**9.68 6709**	3.78	**9.94 1469**	1.18	**9.74 5240**	4.97	**10.25 4760**	**55**
6	6936	3.78	1398	1.17	5538	4.95	4462	54
7	7163	3.77	1328	1.17	5835	4.95	4165	53
8	7389	3.78	1258	1.18	6132	4.95	3868	52
9	7616	3.78	1187	1.17	6429	4.95	3571	51
10	**9.68 7843**	3.77	**9.94 1117**	1.18	**9.74 6726**	4.95	**10.25 3274**	**50**
11	8069	3.77	1046	1.18	7023	4.93	2977	49
12	8295	3.77	0975	1.17	7319	4.95	2681	48
13	8521	3.77	0905	1.18	7616	4.95	2384	47
14	8747	3.75	0834	1.18	7913	4.93	2087	46
15	**9.68 8972**	3.77	**9.94 0763**	1.17	**9.74 8209**	4.93	**10.25 1791**	**45**
16	9198	3.75	0693	1.18	8505	4.93	1495	44
17	9423	3.75	0622	1.18	8801	4.93	1199	43
18	9648	3.75	0551	1.18	9097	4.93	0903	42
19	.68 9873	3.75	0480	1.18	9393	4.93	0607	41
20	**9.69 0098**	3.75	**9.94 0409**	1.18	**9.74 9689**	4.93	**10.25 0311**	**40**
21	0323	3.75	0338	1.18	.74 9985	4.93	.25 0015	39
22	0548	3.73	0267	1.18	.75 0281	4.92	.24 9719	38
23	0772	3.73	0196	1.18	0576	4.93	9424	37
24	0996	3.73	0125	1.18	0872	4.92	9128	36
25	**9.69 1220**	3.73	**9.94 0054**	1.20	**9.75 1167**	4.92	**10.24 8833**	**35**
26	1444	3.73	.93 9982	1.18	1462	4.92	8538	34
27	1668	3.73	9911	1.18	1757	4.92	8243	33
28	1892	3.72	9840	1.20	2052	4.92	7948	32
29	2115	3.73	9768	1.18	2347	4.92	7653	31
30	**9.69 2339**	3.72	**9.93 9697**	1.20	**9.75 2642**	4.92	**10.24 7358**	**30**
31	2562	3.72	9625	1.18	2937	4.90	7063	29
32	2785	3.72	9554	1.20	3231	4.92	6769	28
33	3008	3.72	9482	1.20	3526	4.90	6474	27
34	3231	3.70	9410	1.18	3820	4.92	6180	26
35	**9.69 3453**	3.72	**9.93 9339**	1.20	**9.75 4115**	4.90	**10.24 5885**	**25**
36	3676	3.70	9267	1.20	4409	4.90	5591	24
37	3898	3.70	9195	1.20	4703	4.90	5297	23
38	4120	3.70	9123	1.18	4997	4.90	5003	22
39	4342	3.70	9052	1.20	5291	4.90	4709	21
40	**9.69 4564**	3.70	**9.93 8980**	1.20	**9.75 5585**	4.88	**10.24 4415**	**20**
41	4786	3.68	8908	1.20	5878	4.90	4122	19
42	5007	3.70	8836	1.22	6172	4.88	3828	18
43	5229	3.68	8763	1.20	6465	4.90	3535	17
44	5450	3.68	8691	1.20	6759	4.88	3241	16
45	**9.69 5671**	3.68	**9.93 8619**	1.20	**9.75 7052**	4.88	**10.24 2948**	**15**
46	5892	3.68	8547	1.20	7345	4.88	2655	14
47	6113	3.68	8475	1.22	7638	4.88	2362	13
48	6334	3.67	8402	1.20	7931	4.88	2069	12
49	6554	3.68	8330	1.20	8224	4.88	1776	11
50	**9.69 6775**	3.67	**9.93 8258**	1.22	**9.75 8517**	4.88	**10.24 1483**	**10**
51	6995	3.67	8185	1.20	8810	4.87	1190	9
52	7215	3.67	8113	1.22	9102	4.88	0898	8
53	7435	3.65	8040	1.22	9395	4.87	0605	7
54	7654	3.67	7967	1.20	9687	4.87	0313	6
55	**9.69 7874**	3.67	**9.93 7895**	1.22	**9.75 9979**	4.88	**10.24 0021**	**5**
56	8094	3.65	7822	1.22	.76 0272	4.87	.23 9728	4
57	8313	3.65	7749	1.22	0564	4.87	9436	3
58	8532	3.65	7676	1.20	0856	4.87	9144	2
59	8751	3.65	7604	1.22	1148	4.85	8852	1
60	**9.69 8970**		**9.93 7531**		**9.76 1439**		**10.23 8561**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

119° 60°

30° TABLE XIX.—*Continued.* 149°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.69 8970**	3.65	**9.93 7531**	1.22	**9.76 1439**	4.87	**10.23 8561**	**60**
1	9189	3.63	7458	1.22	1731	4.87	8269	59
2	9407	3.65	7385	1.22	2023	4.85	7977	58
3	9626	3.63	7312	1.23	2314	4.87	7686	57
4	.69 9844	3.63	7238	1.22	2606	4.85	7394	56
5	**9.70 0062**	3.63	**9.93 7165**	1.22	**9.76 2897**	4.85	**10.23 7103**	**55**
6	0280	3.63	7092	1.22	3188	4.85	6812	54
7	0498	3.63	7019	1.22	3479	4.85	6521	53
8	0716	3.62	6946	1.23	3770	4.85	6230	52
9	0933	3.63	6872	1.22	4061	4.85	5939	51
10	**9.70 1151**	3.62	**9.93 6799**	1.23	**9.76 4352**	4.85	**10.23 5648**	**50**
11	1368	3.62	6725	1.22	4643	4.83	5357	49
12	1585	3.62	6652	1.23	4933	4.85	5067	48
13	1802	3.62	6578	1.22	5224	4.83	4776	47
14	2019	3.62	6505	1.23	5514	4.85	4486	46
15	**9.70 2236**	3.60	**9.93 6431**	1.23	**9.76 5805**	4.83	**10.23 4195**	**45**
16	2452	3.62	6357	1.22	6095	4.83	3905	44
17	2669	3.60	6284	1.23	6385	4.83	3615	43
18	2885	3.60	6210	1.23	6675	4.83	3325	42
19	3101	3.60	6136	1.23	6965	4.83	3035	41
20	**9.70 3317**	3.60	**9.93 6062**	1.23	**9.76 7255**	4.83	**10.23 2745**	**40**
21	3533	3.60	5988	1.23	7545	4.82	2455	39
22	3749	3.58	5914	1.23	7834	4.83	2166	38
23	3964	3.58	5840	1.23	8124	4.83	1876	37
24	4179	3.60	5766	1.23	8414	4.82	1586	36
25	**9.70 4395**	3.58	**9.93 5692**	1.23	**9.76 8703**	4.82	**10.23 1297**	**35**
26	4610	3.58	5618	1.25	8992	4.82	1008	34
27	4825	3.58	5543	1.23	9281	4.83	0719	33
28	5040	3.57	5469	1.23	9571	4.82	0429	32
29	5254	3.58	5395	1.25	.76 9860	4.80	.23 0140	31
30	**9.70 5469**	3.57	**9.93 5320**	1.23	**9.77 0148**	4.82	**10.22 9852**	**30**
31	5683	3.58	5246	1.25	0437	4.82	9563	29
32	5898	3.57	5171	1.23	0726	4.82	9274	28
33	6112	3.57	5097	1.25	1015	4.80	8985	27
34	6326	3.55	5022	1.23	1303	4.82	8697	26
35	**9.70 6539**	3.57	**9.93 4948**	1.25	**9.77 1592**	4.80	**10.22 8408**	**25**
36	6753	3.57	4873	1.25	1880	4.80	8120	24
37	6967	3.55	4798	1.25	2168	4.82	7832	23
38	7180	3.55	4723	1.23	2457	4.80	7543	22
39	7393	3.55	4649	1.25	2745	4.80	7255	21
40	**9.70 7606**	3.55	**9.93 4574**	1.25	**9.77 3033**	4.80	**10.22 6967**	**20**
41	7819	3.55	4499	1.25	3321	4.78	6679	19
42	8032	3.55	4424	1.25	3608	4.80	6392	18
43	8245	3.55	4349	1.25	3896	4.80	6104	17
44	8458	3.53	4274	1.25	4184	4.78	5816	16
45	**9.70 8670**	3.53	**9.93 4199**	1.27	**9.77 4471**	4.80	**10.22 5529**	**15**
46	8882	3.53	4123	1.25	4759	4.78	5241	14
47	9094	3.53	4048	1.25	5046	4.78	4954	13
48	9306	3.53	3973	1.25	5333	4.80	4667	12
49	9518	3.53	3898	1.27	5621	4.78	4379	11
50	**9.70 9730**	3.52	**9.93 3822**	1.25	**9.77 5908**	4.78	**10.22 4092**	**10**
51	.70 9941	3.53	3747	1.27	6195	4.78	3805	9
52	.71 0153	3.52	3671	1.25	6482	4.77	3518	8
53	0364	3.52	3596	1.27	6768	4.78	3232	7
54	0575	3.52	3520	1.25	7055	4.78	2945	6
55	**9.71 0786**	3.52	**9.93 3445**	1.27	**9.77 7342**	4.77	**10.22 2658**	**5**
56	0997	3.52	3369	1.27	7628	4.78	2372	4
57	1208	3.52	3293	1.27	7915	4.77	2085	3
58	1419	3.50	3217	1.27	8201	4.78	1799	2
59	1629	3.50	3141	1.25	8488	4.77	1512	1
60	**9.71 1839**		**9.93 3066**		**9.77 8774**		**10.22 1226**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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31° TABLE XIX.—*Continued.* 148°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.71 1839**	3.52	**9.93 3066**	1.27	**9.77 8774**	4.77	**10.22 1226**	**60**
1	2050	3.50	2990	1.27	9060	4.77	0940	59
2	2260	3.48	2914	1.27	9346	4.77	0654	58
3	2469	3.50	2838	1.27	9632	4.77	0368	57
4	2679	3.50	2762	1.28	.77 9918	4.75	.22 0082	56
5	**9.71 2889**	3.48	**9.93 2685**	1.27	**9.78 0203**	4.77	**10.21 9797**	**55**
6	3098	3.50	2609	1.27	0489	4.77	9511	54
7	3308	3.48	2533	1.27	0775	4.75	9225	53
8	3517	3.48	2457	1.28	1060	4.77	8940	52
9	3726	3.48	2380	1.27	1346	4.75	8654	51
10	**9.71 3935**	3.48	**9.93 2304**	1.27	**9.78 1631**	4.75	**10.21 8369**	**50**
11	4144	3.47	2228	1.28	1916	4.75	8084	49
12	4352	3.48	2151	1.27	2201	4.75	7799	48
13	4561	3.47	2075	1.28	2486	4.75	7514	47
14	4769	3.48	1998	1.28	2771	4.75	7229	46
15	**9.71 4978**	3.47	**9.93 1921**	1.27	**9.78 3056**	4.75	**10.21 6944**	**45**
16	5186	3.47	1845	1.28	3341	4.75	6659	44
17	5394	3.47	1768	1.28	3626	4.73	6374	43
18	5602	3.45	1691	1.28	3910	4.75	6090	42
19	5809	3.47	1614	1.28	4195	4.73	5805	41
20	**9.71 6017**	3.45	**9.93 1537**	1.28	**9.78 4479**	4.75	**10.21 5521**	**40**
21	6224	3.47	1460	1.28	4764	4.73	5236	39
22	6432	3.45	1383	1.28	5048	4.73	4952	38
23	6639	3.45	1306	1.28	5332	4.73	4668	37
24	6846	3.45	1229	1.28	5616	4.73	4384	36
25	**9.71 7053**	3.43	**9.93 1152**	1.28	**9.78 5900**	4.73	**10.21 4100**	**35**
26	7259	3.45	1075	1.28	6184	4.73	3816	34
27	7466	3.45	0998	1.28	6468	4.73	3532	33
28	7673	3.43	0921	1.30	6752	4.73	3248	32
29	7879	3.43	0843	1.28	7036	4.72	2964	31
30	**9.71 8085**	3.43	**9.93 0766**	1.30	**9.78 7319**	4.73	**10.21 2681**	**30**
31	8291	3.43	0688	1.28	7603	4.72	2397	29
32	8497	3.43	0611	1.30	7886	4.73	2114	28
33	8703	3.43	0533	1.28	8170	4.72	1830	27
34	8909	3.42	0456	1.30	8453	4.72	1547	26
35	**9.71 9114**	3.43	**9.93 0378**	1.30	**9.78 8736**	4.72	**10.21 1264**	**25**
36	9320	3.42	0300	1.28	9019	4.72	0981	24
37	9525	3.42	0223	1.30	9302	4.72	0698	23
38	9730	3.42	0145	1.30	9585	4.72	0415	22
39	.71 9935	3.42	.93 0067	1.30	.78 9868	4.72	.21 0132	21
40	**9.72 0140**	3.42	**9.92 9989**	1.30	**9.79 0151**	4.72	**10.20 9849**	**20**
41	0345	3.40	9911	1.30	0434	4.70	9566	19
42	0549	3.42	9833	1.30	0716	4.72	9284	18
43	0754	3.40	9755	1.30	0999	4.70	9001	17
44	0958	3.40	9677	1.30	1281	4.70	8719	16
45	**9.72 1162**	3.40	**9.92 9599**	1.30	**9.79 1563**	4.72	**10.20 8437**	**15**
46	1366	3.40	9521	1.32	1846	4.70	8154	14
47	1570	3.40	9442	1.30	2128	4.70	7872	13
48	1774	3.40	9364	1.30	2410	4.70	7590	12
49	1978	3.38	9286	1.32	2692	4.70	7308	11
50	**9.72 2181**	3.40	**9.92 9207**	1.30	**9.79 2974**	4.70	**10.20 7026**	**10**
51	2385	3.38	9129	1.32	3256	4.70	6744	9
52	2588	3.38	9050	1.30	3538	4.68	6462	8
53	2791	3.38	8972	1.32	3819	4.70	6181	7
54	2994	3.38	8893	1.30	4101	4.70	5899	6
55	**9.72 3197**	3.38	**9.92 8815**	1.32	**9.79 4383**	4.68	**10.20 5617**	**5**
56	3400	3.38	8736	1.32	4664	4.70	5336	4
57	3603	3.37	8657	1.32	4946	4.68	5054	3
58	3805	3.37	8578	1.32	5227	4.68	4773	2
59	4007	3.38	8499	1.32	5508	4.68	4492	1
60	**9.72 4210**		**9.92 8420**		**9.79 5789**		**10.20 4211**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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32° TABLE XIX.—*Continued.* 147°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.72 4210**	3.37	**9.92 8420**	1.30	**9.79 5789**	4.68	**10.20 4211**	**60**
1	4412	3.37	8342	1.32	6070	4.68	3930	59
2	4614	3.37	8263	1.33	6351	4.68	3649	58
3	4816	3.35	8183	1.32	6632	4.68	3368	57
4	5017	3.37	8104	1.32	6913	4.68	3087	56
5	**9.72 5219**	3.35	**9.92 8025**	1.32	**9.79 7194**	4.67	**10.20 2806**	**55**
6	5420	3.37	7946	1.32	7474	4.68	2526	54
7	5622	3.35	7867	1.33	7755	4.68	2245	53
8	5823	3.35	7787	1.32	8036	4.67	1964	52
9	6024	3.35	7708	1.32	8316	4.67	1684	51
10	**9.72 6225**	3.35	**9.92 7629**	1.33	**9.79 8596**	4.68	**10.20 1404**	**50**
11	6426	3.33	7549	1.32	8877	4.67	1123	49
12	6626	3.35	7470	1.33	9157	4.67	0843	48
13	6827	3.33	7390	1.33	9437	4.67	0563	47
14	7027	3.35	7310	1.32	9717	4.67	0283	46
15	**9.72 7228**	3.33	**9.92 7231**	1.33	**9.79 9997**	4.67	**10.20 0003**	**45**
16	7428	3.33	7151	1.33	.80 0277	4.67	.19 9723	44
17	7628	3.33	7071	1.33	0557	4.65	9443	43
18	7828	3.32	6991	1.33	0836	4.67	9164	42
19	8027	3.33	6911	1.33	1116	4.67	8884	41
20	**9.72 8227**	3.33	**9.92 6831**	1.33	**9.80 1396**	4.65	**10.19 8604**	**40**
21	8427	3.32	6751	1.33	1675	4.67	8325	39
22	8626	3.32	6671	1.33	1955	4.65	8045	38
23	8825	3.32	6591	1.33	2234	4.65	7766	37
24	9024	3.32	6511	1.33	2513	4.65	7487	36
25	**9.72 9223**	3.32	**9.92 6431**	1.33	**9.80 2792**	4.67	**10.19 7208**	**35**
26	9422	3.32	6351	1.35	3072	4.65	6928	34
27	9621	3.32	6270	1.33	3351	4.65	6649	33
28	.72 9820	3.30	6190	1.33	3630	4.65	6370	32
29	.73 0018	3.32	6110	1.35	3909	4.63	6091	31
30	**9.73 0217**	3.30	**9.92 6029**	1.33	**9.80 4187**	4.65	**10.19 5813**	**30**
31	0415	3.30	5949	1.35	4466	4.65	5534	29
32	0613	3.30	5868	1.33	4745	4.63	5255	28
33	0811	3.30	5788	1.35	5023	4.65	4977	27
34	1009	3.28	5707	1.35	5302	4.63	4698	26
35	**9.73 1206**	3.30	**9.92 5626**	1.35	**9.80 5580**	4.65	**10.19 4420**	**25**
36	1404	3.30	5545	1.33	5859	4.63	4141	24
37	1602	3.28	5465	1.35	6137	4.63	3863	23
38	1799	3.28	5384	1.35	6415	4.63	3585	22
39	1996	3.28	5303	1.35	6693	4.63	3307	21
40	**9.73 2193**	3.28	**9.92 5222**	1.35	**9.80 6971**	4.63	**10.19 3029**	**20**
41	2390	3.28	5141	1.35	7249	4.63	2751	19
42	2587	3.28	5060	1.35	7527	4.63	2473	18
43	2784	3.27	4979	1.37	7805	4.63	2195	17
44	2980	3.28	4897	1.35	8083	4.63	1917	16
45	**9.73 3177**	3.27	**9.92 4816**	1.35	**9.80 8361**	4.62	**10.19 1639**	**15**
46	3373	3.27	4735	1.35	8638	4.63	1362	14
47	3569	3.27	4654	1.37	8916	4.62	1084	13
48	3765	3.27	4572	1.35	9193	4.63	0807	12
49	3961	3.27	4491	1.37	9471	4.62	0529	11
50	**9.73 4157**	3.27	**9.92 4409**	1.35	**9.80 9748**	4.62	**10.19 0252**	**10**
51	4353	3.27	4328	1.37	.81 0025	4.62	.18 9975	9
52	4549	3.25	4246	1.37	0302	4.63	9698	8
53	4744	3.25	4164	1.35	0580	4.62	9420	7
54	4939	3.27	4083	1.37	0857	4.62	9143	6
55	**9.73 5135**	3.25	**9.92 4001**	1.37	**9.81 1134**	4.60	**10.18 8866**	**5**
56	5330	3.25	3919	1.37	1410	4.62	8590	4
57	5525	3.23	3837	1.37	1687	4.62	8313	3
58	5719	3.25	3755	1.37	1964	4.62	8036	2
59	5914	3.25	3673	1.37	2241	4.60	7759	1
60	**9.73 6109**		**9.92 3591**		**9.81 2517**		**10.18 7483**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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33° TABLE XIX.—*Continued.* 146°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.73 6109**	3.23	**9.92 3591**	1.37	**9.81 2517**	4.62	**10.18 7483**	**60**
1	6303	3.25	3509	1.37	2794	4.60	7206	59
2	6498	3.23	3427	1.37	3070	4.62	6930	58
3	6692	3.23	3345	1.37	3347	4.60	6653	57
4	6886	3.23	3263	1.37	3623	4.60	6377	56
5	**9.73 7080**	3.23	**9.92 3181**	1.38	**9.81 3899**	4.62	**10.18 6101**	**55**
6	7274	3.22	3098	1.37	4176	4.60	5824	54
7	7467	3.23	3016	1.38	4452	4.60	5548	53
8	7661	3.23	2933	1.37	4728	4.60	5272	52
9	7855	3.22	2851	1.38	5004	4.60	4996	51
10	**9.73 8048**	3.22	**9.92 2768**	1.37	**9.81 5280**	4.58	**10.18 4720**	**50**
11	8241	3.22	2686	1.38	5555	4.60	4445	49
12	8434	3.22	2603	1.38	5831	4.60	4169	48
13	8627	3.22	2520	1.37	6107	4.58	3893	47
14	8820	3.22	2438	1.38	6382	4.60	3618	46
15	**9.73 9013**	3.22	**9.92 2355**	1.38	**9.81 6658**	4.58	**10.18 3342**	**45**
16	9206	3.20	2272	1.38	6933	4.60	3067	44
17	9398	3.20	2189	1.38	7209	4.58	2791	43
18	9590	3.22	2106	1.38	7484	4.58	2516	42
19	9783	3.20	2023	1.38	7759	4.60	2241	41
20	**9.73 9975**	3.20	**9.92 1940**	1.38	**9.81 8035**	4.58	**10.18 1965**	**40**
21	.74 0167	3.20	1857	1.38	8310	4.58	1690	39
22	0359	3.18	1774	1.38	8585	4.58	1415	38
23	0550	3.20	1691	1.40	8860	4.58	1140	37
24	0742	3.20	1607	1.38	9135	4.58	0865	36
25	**9.74 0934**	3.18	**9.92 1524**	1.38	**9.81 9410**	4.57	**10.18 0590**	**35**
26	1125	3.18	1441	1.40	9684	4.58	0316	34
27	1316	3.20	1357	1.38	.81 9959	4.58	.18 0041	33
28	1508	3.18	1274	1.40	.82 0234	4.57	.17 9766	32
29	1699	3.17	1190	1.38	0508	4.58	9492	31
30	**9.74 1889**	3.18	**9.92 1107**	1.40	**9.82 0783**	4.57	**10.17 9217**	**30**
31	2080	3.18	1023	1.40	1057	4.58	8943	29
32	2271	3.18	0939	1.38	1332	4.57	8668	28
33	2462	3.17	0856	1.40	1606	4.57	8394	27
34	2652	3.17	0772	1.40	1880	4.57	8120	26
35	**9.74 2842**	3.18	**9.92 0688**	1.40	**9.82 2154**	4.58	**10.17 7846**	**25**
36	3033	3.17	0604	1.40	2429	4.57	7571	24
37	3223	3.17	0520	1.40	2703	4.57	7297	23
38	3413	3.15	0436	1.40	2977	4.57	7023	22
39	3602	3.17	0352	1.40	3251	4.55	6749	21
40	**9.74 3792**	3.17	**9.92 0268**	1.40	**9.82 3524**	4.57	**10.17 6476**	**20**
41	3982	3.15	0184	1.42	3798	4.57	6202	19
42	4171	3.17	0099	1.40	4072	4.55	5928	18
43	4361	3.15	.92 0015	1.40	4345	4.57	5655	17
44	4550	3.15	.91 9931	1.42	4619	4.57	5381	16
45	**9.74 4739**	3.15	**9.91 9846**	1.40	**9.82 4893**	4.55	**10.17 5107**	**15**
46	4928	3.15	9762	1.42	5166	4.55	4834	14
47	5117	3.15	9677	1.40	5439	4.57	4561	13
48	5306	3.13	9593	1.42	5713	4.55	4287	12
49	5494	3.15	9508	1.40	5986	4.55	4014	11
50	**9.74 5683**	3.13	**9.91 9424**	1.42	**9.82 6254**	4.55	**10.17 3741**	**10**
51	5871	3.15	9339	1.42	6532	4.55	3468	9
52	6060	3.13	9254	1.42	6805	4.55	3195	8
53	6248	3.13	9169	1.40	7078	4.55	2922	7
54	6436	3.13	9085	1.42	7351	4.55	2649	6
55	**9.74 6624**	3.13	**9.91 9000**	1.42	**9.82 7624**	4.55	**10.17 2376**	**5**
56	6812	3.12	8915	1.42	7897	4.55	2103	4
57	6999	3.13	8830	1.42	8170	4.53	1830	3
58	7187	3.12	8745	1.43	8442	4.55	1558	2
59	7374	3.13	8659	1.42	8715	4.53	1285	1
60	**9.74 7562**		**9.91 8574**		**9.82 8987**		**10.17 1013**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

123° 56°

34° TABLE XIX.—*Continued.* 145°

′	Sine.	D.1″	Cosine.	D.1″	Tang.	D.1″	Cotang.	′
0	9.74 7562	3.12	9.91 8574	1.42	9.82 8987	4.55	10.17 1013	60
1	7749	3.12	8489	1.42	9260	4.53	0740	59
2	7936	3.12	8404	1.43	9532	4.55	0468	58
3	8123	3.12	8318	1.42	.82 9805	4.53	.17 0195	57
4	8310	3.12	8233	1.43	.83 0077	4.53	.16 9923	56
5	9.74 8497	3.10	9.91 8147	1.42	9.83 0349	4.53	10.16 9651	55
6	8683	3.12	8062	1.43	0621	4.53	9379	54
7	8870	3.10	7976	1.42	0893	4.53	9107	53
8	9056	3.12	7891	1.43	1165	4.53	8835	52
9	9243	3.10	7805	1.43	1437	4.53	8563	51
10	9.74 9429	3.10	9.91 7719	1.42	9.83 1709	4.53	10.16 8291	50
11	9615	3.10	7634	1.43	1981	4.53	8019	49
12	9801	3.10	7548	1.43	2253	4.53	7747	48
13	.74 9987	3.08	7462	1.43	2525	4.52	7475	47
14	.75 0172	3.10	7376	1.43	2796	4.53	7204	46
15	9.75 0358	3.08	9.91 7290	1.43	9.83 3068	4.52	10.16 6932	45
16	0543	3.10	7204	1.43	3339	4.53	6661	44
17	0729	3.08	7118	1.43	3611	4.52	6389	43
18	0914	3.08	7032	1.43	3882	4.53	6118	42
19	1099	3.08	6946	1.45	4154	4.52	5846	41
20	9.75 1284	3.08	9.91 6859	1.43	9.83 4425	4.52	10.16 5575	40
21	1469	3.08	6773	1.43	4696	4.52	5304	39
22	1654	3.08	6687	1.45	4967	4.52	5033	38
23	1839	3.07	6600	1.43	5238	4.52	4762	37
24	2023	3.08	6514	1.45	5509	4.52	4491	36
25	9.75 2208	3.07	9.91 6427	1.43	9.83 5780	4.52	10.16 4220	35
26	2392	3.07	6341	1.45	6051	4.52	3949	34
27	2576	3.07	6254	1.45	6322	4.52	3678	33
28	2760	3.07	6167	1.43	6593	4.52	3407	32
29	2944	3.07	6081	1.45	6864	4.50	3136	31
30	9.75 3128	3.07	9.91 5994	1.45	9.83 7134	4.52	10.16 2866	30
31	3312	3.05	5907	1.45	7405	4.50	2595	29
32	3495	3.07	5820	1.45	7675	4.52	2325	28
33	3679	3.07	5733	1.45	7946	4.50	2054	27
34	3862	3.07	5646	1.45	8216	4.52	1784	26
35	9.75 4046	3.05	9.91 5559	1.45	9.83 8487	4.50	10.16 1513	25
36	4229	3.05	5472	1.45	8757	4.50	1243	24
37	4412	3.05	5385	1.47	9027	4.50	0973	23
38	4595	3.05	5297	1.45	9297	4.52	0703	22
39	4778	3.03	5210	1.45	9568	4.50	0432	21
40	9.75 4960	3.05	9.91 5123	1.47	9.83 9838	4.50	10.16 0162	20
41	5143	3.05	5035	1.45	.84 0108	4.50	.15 9892	19
42	5326	3.03	4948	1.47	0378	4.50	9622	18
43	5508	3.03	4860	1.45	0648	4.48	9352	17
44	5690	3.03	4773	1.47	0917	4.50	9083	16
45	9.75 5872	3.03	9.91 4685	1.45	9.84 1187	4.50	10.15 8813	15
46	6054	3.03	4598	1.47	1457	4.50	8543	14
47	6236	3.03	4510	1.47	1727	4.48	8273	13
48	6418	3.03	4422	1.47	1996	4.50	8004	12
49	6600	3.03	4334	1.47	2266	4.48	7734	11
50	9.75 6782	3.02	9.91 4246	1.47	9.84 2535	4.50	10.15 7465	10
51	6963	3.02	4158	1.47	2805	4.48	7195	9
52	7144	3.03	4070	1.47	3074	4.48	6926	8
53	7326	3.02	3982	1.47	3343	4.48	6657	7
54	7507	3.02	3894	1.47	3612	4.50	6388	6
55	9.75 7688	3.02	9.91 3806	1.47	9.84 3882	4.48	10.15 6118	5
56	7869	3.02	3718	1.47	4151	4.48	5849	4
57	8050	3.00	3630	1.48	4420	4.48	5580	3
58	8230	3.02	3541	1.47	4689	4.48	5311	2
59	8411	3.00	3453	1.47	4958	4.48	5042	1
60	9.75 8591		9.91 3365		9.84 5227		10.15 4773	0
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

124° 55°

35° TABLE XIX.—*Continued.* 144°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.75 8591**	3.02	**9.91 3365**	1.48	**9.84 5227**	4.48	**10.15 4773**	**60**
1	8772	3.00	3276	1.48	5496	4.47	4504	59
2	8952	3.00	3187	1.47	5764	4.48	4236	58
3	9132	3.00	3099	1.48	6033	4.48	3967	57
4	9312	3.00	3010	1.47	6302	4.47	3698	56
5	**9.75 9492**	3.00	**9.91 2922**	1.48	**9.84 6570**	4.48	**10.15 3430**	**55**
6	9672	3.00	2833	1.48	6839	4.48	3161	54
7	.75 9852	2.98	2744	1.48	7108	4.47	2892	53
8	.76 0031	3.00	2655	1.48	7376	4.47	2624	52
9	0211	2.98	2566	1.48	7644	4.48	2356	51
10	**9.76 0390**	2.98	**9.91 2477**	1.48	**9.84 7913**	4.47	**10.15 2087**	**50**
11	0569	2.98	2388	1.48	8181	4.47	1819	49
12	0748	2.98	2299	1.48	8449	4.47	1551	48
13	0927	2.98	2210	1.48	8717	4.48	1283	47
14	1106	2.98	2121	1.50	8986	4.47	1014	46
15	**9.76 1285**	2.98	**9.91 2031**	1.48	**9.84 9254**	4.47	**10.15 0746**	**45**
16	1464	2.97	1942	1.48	9522	4.47	0478	44
17	1642	2.98	1853	1.50	.84 9790	4.45	.15 0210	43
18	1821	2.97	1763	1.48	.85 0057	4.47	.14 9943	42
19	1999	2.97	1674	1.50	0325	4.47	9675	41
20	**9.76 2177**	2.98	**9.91 1584**	1.48	**9.85 0593**	4.47	**10.14 9407**	**40**
21	2356	2.97	1495	1.50	0861	4.47	9139	39
22	2534	2.97	1405	1.50	1129	4.45	8871	38
23	2712	2.95	1315	1.48	1396	4.47	8604	37
24	2889	2.97	1226	1.50	1664	4.45	8336	36
25	**9.76 3067**	2.97	**9.91 1136**	1.50	**9.85 1931**	4.47	**10.14 8069**	**35**
26	3245	2.95	1046	1.50	2199	4.45	7801	34
27	3422	2.97	0956	1.50	2466	4.45	7534	33
28	3600	2.95	0866	1.50	2733	4.47	7267	32
29	3777	2.95	0776	1.50	3001	4.45	6999	31
30	**9.76 3954**	2.95	**9.91 0686**	1.50	**9.85 3268**	4.45	**10.14 6732**	**30**
31	4131	2.95	0596	1.50	3535	4.45	6465	29
32	4308	2.95	0506	1.52	3802	4.45	6198	28
33	4485	2.95	0415	1.50	4069	4.45	5931	27
34	4662	2.93	0325	1.50	4336	4.45	5664	26
35	**9.76 4838**	2.95	**9.91 0235**	1.52	**9.85 4603**	4.45	**10.14 5397**	**25**
36	5015	2.93	0144	1.50	4870	4.45	5130	24
37	5191	2.93	.91 0054	1.52	5137	4.45	4863	23
38	5367	2.95	.90 9963	1.50	5404	4.45	4596	22
39	5544	2.93	9873	1.52	5671	4.45	4329	21
40	**9.76 5720**	2.93	**9.90 9782**	1.52	**9.85 5938**	4.43	**10.14 4062**	**20**
41	5896	2.93	9691	1.50	6204	4.45	3796	19
42	6072	2.92	9601	1.52	6471	4.43	3529	18
43	6247	2.93	9510	1.52	6737	4.45	3263	17
44	6423	2.92	9419	1.52	7004	4.43	2996	16
45	**9.76 6598**	2.93	**9.90 9328**	1.52	**9.85 7270**	4.45	**10.14 2730**	**15**
46	6774	2.92	9237	1.52	7537	4.43	2463	14
47	6949	2.92	9146	1.52	7803	4.43	2197	13
48	7124	2.93	9055	1.52	8069	4.45	1931	12
49	7300	2.92	8964	1.52	8336	4.43	1664	11
50	**9.76 7475**	2.90	**9.90 8873**	1.53	**9.85 8602**	4.43	**10.14 1398**	**10**
51	7649	2.92	8781	1.52	8868	4.43	1132	9
52	7824	2.92	8690	1.52	9134	4.43	0866	8
53	7999	2.90	8599	1.53	9400	4.43	0600	7
54	8173	2.92	8507	1.52	9666	4.43	0334	6
55	**9.76 8348**	2.90	**9.90 8416**	1.53	**9.85 9932**	4.43	**10.14 0068**	**5**
56	8522	2.92	8324	1.52	.86 0198	4.43	.13 9802	4
57	8697	2.90	8233	1.53	0464	4.43	9536	3
58	8871	2.90	8141	1.53	0730	4.42	9270	2
59	9045	2.90	8049	1.52	0995	4.43	9005	1
60	**9.76 9219**		**9.90 7958**		**9.86 1261**		**10.13 8739**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

125° 54°

36° TABLE XIX.—*Continued.* 143°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.76 9219**	2.90	**9.90 7958**	1.53	**9.86 1261**	4.43	**10.13 8739**	**60**
1	9393	2.88	7866	1.53	1527	4.42	8473	59
2	9566	2.90	7774	1.53	1792	4.43	8208	58
3	9740	2.88	7682	1.53	2058	4.42	7942	57
4	.76 9913	2.90	7590	1.53	2323	4.43	7677	56
5	**9.77 0087**	2.88	**9.90 7498**	1.53	**9.86 2589**	4.42	**10.13 7411**	**55**
6	0260	2.88	7406	1.53	2854	4.42	7146	54
7	0433	2.88	7314	1.53	3119	4.43	6881	53
8	0606	2.88	7222	1.55	3385	4.42	6615	52
9	0779	2.88	7129	1.53	3650	4.42	6350	51
10	**9.77 0952**	2.88	**9.90 7037**	1.53	**9.86 3915**	4.42	**10.13 6085**	**50**
11	1125	2.88	6945	1.55	4180	4.42	5820	49
12	1298	2.87	6852	1.53	4445	4.42	5555	48
13	1470	2.88	6760	1.55	4710	4.42	5290	47
14	1643	2.87	6667	1.53	4975	4.42	5025	46
15	**9.77 181**[illegible]	2.87	**9.90 6575**	1.55	**9.86 5240**	4.42	**10.13 4760**	**45**
16	198[illegible]	2.87	6482	1.55	5505	4.42	4495	44
17	2159	2.87	6389	1.55	5770	4.42	4230	43
18	2331	2.87	6296	1.53	6035	4.42	3965	42
19	2503	2.87	6204	1.55	6300	4.40	3700	41
20	**9.77 2675**	2.87	**9.90 6111**	1.55	**9.86 6564**	4.42	**10.13 3436**	**40**
21	2847	2.85	6018	1.55	6829	4.42	3171	39
22	3018	2.87	5925	1.55	7094	4.40	2906	38
23	3190	2.85	5832	1.55	7358	4.42	2642	37
24	3361	2.87	5739	1.57	7623	4.40	2377	36
25	**9.77 3533**	2.85	**9.90 5645**	1.55	**9.86 7887**	4.42	**10.13 2113**	**35**
26	3704	2.85	5552	1.55	8152	4.40	1848	34
27	3875	2.85	5459	1.55	8416	4.40	1584	33
28	4046	2.85	5366	1.57	8680	4.42	1320	32
29	4217	2.85	5272	1.55	8945	4.40	1055	31
30	**9.77 4388**	2.83	**9.90 5179**	1.57	**9.86 9209**	4.40	**10.13 0791**	**30**
31	4558	2.85	5085	1.55	9473	4.40	0527	29
32	4729	2.83	4992	1.57	.86 9737	4.40	.13 0263	28
33	4899	2.85	4898	1.57	.87 0001	4.40	.12 9999	27
34	5070	2.83	4804	1.55	0265	4.40	9735	26
35	**9.77 5240**	2.83	**9.90 4711**	1.57	**9.87 0529**	4.40	**10.12 9471**	**25**
36	5410	2.83	4617	1.57	0793	4.40	9207	24
37	5580	2.83	4523	1.57	1057	4.40	8943	23
38	5750	2.83	4429	1.57	1321	4.40	8679	22
39	5920	2.83	4335	1.57	1585	4.40	8415	21
40	**9.77 6090**	2.82	**9.90 4241**	1.57	**9.87 1849**	4.38	**10.12 8151**	**20**
41	6259	2.83	4147	1.57	2112	4.40	7888	19
42	6429	2.82	4053	1.57	2376	4.40	7624	18
43	6598	2.83	3959	1.58	2640	4.38	7360	17
44	6768	2.82	3864	1.57	2903	4.40	7097	16
45	**9.77 6937**	2.82	**9.90 3770**	1.57	**9.87 3167**	4.38	**10.12 6833**	**15**
46	7106	2.82	3676	1.58	3430	4.40	6570	14
47	7275	2.82	3581	1.57	3694	4.38	6306	13
48	7444	2.82	3487	1.58	3957	4.38	6043	12
49	7613	2.80	3392	1.57	4220	4.40	5780	11
50	**9.77 7781**	2.82	**9.90 3298**	1.58	**9.87 4484**	4.38	**10.12 5516**	**10**
51	7950	2.82	3203	1.58	4747	4.38	5253	9
52	8119	2.80	3108	1.57	5010	4.38	4990	8
53	8287	2.80	3014	1.58	5273	4.40	4727	7
54	8455	2.82	2919	1.58	5537	4.38	4463	6
55	**9.77 8624**	2.80	**9.90 2824**	1.58	**9.87 5800**	4.38	**10.12 4200**	**5**
56	8792	2.80	2729	1.58	6063	4.38	3937	4
57	8960	2.80	2634	1.58	6326	4.38	3674	3
58	9128	2.78	2539	1.58	6589	4.38	3411	2
59	9295	2.80	2444	1.58	6852	4.37	3148	1
60	**9.77 9463**		**9.90 2349**		**9.87 7114**		**10.12 2886**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

126° 53°

37° TABLE XIX.—*Continued.* 142°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.77 9463**	2.80	**9.90 2349**	1.60	**9.87 7114**	4.38	**10.12 2886**	60
1	9631	2.78	2253	1.58	7377	4.38	2623	59
2	9798	2.80	2158	1.58	7640	4.38	2360	58
3	.77 9966	2.78	2063	1.60	7903	4.37	2097	57
4	.78 0133	2.78	1967	1.58	8165	4.38	1835	56
5	**9.78 0300**	2.78	**9.90 1872**	1.60	**9.87 8428**	4.38	**10.12 1572**	**55**
6	0467	2.78	1776	1.58	8691	4.37	1309	54
7	0634	2.78	1681	1.60	8953	4.38	1047	53
8	0801	2.78	1585	1.58	9216	4.37	0784	52
9	0968	2.77	1490	1.60	9478	4.38	0522	51
10	**9.78 1134**	2.78	**9.90 1394**	1.60	**9.87 9741**	4.37	**10.12 0259**	**50**
11	1301	2.78	1298	1.60	.88 0003	4.37	.11 9997	49
12	1468	2.77	1202	1.60	0265	4.38	9735	48
13	1634	2.77	1106	1.60	0528	4.37	9472	47
14	1800	2.77	1010	1.60	0790	4.37	9210	46
15	**9.78 1966**	2.77	**9.90 0914**	1.60	**9.88 1052**	4.37	**10.11 8948**	**45**
16	2132	2.77	0818	1.60	1314	4.38	8686	44
17	2298	2.77	0722	1.60	1577	4.37	8423	43
18	2464	2.77	0626	1.62	1839	4.37	8161	42
19	2630	2.77	0529	1.60	2101	4.37	7899	41
20	**9.78 2796**	2.75	**9.90 0433**	1.60	**9.88 2363**	4.37	**10.11 7637**	**40**
21	2961	2.77	0337	1.62	2625	4.37	7375	39
22	3127	2.75	0240	1.60	2887	4.35	7113	38
23	3292	2.77	0144	1.62	3148	4.37	6852	37
24	3458	2.75	.90 0047	1.60	3410	4.37	6590	36
25	**9.78 3623**	2.75	**9.89 9951**	1.62	**9.88 3672**	4.37	**10.11 6328**	**35**
26	3788	2.75	9854	1.62	3934	4.37	6066	34
27	3953	2.75	9757	1.62	4196	4.35	5804	33
28	4118	2.73	9660	1.60	4457	4.37	5543	32
29	4282	2.75	9564	1.62	4719	4.35	5281	31
30	**9.78 4447**	2.75	**9.89 9467**	1.62	**9.88 4980**	4.37	**10.11 5020**	**30**
31	4612	2.73	9370	1.62	5242	4.37	4758	29
32	4776	2.75	9273	1.62	5504	4.35	4496	28
33	4941	2.73	9176	1.63	5765	4.35	4235	27
34	5105	2.73	9078	1.62	6026	4.37	3974	26
35	**9.78 5269**	2.73	**9.89 8981**	1.62	**9.88 6288**	4.35	**10.11 3712**	**25**
36	5433	2.73	8884	1.62	6549	4.37	3451	24
37	5597	2.73	8787	1.63	6811	4.35	3189	23
38	5761	2.73	8689	1.62	7072	4.35	2928	22
39	5925	2.73	8592	1.63	7333	4.35	2667	21
40	**9.78 6089**	2.72	**9.89 8494**	1.62	**9.88 7594**	4.35	**10.11 2406**	**20**
41	6252	2.73	8397	1.63	7855	4.35	2145	19
42	6416	2.72	8299	1.62	8116	4.37	1884	18
43	6579	2.72	8202	1.63	8378	4.35	1622	17
44	6742	2.73	8104	1.63	8639	4.35	1361	16
45	**9.78 6906**	2.72	**9.89 8006**	1.63	**9.88 8900**	4.35	**10.11 1100**	**15**
46	7069	2.72	7908	1.63	9161	4.33	0839	14
47	7232	2.72	7810	1.63	9421	4.35	0579	13
48	7395	2.70	7712	1.63	9682	4.35	0318	12
49	7557	2.72	7614	1.63	.88 9943	4.35	.11 0057	11
50	**9.78 7720**	2.72	**9.89 7516**	1.63	**9.89 0204**	4.35	**10.10 9796**	**10**
51	7883	2.70	7418	1.63	0465	4.33	9535	9
52	8045	2.72	7320	1.63	0725	4.35	9275	8
53	8208	2.70	7222	1.65	0986	4.35	9014	7
54	8370	2.70	7123	1.63	1247	4.33	8753	6
55	**9.78 8532**	2.70	**9.89 7025**	1.65	**9.89 1507**	4.35	**10.10 8493**	**5**
56	8694	2.70	6926	1.63	1768	4.33	8232	4
57	8856	2.70	6828	1.65	2028	4.35	7972	3
58	9018	2.70	6729	1.63	2289	4.33	7711	2
59	9180	2.70	6631	1.65	2549	4.35	7451	1
60	**9.78 9342**		**9.89 6532**		**9.89 2810**		**10.10 7190**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

127° 52°

38° Table XIX.—*Continued.* 141°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.78 9342**	2.70	**9.89 6532**	1.65	**9.89 2810**	4.33	**10.10 7190**	**60**
1	9504	2.68	6433	1.63	3070	4.35	6930	59
2	9665	2.70	6335	1.65	3331	4.33	6669	58
3	9827	2.68	6236	1.65	3591	4.33	6409	57
4	.78 9988	2.68	6137	1.65	3851	4.33	6149	56
5	**9.79 0149**	2.68	**9.89 6038**	1.65	**9.89 4111**	4.35	**10.10 5889**	**55**
6	0310	2.68	5939	1.65	4372	4.33	5628	54
7	0471	2.68	5840	1.65	4632	4.33	5368	53
8	0632	2.68	5741	1.67	4892	4.33	5108	52
9	0793	2.68	5641	1.65	5152	4.33	4848	51
10	**9.79 0954**	2.68	**9.89 5542**	1.65	**9.89 5412**	4.33	**10.10 4588**	**50**
11	1115	2.67	5443	1.67	5672	4.33	4328	49
12	1275	2.68	5343	1.65	5932	4.33	4068	48
13	1436	2.67	5244	1.65	6192	4.33	3808	47
14	1596	2.68	5145	1.67	6452	4.33	3548	46
15	**9.79 1757**	2.67	**9.89 5045**	1.67	**9.89 6712**	4.32	**10.10 3288**	**45**
16	1917	2.67	4945	1.65	6971	4.33	3029	44
17	2077	2.67	4846	1.67	7231	4.33	2769	43
18	2237	2.67	4746	1.67	7491	4.33	2509	42
19	2397	2.67	4646	1.67	7751	4.32	2249	41
20	**9.79 2557**	2.65	**9.89 4546**	1.67	**9.89 8010**	4.33	**10.10 1990**	**40**
21	2716	2.67	4446	1.67	8270	4.33	1730	39
22	2876	2.65	4346	1.67	8530	4.32	1470	38
23	3035	2.67	4246	1.67	8789	4.33	1211	37
24	3195	2.65	4146	1.67	9049	4.32	0951	36
25	**9.79 3354**	2.67	**9.89 4046**	1.67	**9.89 9308**	4.33	**10.10 0692**	**35**
26	3514	2.65	3946	1.67	9568	4.32	0432	34
27	3673	2.65	3846	1.68	.89 9827	4.33	.10 0173	33
28	3832	2.65	3745	1.67	.90 0087	4.32	.09 9913	32
29	3991	2.65	3645	1.68	0346	4.32	9654	31
30	**9.79 4150**	2.63	**9.89 3544**	1.67	**9.90 0605**	4.32	**10.09 9395**	**30**
31	4308	2.65	3444	1.68	0864	4.33	9136	29
32	4467	2.65	3343	1.67	1124	4.32	8876	28
33	4626	2.63	3243	1.68	1383	4.32	8617	27
34	4784	2.63	3142	1.68	1642	4.32	8358	26
35	**9.79 4942**	2.65	**9.89 3041**	1.68	**9.90 1901**	4.32	**10.09 8099**	**25**
36	5101	2.63	2940	1.68	2160	4.33	7840	24
37	5259	2.63	2839	1.67	2420	4.32	7580	23
38	5417	2.63	2739	1.68	2679	4.32	7321	22
39	5575	2.63	2638	1.70	2938	4.32	7062	21
40	**9.79 5733**	2.63	**9.89 2536**	1.68	**9.90 3197**	4.32	**10.09 6803**	**20**
41	5891	2.63	2435	1.68	3456	4.30	6544	19
42	6049	2.62	2334	1.68	3714	4.32	6286	18
43	6206	2.63	2233	1.68	3973	4.32	6027	17
44	6364	2.62	2132	1.70	4232	4.32	5768	16
45	**9.79 6521**	2.63	**9.89 2030**	1.68	**9.90 4491**	4.32	**10.09 5509**	**15**
46	6679	2.62	1929	1.70	4750	4.30	5250	14
47	6836	2.62	1827	1.68	5008	4.32	4992	13
48	6993	2.62	1726	1.70	5267	4.32	4733	12
49	7150	2.62	1624	1.68	5526	4.32	4474	11
50	**9.79 7307**	2.62	**9.89 1523**	1.70	**9.90 5785**	4.30	**10.09 4215**	**10**
51	7464	2.62	1421	1.70	6043	4.32	3957	9
52	7621	2.60	1319	1.70	6302	4.30	3698	8
53	7777	2.62	1217	1.70	6560	4.32	3440	7
54	7934	2.62	1115	1.70	6819	4.30	3181	6
55	**9.79 8091**	2.60	**9.89 1013**	1.70	**9.90 7077**	4.32	**10.09 2923**	**5**
56	8247	2.60	0911	1.70	7336	4.30	2664	4
57	8403	2.62	0809	1.70	7594	4.32	2406	3
58	8560	2.60	0707	1.70	7853	4.30	2147	2
59	8716	2.60	0605	1.70	8111	4.30	1889	1
60	**9.79 8872**		**9.89 0503**		**9.90 8369**		**10.09 1631**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

128° 51°

39° TABLE XIX.—*Continued.* 140°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.79 8872**	2.60	**9.89 0503**	1.72	**9.90 8369**	4.32	**10.09 1631**	**60**
1	9028	2.60	0400	1.70	8628	4.30	1372	59
2	9184	2.58	0298	1.72	8886	4.30	1114	58
3	9339	2.60	0195	1.70	9144	4.30	0856	57
4	9495	2.60	.89 0093	1.72	9402	4.30	0598	56
5	**9.79 9651**	2.58	**9.88 9990**	1.70	**9.90 9660**	4.30	**10.09 0340**	**55**
6	9806	2.60	9888	1.72	.90 9918	4.32	.09 0082	54
7	.79 9962	2.58	9785	1.72	.91 0177	4.30	.08 9823	53
8	.80 0117	2.58	9682	1.72	0435	4.30	9565	52
9	0272	2.58	9579	1.70	0693	4.30	9307	51
10	**9.80 0427**	2.58	**9.88 9477**	1.72	**9.91 0951**	4.30	**10.08 9049**	**50**
11	0582	2.58	9374	1.72	1209	4.30	8791	49
12	0737	2.58	9271	1.72	1467	4.30	8533	48
13	0892	2.58	9168	1.73	1725	4.28	8275	47
14	1047	2.57	9064	1.72	1982	4.30	8018	46
15	**9.80 1201**	2.58	**9.88 8961**	1.72	**9.91 2240**	4.30	**10.08 7760**	**45**
16	1356	2.58	8858	1.72	2498	4.30	7502	44
17	1511	2.57	8755	1.73	2756	4.30	7244	43
18	1665	2.57	8651	1.72	3014	4.28	6986	42
19	1819	2.57	8548	1.73	3271	4.30	6729	41
20	**9.80 1973**	2.58	**9.88 8444**	1.72	**9.91 3529**	4.30	**10.08 6471**	**40**
21	2128	2.57	8341	1.73	3787	4.28	6213	39
22	2282	2.57	8237	1.72	4044	4.30	5956	38
23	2436	2.55	8134	1.73	4302	4.30	5698	37
24	2589	2.57	8030	1.73	4560	4.28	5440	36
25	**9.80 2743**	2.57	**9.88 7926**	1.73	**9.91 4817**	4.30	**10.08 5183**	**35**
26	2897	2.55	7822	1.73	5075	4.28	4925	34
27	3050	2.57	7718	1.73	5332	4.30	4668	33
28	3204	2.55	7614	1.73	5590	4.28	4410	32
29	3357	2.57	7510	1.73	5847	4.28	4153	31
30	**9.80 3511**	2.55	**9.88 7406**	1.73	**9.91 6104**	4.30	**10.08 3896**	**30**
31	3664	2.55	7302	1.73	6362	4.28	3638	29
32	3817	2.55	7198	1.75	6619	4.30	3381	28
33	3970	2.55	7093	1.73	6877	4.28	3123	27
34	4123	2.55	6989	1.73	7134	4.28	2866	26
35	**9.80 4276**	2.55	**9.88 6885**	1.75	**9.91 7391**	4.28	**10.08 2609**	**25**
36	4428	2.55	6780	1.73	7648	4.30	2352	24
37	4581	2.55	6676	1.75	7906	4.28	2094	23
38	4734	2.53	6571	1.75	8163	4.28	1837	22
39	4886	2.55	6466	1.73	8420	4.28	1580	21
40	**9.80 5039**	2.53	**9.88 6362**	1.75	**9.91 8677**	4.28	**10.08 1323**	**20**
41	5191	2.53	6257	1.75	8934	4.28	1066	19
42	5343	2.53	6152	1.75	9191	4.28	0809	18
43	5495	2.53	6047	1.75	9448	4.28	0552	17
44	5647	2.53	5942	1.75	9705	4.28	0295	16
45	**9.80 5799**	2.53	**9.88 5837**	1.75	**9.91 9962**	4.28	**10.08 0038**	**15**
46	5951	2.53	5732	1.75	.92 0219	4.28	.07 9781	14
47	6103	2.52	5627	1.75	0476	4.28	9524	13
48	6254	2.53	5522	1.77	0733	4.28	9267	12
49	6406	2.52	5416	1.75	0990	4.28	9010	11
50	**9.80 6557**	2.53	**9.88 5311**	1.77	**9.92 1247**	4.27	**10.07 8753**	**10**
51	6709	2.52	5205	1.75	1503	4.28	8497	9
52	6860	2.52	5100	1.77	1760	4.28	8240	8
53	7011	2.53	4994	1.75	2017	4.28	7983	7
54	7163	2.52	4889	1.77	2274	4.27	7726	6
55	**9.80 7314**	2.52	**9.88 4783**	1.77	**9.92 2530**	4.28	**10.07 7470**	**5**
56	7465	2.50	4677	1.75	2787	4.28	7213	4
57	7615	2.52	4572	1.77	3044	4.27	6956	3
58	7766	2.52	4466	1.77	3300	4.28	6700	2
59	7917	2.50	4360	1.77	3557	4.28	6443	1
60	**9.80 8067**		**9.88 4254**		**9.92 3814**		**10.07 6186**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

129° 50°

40° Table XIX.—*Continued.* 139°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.80 8067**	2.52	**9.88 4254**	1.77	**9.92 3814**	4.27	**10.07 6186**	**60**
1	8218	2.50	4148	1.77	4070	4.28	5930	59
2	8368	2.52	4042	1.77	4327	4.27	5673	58
3	8519	2.50	3936	1.78	4583	4.28	5417	57
4	8669	2.50	3829	1.77	4840	4.27	5160	56
5	**9.80 8819**	2.50	**9.88 3723**	1.77	**9.92 5096**	4.27	**10.07 4904**	**55**
6	8969	2.50	3617	1.78	5352	4.28	4648	54
7	9119	2.50	3510	1.77	5609	4.27	4391	53
8	9269	2.50	3404	1.78	5865	4.28	4135	52
9	9419	2.50	3297	1.77	6122	4.27	3878	51
10	**9.80 9569**	2.48	**9.88 3191**	1.78	**9.92 6378**	4.27	**10.07 3622**	**50**
11	9718	2.50	3084	1.78	6634	4.27	3366	49
12	.80 9868	2.48	2977	1.77	6890	4.28	3110	48
13	.81 0017	2.50	2871	1.78	7147	4.27	2853	47
14	0167	2.48	2764	1.78	7403	4.27	2597	46
15	**9.81 0316**	2.48	**9.88 2657**	1.78	**9.92 7659**	4.27	**10.07 2341**	**45**
16	0465	2.48	2550	1.78	7915	4.27	2085	44
17	0614	2.48	2443	1.78	8171	4.27	1829	43
18	0763	2.48	2336	1.78	8427	4.28	1573	42
19	0912	2.48	2229	1.80	8684	4.27	1316	41
20	**9.81 1061**	2.48	**9.88 2121**	1.78	**9.92 8940**	4.27	**10.07 1060**	**40**
21	1210	2.47	2014	1.78	9196	4.27	0804	39
22	1358	2.48	1907	1.80	9452	4.27	0548	38
23	1507	2.47	1799	1.78	9708	4.27	0292	37
24	1655	2.48	1692	1.80	.92 9964	4.27	.07 0036	36
25	**9.81 1804**	2.47	**9.88 1584**	1.78	**9.93 0220**	4.25	**10.06 9780**	**35**
26	1952	2.47	1477	1.80	0475	4.27	9525	34
27	2100	2.47	1369	1.80	0731	4.27	9269	33
28	2248	2.47	1261	1.80	0987	4.27	9013	32
29	2396	2.47	1153	1.78	1243	4.27	8757	31
30	**9.81 2544**	2.47	**9.88 1046**	1.80	**9.93 1499**	4.27	**10.06 8501**	**30**
31	2692	2.47	0938	1.80	1755	4.25	8245	29
32	2840	2.47	0830	1.80	2010	4.27	7990	28
33	2988	2.45	0722	1.82	2266	4.27	7734	27
34	3135	2.47	0613	1.80	2522	4.27	7478	26
35	**9.81 3283**	2.45	**9.88 0505**	1.80	**9.93 2778**	4.25	**10.06 7222**	**25**
36	3430	2.47	0397	1.80	3033	4.27	6967	24
37	3578	2.45	0289	1.82	3289	4.27	6711	23
38	3725	2.45	0180	1.80	3545	4.25	6455	22
39	3872	2.45	.88 0072	1.82	3800	4.27	6200	21
40	**9.81 4019**	2.45	**9.87 9963**	1.80	**9.93 4056**	4.25	**10.06 5944**	**20**
41	4166	2.45	9855	1.82	4311	4.27	5689	19
42	4313	2.45	9746	1.82	4567	4.25	5433	18
43	4460	2.45	9637	1.80	4822	4.27	5178	17
44	4607	2.43	9529	1.82	5078	4.25	4922	16
45	**9.81 4753**	2.45	**9.87 9420**	1.82	**9.93 5333**	4.27	**10.06 4667**	**15**
46	4900	2.43	9311	1.82	5589	4.25	4411	14
47	5046	2.45	9202	1.82	5844	4.27	4156	13
48	5193	2.43	9093	1.82	6100	4.25	3900	12
49	5339	2.43	8984	1.82	6355	4.27	3645	11
50	**9.81 5485**	2.45	**9.87 8875**	1.82	**9.93 6611**	4.25	**10.06 3389**	**10**
51	5632	2.43	8766	1.83	6866	4.25	3134	9
52	5778	2.43	8656	1.82	7121	4.27	2879	8
53	5924	2.42	8547	1.82	7377	4.25	2623	7
54	6069	2.43	8438	1.83	7632	4.25	2368	6
55	**9.81 6215**	2.43	**9.87 8328**	1.82	**9.93 7887**	4.25	**10.06 2113**	**5**
56	6361	2.43	8219	1.83	8142	4.27	1858	4
57	6507	2.42	8109	1.83	8398	4.25	1602	3
58	6652	2.43	7999	1.82	8653	4.25	1347	2
59	6798	2.42	7890	1.83	8908	4.25	1092	1
60	**9.81 6943**		**9.87 7780**		**9.93 9163**		**10.06 0837**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

130° 49°

41° TABLE XIX.—*Continued.* 138°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.81 6943**	2.42	**9.87 7780**	1.83	**9.93 9163**	4.25	**10.06 0837**	**60**
1	7088	2.42	7670	1.83	9418	4.25	0582	59
2	7233	2.43	7560	1.83	9673	4.25	0327	58
3	7379	2.42	7450	1.83	.93 9928	4.25	.06 0072	57
4	7524	2.40	7340	1.83	.94 0183	4.27	.05 9817	56
5	**9.81 7668**	2.42	**9.87 7230**	1.83	**9.94 0439**	4.25	**10.05 9561**	**55**
6	7813	2.42	7120	1.83	0694	4.25	9306	54
7	7958	2.42	7010	1.85	0949	4.25	9051	53
8	8103	2.40	6899	1.83	1204	4.25	8796	52
9	8247	2.42	6789	1.85	1459	4.23	8541	51
10	**9.81 8392**	2.40	**9.87 6678**	1.83	**9.94 1713**	4.25	**10.05 8287**	**50**
11	8536	2.42	6568	1.85	1968	4.25	8032	49
12	8681	2.40	6457	1.83	2223	4.25	7777	48
13	8825	2.40	6347	1.85	2478	4.25	7522	47
14	8969	2.40	6236	1.85	2733	4.25	7267	46
15	**9.81 9113**	2.40	**9.87 6125**	1.85	**9.94 2988**	4.25	**10.05 7012**	**45**
16	9257	2.40	6014	1.83	3243	4.25	6757	44
17	9401	2.40	5904	1.85	3498	4.23	6502	43
18	9545	2.40	5793	1.85	3752	4.25	6248	42
19	9689	2.38	5682	1.85	4007	4.25	5993	41
20	**9.81 9832**	2.40	**9.87 5571**	1.87	**9.94 4262**	4.25	**10.05 5738**	**40**
21	.81 9976	2.40	5459	1.85	4517	4.23	5483	39
22	.82 0120	2.38	5348	1.85	4771	4.25	5229	38
23	0263	2.38	5237	1.85	5026	4.25	4974	37
24	0406	2.40	5126	1.87	5281	4.23	4719	36
25	**9.82 0550**	2.38	**9.87 5014**	1.85	**9.94 5535**	4.25	**10.05 4465**	**35**
26	0693	2.38	4903	1.87	5790	4.25	4210	34
27	0836	2.38	4791	1.85	6045	4.23	3955	33
28	0979	2.38	4680	1.87	6299	4.25	3701	32
29	1122	2.38	4568	1.87	6554	4.23	3446	31
30	**9.82 1265**	2.37	**9.87 4456**	1.87	**9.94 6808**	4.25	**10.05 3192**	**30**
31	1407	2.38	4344	1.87	7063	4.25	2937	29
32	1550	2.38	4232	1.85	7318	4.23	2682	28
33	1693	2.37	4121	1.87	7572	4.25	2428	27
34	1835	2.37	4009	1.88	7827	4.23	2173	26
35	**9.82 1977**	2.38	**9.87 3896**	1.87	**9.94 8081**	4.23	**10.05 1919**	**25**
36	2120	2.37	3784	1.87	8335	4.25	1665	24
37	2262	2.37	3672	1.87	8590	4.23	1410	23
38	2404	2.37	3560	1.87	8844	4.25	1156	22
39	2546	2.37	3448	1.88	9099	4.23	0901	21
40	**9.82 2688**	2.37	**9.87 3335**	1.87	**9.94 9353**	4.25	**10.05 0647**	**20**
41	2830	2.37	3223	1.88	9608	4.23	0392	19
42	2972	2.37	3110	1.87	.94 9862	4.23	.05 0138	18
43	3114	2.35	2998	1.88	.95 0116	4.25	.04 9884	17
44	3255	2.37	2885	1.88	0371	4.23	9629	16
45	**9.82 3397**	2.37	**9.87 2772**	1.88	**9.95 0625**	4.23	**10.04 9375**	**15**
46	3539	2.35	2659	1.87	0879	4.23	9121	14
47	3680	2.35	2547	1.88	1133	4.25	8867	13
48	3821	2.37	2434	1.88	1388	4.23	8612	12
49	3963	2.35	2321	1.88	1642	4.23	8358	11
50	**9.82 4104**	2.35	**9.87 2208**	1.88	**9.95 1896**	4.23	**10.04 8104**	**10**
51	4245	2.35	2095	1.90	2150	4.25	7850	9
52	4386	2.35	1981	1.88	2405	4.23	7595	8
53	4527	2.35	1868	1.88	2659	4.23	7341	7
54	4668	2.33	1755	1.90	2913	4.23	7087	6
55	**9.82 4808**	2.35	**9.87 1641**	1.88	**9.95 3167**	4.23	**10.04 6833**	**5**
56	4949	2.35	1528	1.90	3421	4.23	6579	4
57	5090	2.33	1414	1.88	3675	4.23	6325	3
58	5230	2.35	1301	1.90	3929	4.23	6071	2
59	5371	2.33	1187	1.90	4183	4.23	5817	1
60	**9.82 5511**		**9.87 1073**		**9.95 4437**		**10.04 5563**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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42° TABLE XIX.—*Continued.* 137°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.82 5511**	2.33	**9.87 1073**	1.88	**9.95 4437**	4.23	**10.04 5563**	**60**
1	5651	2.33	0960	1.90	4691	4.25	5309	59
2	5791	2.33	0846	1.90	4946	4.23	5054	58
3	5931	2.33	0732	1.90	5200	4.23	4800	57
4	6071	2.33	0618	1.90	5454	4.23	4546	56
5	**9.82 6211**	2.33	**9.87 0504**	1.90	**9.95 5708**	4.22	**10.04 4292**	**55**
6	6351	2.33	0390	1.90	5961	4.23	4039	54
7	6491	2.33	0276	1.92	6215	4.23	3785	53
8	6631	2.32	0161	1.90	6469	4.23	3531	52
9	6770	2.33	.87 0047	1.90	6723	4.23	3277	51
10	**9.82 6910**	2.32	**9.86 9933**	1.92	**9.95 6977**	4.23	**10.04 3023**	**50**
11	7049	2.33	9818	1.90	7231	4.23	2769	49
12	7189	2.32	9704	1.92	7485	4.23	2515	48
13	7328	2.32	9589	1.92	7739	4.23	2261	47
14	7467	2.32	9474	1.90	7993	4.23	2007	46
15	**9.82 7606**	2.32	**9.86 9360**	1.92	**9.95 8247**	4.22	**10.04 1753**	**45**
16	7745	2.32	9245	1.92	8500	4.23	1500	44
17	7884	2.32	9130	1.92	8754	4.23	1246	43
18	8023	2.32	9015	1.92	9008	4.23	0992	42
19	8162	2.32	8900	1.92	9262	4.23	0738	41
20	**9.82 8301**	2.30	**9.86 8785**	1.92	**9.95 9516**	4.22	**10.04 0484**	**40**
21	8439	2.32	8670	1.92	.95 9769	4.23	.04 0231	39
22	8578	2.30	8555	1.92	.96 0023	4.23	.03 9977	38
23	8716	2.32	8440	1.93	0277	4.22	9723	37
24	8855	2.30	8324	1.92	0530	4.23	9470	36
25	**9.82 8993**	2.30	**9.86 8209**	1.93	**9.96 0784**	4.23	**10.03 9216**	**35**
26	9131	2.30	8093	1.92	1038	4.23	8962	34
27	9269	2.30	7978	1.93	1292	4.22	8708	33
28	9407	2.30	7862	1.92	1545	4.23	8455	32
29	9545	2.30	7747	1.93	1799	4.22	8201	31
30	**9.82 9683**	2.30	**9.86 7631**	1.93	**9.96 2052**	4.23	**10.03 7948**	**30**
31	9821	2.30	7515	1.93	2306	4.23	7694	29
32	.82 9959	2.30	7399	1.93	2560	4.22	7440	28
33	.83 0097	2.28	7283	1.93	2813	4.23	7187	27
34	0234	2.30	7167	1.93	3067	4.22	6933	26
35	**9.83 0372**	2.28	**9.86 7051**	1.93	**9.96 3320**	4.23	**10.03 6680**	**25**
36	0509	2.28	6935	1.93	3574	4.23	6426	24
37	0646	2.30	6819	1.93	3828	4.22	6172	23
38	0784	2.28	6703	1.95	4081	4.23	5919	22
39	0921	2.28	6586	1.93	4335	4.22	5665	21
40	**9.83 1058**	2.28	**9.86 6470**	1.95	**9.96 4588**	4.23	**10.03 5412**	**20**
41	1195	2.28	6353	1.93	4842	4.22	5158	19
42	1332	2.28	6237	1.95	5095	4.23	4905	18
43	1469	2.28	6120	1.93	5349	4.22	4651	17
44	1606	2.27	6004	1.95	5602	4.22	4398	16
45	**9.83 1742**	2.28	**9.86 5887**	1.95	**9.96 5855**	4.23	**10.03 4145**	**15**
46	1879	2.27	5770	1.95	6109	4.22	3891	14
47	2015	2.28	5653	1.95	6362	4.23	3638	13
48	2152	2.27	5536	1.95	6616	4.22	3384	12
49	2288	2.28	5419	1.95	6869	4.23	3131	11
50	**9.83 2425**	2.27	**9.86 5302**	1.95	**9.96 7123**	4.22	**10.03 2877**	**10**
51	2561	2.27	5185	1.95	7376	4.22	2624	9
52	2697	2.27	5068	1.97	7629	4.23	2371	8
53	2833	2.27	4950	1.95	7883	4.22	2117	7
54	2969	2.27	4833	1.95	8136	4.22	1864	6
55	**9.83 3105**	2.27	**9.86 4716**	1.97	**9.96 8389**	4.23	**10.03 1611**	**5**
56	3241	2.27	4598	1.95	8643	4.22	1357	4
57	3377	2.25	4481	1.97	8896	4.22	1104	3
58	3512	2.27	4363	1.97	9149	4.23	0851	2
59	3648	2.25	4245	1.97	9403	4.22	0597	1
60	**9.83 3783**		**9.86 4127**		**9.96 9656**		**10.03 0344**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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43° TABLE XIX.—*Continued.* 136°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.83 3783**	2.27	**9.86 4127**	1.95	**9.96 9656**	4.22	**10.03 0344**	**60**
1	3919	2.25	4010	1.97	.96 9909	4.22	.03 0091	59
2	4054	2.25	3892	1.97	.97 0162	4.23	.02 9838	58
3	4189	2.27	3774	1.97	0416	4.22	9584	57
4	4325	2.25	3656	1.97	0669	4.22	9331	56
5	**9.83 4460**	2.25	**9.86 3538**	1.98	**9.97 0922**	4.22	**10.02 9078**	**55**
6	4595	2.25	3419	1.97	1175	4.23	8825	54
7	4730	2.25	3301	1.97	1429	4.22	8571	53
8	4865	2.23	3183	1.98	1682	4.22	8318	52
9	4999	2.25	3064	1.97	1935	4.22	8065	51
10	**9.83 5134**	2.25	**9.86 2946**	1.98	**9.97 2188**	4.22	**10.02 7812**	**50**
11	5269	2.23	2827	1.97	2441	4.23	7559	49
12	5403	2.25	2709	1.98	2695	4.22	7305	48
13	5538	2.23	2590	1.98	2948	4.22	7052	47
14	5672	2.25	2471	1.97	3201	4.22	6799	46
15	**9.83 5807**	2.23	**9.86 2353**	1.98	**9.97 3454**	4.22	**10.02 6546**	**45**
16	5941	2.23	2234	1.98	3707	4.22	6293	44
17	6075	2.23	2115	1.98	3960	4.22	6040	43
18	6209	2.23	1996	1.98	4213	4.22	5787	42
19	6343	2.23	1877	1.98	4466	4.23	5534	41
20	**9.83 6477**	2.23	**9.86 1758**	2.00	**9.97 4720**	4.22	**10.02 5280**	**40**
21	6611	2.23	1638	1.98	4973	4.22	5027	39
22	6745	2.22	1519	1.98	5226	4.22	4774	38
23	6878	2.23	1400	2.00	5479	4.22	4521	37
24	7012	2.23	1280	1.98	5732	4.22	4268	36
25	**9.83 7146**	2.22	**9.86 1161**	2.00	**9.97 5985**	4.22	**10.02 4015**	**35**
26	7279	2.22	1041	1.98	6238	4.22	3762	34
27	7412	2.23	0922	2.00	6491	4.22	3509	33
28	7546	2.22	0802	2.00	6744	4.22	3256	32
29	7679	2.22	0682	2.00	6997	4.22	3003	31
30	**9.83 7812**	2.22	**9.86 0562**	2.00	**9.97 7250**	4.22	**10.02 2750**	**30**
31	7945	2.22	0442	2.00	7503	4.22	2497	29
32	8078	2.22	0322	2.00	7756	4.23	2244	28
33	8211	2.22	0202	2.00	8009	4.22	1991	27
34	8344	2.22	.86 0082	2.00	8262	4.22	1738	26
35	**9.83 8477**	2.22	**9.85 9962**	2.00	**9.97 8515**	4.22	**10.02 1485**	**25**
36	8610	2.20	9842	2.02	8768	4.22	1232	24
37	8742	2.22	9721	2.00	9021	4.22	0979	23
38	8875	2.20	9601	2.02	9274	4.22	0726	22
39	9007	2.22	9480	2.00	9527	4.22	0473	21
40	**9.83 9140**	2.20	**9.85 9360**	2.02	**9.97 9780**	4.22	**10.02 0220**	**20**
41	9272	2.20	9239	2.00	.98 0033	4.22	.01 9967	19
42	9404	2.20	9119	2.02	0286	4.20	9714	18
43	9536	2.20	8998	2.02	0538	4.22	9462	17
44	9668	2.20	8877	2.02	0791	4.22	9209	16
45	**9.83 9800**	2.20	**9.85 8756**	2.02	**9.98 1044**	4.22	**10.01 8956**	**15**
46	.83 9932	2.20	8635	2.02	1297	4.22	8703	14
47	.84 0064	2.20	8514	2.02	1550	4.22	8450	13
48	0196	2.20	8393	2.02	1803	4.22	8197	12
49	0328	2.18	8272	2.02	2056	4.22	7944	11
50	**9.84 0459**	2.20	**9.85 8151**	2.03	**9.98 2309**	4.22	**10.01 7691**	**10**
51	0591	2.18	8029	2.02	2562	4.20	7438	9
52	0722	2.20	7908	2.03	2814	4.22	7186	8
53	0854	2.18	7786	2.02	3067	4.23	6933	7
54	0985	2.18	7665	2.03	3320	4.22	6680	6
55	**9.84 1116**	2.18	**9.85 7543**	2.02	**9.98 3573**	4.22	**10.01 6427**	**5**
56	1247	2.18	7422	2.03	3826	4.23	6174	4
57	1378	2.18	7300	2.03	4079	4.22	5921	3
58	1509	2.18	7178	2.03	4332	4.20	5668	2
59	1640	2.18	7056	2.03	4584	4.22	5416	1
60	**9.84 1771**		**9.85 6934**		**9.98 4837**		**10.01 5163**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

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44° **Table XIX.**—*Concluded.* 135°

′	Sine.	D.1″.	Cosine.	D.1″.	Tang.	D.1″.	Cotang.	′
0	**9.84 1771**	2.18	**9.85 6934**	2.03	**9.98 4837**	4.22	**10.01 5163**	**60**
1	1902	2.18	6812	2.03	5090	4.22	4910	59
2	2033	2.17	6690	2.03	5343	4.22	4657	58
3	2163	2.18	6568	2.03	5596	4.20	4404	57
4	2294	2.17	6446	2.05	5848	4.22	4152	56
5	**9.84 2424**	2.18	**9.85 6323**	2.03	**9.98 6101**	4.22	**10.01 3899**	**55**
6	2555	2.17	6201	2.05	6354	4.22	3646	54
7	2685	2.17	6078	2.03	6607	4.22	3393	53
8	2815	2.18	5956	2.05	6860	4.20	3140	52
9	2946	2.17	5833	2.03	7112	4.22	2888	51
10	**9.84 3076**	2.17	**9.85 5711**	2.05	**9.98 7365**	4.22	**10.01 2635**	**50**
11	3206	2.17	5588	2.05	7618	4.22	2382	49
12	3336	2.17	5465	2.05	7871	4.20	2129	48
13	3466	2.15	5342	2.05	8123	4.22	1877	47
14	3595	2.17	5219	2.05	8376	4.22	1624	46
15	**9.84 3725**	2.17	**9.85 5096**	2.05	**9.98 8629**	4.22	**10.01 1371**	**45**
16	3855	2.15	4973	2.05	8882	4.20	1118	44
17	3984	2.17	4850	2.05	9134	4.22	0866	43
18	4114	2.15	4727	2.07	9387	4.22	0613	42
19	4243	2.15	4603	2.05	9640	4.22	0360	41
20	**9.84 4372**	2.17	**9.85 4480**	2.07	**9.98 9893**	4.20	**10.01 0107**	**40**
21	4502	2.15	4356	2.05	.99 0145	4.22	.00 9855	39
22	4631	2.15	4233	2.07	0398	4.22	9602	38
23	4760	2.15	4109	2.05	0651	4.20	9349	37
24	4889	2.15	3986	2.07	0903	4.22	9097	36
25	**9.84 5018**	2.15	**9.85 3862**	2.07	**9.99 1156**	4.22	**10.00 8844**	**35**
26	5127	2.15	3738	2.07	1409	4.22	8591	34
27	5276	2.15	3614	2.07	1662	4.20	8338	33
28	5405	2.13	3490	2.07	1914	4.22	8086	32
29	5533	2.15	3366	2.07	2167	4.22	7833	31
30	**9.84 5662**	2.13	**9.85 3242**	2.07	**9.99 2420**	4.20	**10.00 7580**	**30**
31	5790	2.15	3118	2.07	2672	4.22	7328	29
32	5919	2.13	2994	2.08	2925	4.22	7075	28
33	6047	2.13	2869	2.07	3178	4.22	6822	27
34	6175	2.15	2745	2.08	3431	4.20	6569	26
35	**9.84 6304**	2.13	**9.85 2620**	2.07	**9.99 3683**	4.22	**10.00 6317**	**25**
36	6432	2.13	2496	2.08	3936	4.22	6064	24
37	6560	2.13	2371	2.07	4189	4.20	5811	23
38	6688	2.13	2247	2.08	4441	4.22	5559	22
39	6816	2.13	2122	2.08	4694	4.22	5306	21
40	**9.84 6944**	2.12	**9.85 1997**	2.08	**9.99 4947**	4.20	**10.00 5053**	**20**
41	7071	2.13	1872	2.08	5199	4.22	4801	19
42	7199	2.13	1747	2.08	5452	4.22	4548	18
43	7327	2.12	1622	2.08	5705	4.20	4295	17
44	7454	2.13	1497	2.08	5957	4.22	4043	16
45	**9.84 7582**	2.12	**9.85 1372**	2.10	**9.99 6210**	4.22	**10.00 3790**	**15**
46	7709	2.12	1246	2.08	6463	4.20	3537	14
47	7836	2.13	1121	2.08	6715	4.22	3285	13
48	7964	2.12	0996	2.10	6968	4.22	3032	12
49	8091	2.12	0870	2.08	7221	4.20	2779	11
50	**9.84 8218**	2.12	**9.85 0745**	2.10	**9.99 7473**	4.22	**10.00 2527**	**10**
51	8345	2.12	0619	2.10	7726	4.22	2274	9
52	8472	2.12	0493	2.08	7979	4.20	2021	8
53	8599	2.12	0368	2.10	8231	4.22	1769	7
54	8726	2.10	0242	2.10	8484	4.22	1516	6
55	**9.84 8852**	2.12	**9.85 0116**	2.10	**9.99 8737**	4.20	**10.00 1263**	**5**
56	8979	2.12	84 9990	2.10	8989	4.22	1011	4
57	9106	2.10	9864	2.10	9242	4.22	0758	3
58	9232	2.12	9738	2.12	9495	4.20	0505	2
59	9359	2.10	9611	2.10	9.99 9747	4.22	0253	1
60	**9.84 9485**		**9.84 9485**		**10.00 0000**		**10.00 0000**	**0**
′	Cosine.	D.1″.	Sine.	D.1″.	Cotang.	D.1″.	Tang.	′

134° 45°

TABLE XX.—NATURAL SINES AND COSINES

′	0° Sine	0° Cosine	′	′	0° Sine	0° Cosine	′	′	0° Sine	0° Cosine	′
0	.00000	1	60	21	.00611	.99998	39	41	.01193	.99993	19
1	.00029	1	59	22	.00640	.99998	38	42	.01222	.99993	18
2	.00058	1	58	23	.00669	.99998	37	43	.01251	.99992	17
3	.00087	1	57	24	.00698	.99998	36	44	.01280	.99992	16
4	.00116	1	56	25	.00727	.99997	35	45	.01309	.99991	15
5	.00145	1	55	26	.00756	.99997	34	46	.01338	.99991	14
6	.00175	1	54	27	.00785	.99997	33	47	.01367	.99991	13
7	.00204	1	53	28	.00814	.99997	32	48	.01396	.99990	12
8	.00233	1	52	29	.00844	.99996	31	49	.01425	.99990	11
9	.00262	1	51	30	.00873	.99996	30	50	.01454	.99989	10
10	.00291	1	50	31	.00902	.99996	29	51	.01483	.99989	9
11	.00320	.99999	49	32	.00931	.99996	28	52	.01513	.99989	8
12	.00349	.99999	48	33	.00960	.99995	27	53	.01542	.99988	7
13	.00378	.99999	47	34	.00989	.99995	26	54	.01571	.99988	6
14	.00407	.99999	46	35	.01018	.99995	25	55	.01600	.99987	5
15	.00436	.99999	45	36	.01047	.99995	24	56	.01629	.99987	4
16	.00465	.99999	44	37	.01076	.99994	23	57	.01658	.99986	3
17	.00495	.99999	43	38	.01105	.99994	22	58	.01687	.99986	2
18	.00524	.99999	42	39	.01134	.99994	21	59	.01716	.99985	1
19	.00553	.99998	41	40	.01164	.99993	20	60	.01745	.99985	0
20	.00582	.99998	40								
′	Cosine 89°	Sine 89°	′	′	Cosine 89°	Sine 89°	′	′	Cosine 89°	Sine 89°	′

TABLE XX.—*Continued*

′	1°		2°		3°		4°		′
	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	
0	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
′	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	′
	88°		87°		86°		85°		

TABLE XX.—*Continued*

′	5° Sine	5° Cosine	6° Sine	6° Cosine	7° Sine	7° Cosine	8° Sine	8° Cosine	′
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	60
1	.08745	.99617	.10482	.99449	.12216	.99251	.13946	.99023	59
2	.08774	.99614	.10511	.99446	.12245	.99248	.13975	.99019	58
3	.08803	.99612	.10540	.99443	.12274	.99244	.14004	.99015	57
4	.08831	.99609	.10569	.99440	.12302	.99240	.14033	.99011	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	55
6	.08889	.99604	.10626	.99434	.12360	.99233	.14090	.99002	54
7	.08918	.99602	.10655	.99431	.12389	.99230	.14119	.98998	53
8	.08947	.99599	.10684	.99428	.12418	.99226	.14148	.98994	52
9	.08976	.99596	.10713	.99424	.12447	.99222	.14177	.98990	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	50
11	.09034	.99591	.10771	.99418	.12504	.99215	.14234	.98982	49
12	.09063	.99588	.10800	.99415	.12533	.99211	.14263	.98978	48
13	.09092	.99586	.10829	.99412	.12562	.99208	.14292	.98973	47
14	.09121	.99583	.10858	.99409	.12591	.99204	.14320	.98969	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	45
16	.09179	.99578	.10916	.99402	.12649	.99197	.14378	.98961	44
17	.09208	.99575	.10945	.99399	.12678	.99193	.14407	.98957	43
18	.09237	.99572	.10973	.99396	.12706	.99189	.14436	.98953	42
19	.09266	.99570	.11002	.99393	.12735	.99186	.14464	.98948	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	40
21	.09324	.99564	.11060	.99386	.12793	.99178	.14522	.98940	39
22	.09353	.99562	.11089	.99383	.12822	.99175	.14551	.98936	38
23	.09382	.99559	.11118	.99380	.12851	.99171	.14580	.98931	37
24	.09411	.99556	.11147	.99377	.12880	.99167	.14608	.98927	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	35
26	.09469	.99551	.11205	.99370	.12937	.99160	.14666	.98919	34
27	.09498	.99548	.11234	.99367	.12966	.99156	.14695	.98914	33
28	.09527	.99545	.11263	.99364	.12995	.99152	.14723	.98910	32
29	.09556	.99542	.11291	.99360	.13024	.99148	.14752	.98906	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	30
31	.09614	.99537	.11349	.99354	.13081	.99141	.14810	.98897	29
32	.09642	.99534	.11378	.99351	.13110	.99137	.14838	.98893	28
33	.09671	.99531	.11407	.99347	.13139	.99133	.14867	.98889	27
34	.09700	.99528	.11436	.99344	.13168	.99129	.14896	.98884	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	25
36	.09758	.99523	.11494	.99337	.13226	.99122	.14954	.98876	24
37	.09787	.99520	.11523	.99334	.13254	.99118	.14982	.98871	23
38	.09816	.99517	.11552	.99331	.13283	.99114	.15011	.98867	22
39	.09845	.99514	.11580	.99327	.13312	.99110	.15040	.98863	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	20
41	.09903	.99508	.11638	.99320	.13370	.99102	.15097	.98854	19
42	.09932	.99506	.11667	.99317	.13399	.99098	.15126	.98849	18
43	.09961	.99503	.11696	.99314	.13427	.99094	.15155	.98845	17
44	.09990	.99500	.11725	.99310	.13456	.99091	.15184	.98841	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	15
46	.10048	.99494	.11783	.99303	.13514	.99083	.15241	.98832	14
47	.10077	.99491	.11812	.99300	.13543	.99079	.15270	.98827	13
48	.10106	.99488	.11840	.99297	.13572	.99075	.15299	.98823	12
49	.10135	.99485	.11869	.99293	.13600	.99071	.15327	.98818	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	10
51	.10192	.99479	.11927	.99286	.13658	.99063	.15385	.98809	9
52	.10221	.99476	.11956	.99283	.13687	.99059	.15414	.98805	8
53	.10250	.99473	.11985	.99279	.13716	.99055	.15442	.98800	7
54	.10279	.99470	.12014	.99276	.13744	.99051	.15471	.98796	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	5
56	.10337	.99464	.12071	.99269	.13802	.99043	.15529	.98787	4
57	.10366	.99461	.12100	.99265	.13831	.99039	.15557	.98782	3
58	.10395	.99458	.12129	.99262	.13860	.99035	.15586	.98778	2
59	.10424	.99455	.12158	.99258	.13889	.99031	.15615	.98773	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	0
′	Cosine 84°	Sine 84°	Cosine 83°	Sine 83°	Cosine 82°	Sine 82°	Cosine 81°	Sine 81°	′

TABLE XX.—*Continued*

′	9° Sine	9° Cosine	10° Sine	10° Cosine	11° Sine	11° Cosine	12° Sine	12° Cosine	′
0	.15643	.98769	.17365	.98481	.19081	.98163	.20791	.97815	60
1	.15672	.98764	.17393	.98476	.19109	.98157	.20820	.97809	59
2	.15701	.98760	.17422	.98471	.19138	.98152	.20848	.97803	58
3	.15730	.98755	.17451	.98466	.19167	.98146	.20877	.97797	57
4	.15758	.98751	.17479	.98461	.19195	.98140	.20905	.97791	56
5	.15787	.98746	.17508	.98455	.19224	.98135	.20933	.97784	55
6	.15816	.98741	.17537	.98450	.19252	.98129	.20962	.97778	54
7	.15845	.98737	.17565	.98445	.19281	.98124	.20990	.97772	53
8	.15873	.98732	.17594	.98440	.19309	.98118	.21019	.97766	52
9	.15902	.98728	.17623	.98435	.19338	.98112	.21047	.97760	51
10	.15931	.98723	.17651	.98430	.19366	.98107	.21076	.97754	50
11	.15959	.98718	.17680	.98425	.19395	.98101	.21104	.97748	49
12	.15988	.98714	.17708	.98420	.19423	.98096	.21132	.97742	48
13	.16017	.98709	.17737	.98414	.19452	.98090	.21161	.97735	47
14	.16046	.98704	.17766	.98409	.19481	.98084	.21189	.97729	46
15	.16074	.98700	.17794	.98404	.19509	.98079	.21218	.97723	45
16	.16103	.98695	.17823	.98399	.19538	.98073	.21246	.97717	44
17	.16132	.98690	.17852	.98394	.19566	.98067	.21275	.97711	43
18	.16160	.98686	.17880	.98389	.19595	.98061	.21303	.97705	42
19	.16189	.98681	.17909	.98383	.19623	.98056	.21331	.97698	41
20	.16218	.98676	.17937	.98378	.19652	.98050	.21360	.97692	40
21	.16246	.98671	.17966	.98373	.19680	.98044	.21388	.97686	39
22	.16275	.98667	.17995	.98368	.19709	.98039	.21417	.97680	38
23	.16304	.98662	.18023	.98362	.19737	.98033	.21445	.97673	37
24	.16333	.98657	.18052	.98357	.19766	.98027	.21474	.97667	36
25	.16361	.98652	.18081	.98352	.19794	.98021	.21502	.97661	35
26	.16390	.98648	.18109	.98347	.19823	.98016	.21530	.97655	34
27	.16419	.98643	.18138	.98341	.19851	.98010	.21559	.97648	33
28	.16447	.98638	.18166	.98336	.19880	.98004	.21587	.97642	32
29	.16476	.98633	.18195	.98331	.19908	.97987	.21616	.97636	31
30	.16505	.98629	.18224	.98325	.19937	.97992	.21644	.97630	30
31	.16533	.98624	.18252	.98320	.19965	.97987	.21672	.97623	29
32	.16562	.98619	.18281	.98315	.19994	.97981	.21701	.97617	28
33	.16591	.98614	.18309	.98310	.20022	.97975	.21729	.97611	27
34	.16620	.98609	.18338	.98304	.20051	.97969	.21758	.97604	26
35	.16648	.98604	.18367	.98299	.20079	.97963	.21786	.97598	25
36	.16677	.98600	.18395	.98294	.20108	.97958	.21814	.97592	24
37	.16706	.98595	.18424	.98288	.20136	.97952	.21843	.97585	23
38	.16734	.98590	.18452	.98283	.20165	.97946	.21871	.97579	22
39	.16763	.98585	.18481	.98277	.20193	.97940	.21899	.97573	21
40	.16792	.98580	.18509	.98272	.20222	.97934	.21928	.97566	20
41	.16820	.98575	.18538	.98267	.20250	.97928	.21956	.97560	19
42	.16840	.98570	.18567	.98261	.20279	.97922	.21985	.97553	18
43	.16878	.98565	.18595	.98256	.20307	.97916	.22013	.97547	17
44	.16906	.98561	.18624	.98250	.20336	.97910	.22041	.97541	16
45	.16935	.98556	.18652	.98245	.20364	.97905	.22070	.97534	15
46	.16964	.98551	.18681	.98240	.20393	.97899	.22098	.97528	14
47	.16992	.98546	.18710	.98234	.20421	.97893	.22126	.97521	13
48	.17021	.98541	.18738	.98229	.20450	.97887	.22155	.97515	12
40	.17050	.98536	.18767	.98223	.20478	.97881	.22183	.97508	11
50	.17078	.98531	.18795	.98218	.20507	.97875	.22212	.97502	10
51	.17107	.98526	.18824	.98212	.20535	.97869	.22240	.97496	9
52	.17136	.98521	.18852	.98207	.20563	.97863	.22268	.97489	8
53	.17164	.98516	.18881	.98201	.20592	.97857	.22297	.97483	7
54	.17193	.98511	.18910	.98196	.20620	.97851	.22325	.97476	6
55	.17222	.98506	.18938	.98190	.20649	.97845	.22353	.97470	5
56	.17250	.98501	.18967	.98185	.20677	.97839	.22382	.97463	4
57	.17279	.98496	.18995	.98179	.20706	.97833	.22410	.97457	3
58	.17308	.98491	.19024	.98174	.20734	.97827	.22438	.97450	2
59	.17336	.98486	.19052	.98168	.20763	.97821	.22467	.97444	1
60	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	0
′	80° Cosine	80° Sine	79° Cosine	79° Sine	78° Cosine	78° Sine	77° Cosine	77° Sine	′

TABLE XX.—*Continued*

′	13° Sine	13° Cosine	14° Sine	14° Cosine	15° Sine	15° Cosine	16° Sine	16° Cosine	′
0	.22495	.97437	.24192	.97030	.25882	.96593	.27564	.96126	60
1	.22523	.97430	.24220	.97023	.25910	.96585	.27592	.96118	59
2	.22552	.97424	.24249	.97015	.25938	.96578	.27620	.96110	58
3	.22580	.97417	.24277	.97008	.25966	.96570	.27648	.96102	57
4	.22608	.97411	.24305	.97001	.25994	.96562	.27676	.96094	56
5	.22637	.97404	.24333	.96994	.26022	.96555	.27704	.96086	55
6	.22665	.97398	.24362	.96987	.26050	.96547	.27731	.96078	54
7	.22693	.97391	.24390	.96980	.26079	.96540	.27759	.96070	53
8	.22722	.97384	.24418	.96973	.26107	.96532	.27787	.96062	52
9	.22750	.97378	.24446	.96966	.26135	.96524	.27815	.96054	51
10	.22778	.97371	.24474	.96959	.26163	.96517	.27843	.96046	50
11	.22807	.97365	.24503	.96952	.26191	.96509	.27871	.96037	49
12	.22835	.97358	.24531	.96945	.26219	.96502	.27899	.96029	48
13	.22863	.97351	.24559	.96937	.26247	.96494	.27927	.96021	47
14	.22892	.97345	.24587	.96930	.26275	.96486	.27955	.96013	46
15	.22920	.97338	.24615	.96923	.26303	.96479	.27983	.96005	45
16	.22948	.97331	.24644	.96916	.26331	.96471	.28011	.95997	44
17	.22977	.97325	.24672	.96909	.26359	.96463	.28039	.95989	43
18	.23005	.97318	.24700	.96902	.26387	.96456	.28067	.95981	42
19	.23033	.97311	.24728	.96894	.26415	.96448	.28095	.95972	41
20	.23062	.97304	.24756	.96887	.26443	.96440	.28123	.95964	40
21	.23090	.97298	.24784	.96880	.26471	.96433	.28150	.95956	39
22	.23118	.97291	.24813	.96873	.26500	.96425	.28178	.95948	38
23	.23146	.97284	.24841	.96866	.26528	.96417	.28206	.95940	37
24	.23175	.97278	.24869	.96858	.26556	.96410	.28234	.95931	36
25	.23203	.97271	.24897	.96851	.26584	.96402	.28262	.95923	35
26	.23231	.97264	.24925	.96844	.26612	.96394	.28290	.95915	34
27	.23260	.97257	.24954	.96837	.26640	.96386	.28318	.95907	33
28	.23288	.97251	.24982	.96829	.26668	.96379	.28346	.95898	32
29	.23316	.97244	.25010	.96822	.26696	.96371	.28374	.95890	31
30	.23345	.97237	.25038	.96815	.26724	.96363	.28402	.95882	30
31	.23373	.97230	.25066	.96807	.26752	.96355	.28429	.95874	29
32	.23401	.97223	.25094	.96800	.26780	.96347	.28457	.95865	28
33	.23429	.97217	.25122	.96793	.26808	.96340	.28485	.95857	27
34	.23458	.97210	.25151	.96786	.26836	.96332	.28513	.95849	26
35	.23486	.97203	.25179	.96778	.26864	.96324	.28541	.95841	25
36	.23514	.97196	.25207	.96771	.26892	.96316	.28569	.95832	24
37	.23542	.97189	.25235	.96764	.26920	.96308	.28597	.95824	23
38	.23571	.97182	.25263	.96756	.26948	.96301	.28625	.95816	22
39	.23599	.97176	.25291	.96749	.26976	.96293	.28652	.95807	21
40	.23627	.97169	.25320	.96742	.27004	.96285	.28680	.95799	20
41	.23656	.97162	.25348	.96734	.27032	.96277	.28708	.95791	19
42	.23684	.97155	.25376	.96727	.27060	.96269	.28736	.95782	18
43	.23712	.97148	.25404	.96719	.27088	.96261	.28764	.95774	17
44	.23740	.97141	.25432	.96712	.27116	.96253	.28792	.95766	16
45	.23769	.97134	.25460	.96705	.27144	.96246	.28820	.95757	15
46	.23797	.97127	.25488	.96697	.27172	.96238	.28847	.95749	14
47	.23825	.97120	.25516	.96690	.27200	.96230	.28875	.95740	13
48	.23853	.97113	.25545	.96682	.27228	.96222	.28903	.95732	12
49	.23882	.97106	.25573	.96675	.27256	.96214	.28931	.95724	11
50	.23910	.97100	.25601	.96667	.27284	.96206	.28959	.95715	10
51	.23938	.97093	.25629	.96660	.27312	.96198	.28987	.95707	9
52	.23966	.97086	.25657	.96653	.27340	.96190	.29015	.95698	8
53	.23995	.97079	.25685	.96645	.27368	.96182	.29042	.95690	7
54	.24023	.97072	.25713	.96638	.27396	.96174	.29070	.95681	6
55	.24051	.97065	.25741	.96630	.27424	.96166	.29098	.95673	5
56	.24079	.97058	.25769	.96623	.27452	.96158	.29126	.95664	4
57	.24108	.97051	.25798	.96615	.27480	.96150	.29154	.95656	3
58	.24136	.97044	.25826	.96608	.27508	.96142	.29182	.95647	2
59	.24164	.97037	.25854	.96600	.27536	.96134	.29209	.95639	1
60	.24192	.97030	.25882	.96593	.27564	.96126	.29237	.95630	0
′	76° Cosine	76° Sine	75° Cosine	75° Sine	74° Cosine	74° Sine	73° Cosine	73° Sine	′

TABLE XX.—*Continued*

′	17° Sine	17° Cosine	18° Sine	18° Cosine	19° Sine	19° Cosine	20° Sine	20° Cosine	′
0	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	60
1	.29265	.95622	.30929	.95097	.32584	.94542	.34229	.93959	59
2	.29293	.95613	.30957	.95088	.32612	.94533	.34257	.93949	58
3	.29321	.95605	.30985	.95079	.32639	.94523	.34284	.93939	57
4	.29348	.95596	.31012	.95070	.32667	.94514	.34311	.93929	56
5	.29376	.95588	.31040	.95061	.32694	.94504	.34339	.93919	55
6	.29404	.95579	.31068	.95052	.32722	.94495	.34366	.93909	54
7	.29432	.95571	.31095	.95043	.32749	.94485	.34393	.93899	53
8	.29460	.95562	.31123	.95033	.32777	.94476	.34421	.93889	52
9	.29487	.95554	.31151	.95024	.32804	.94466	.34448	.93879	51
10	.29515	.95545	.31178	.95015	.32832	.94457	.34475	.93869	50
11	.29543	.95536	.31206	.95006	.32859	.94447	.34503	.93859	49
12	.29571	.95528	.31233	.94997	.32887	.94438	.34530	.93849	48
13	.29599	.95519	.31261	.94988	.32914	.94428	.34557	.93839	47
14	.29626	.95511	.31289	.94979	.32942	.94418	.34584	.93829	46
15	.29654	.95502	.31316	.94970	.32969	.94409	.34612	.93819	45
16	.29682	.95493	.31344	.94961	.32997	.94399	.34639	.93809	44
17	.29710	.95485	.31372	.94952	.33024	.94390	.34666	.93799	43
18	.29737	.95476	.31399	.94943	.33051	.94380	.34694	.93789	42
19	.29765	.95467	.31427	.94933	.33079	.94370	.34721	.93779	41
20	.29793	.95459	.31454	.94924	.33106	.94361	.34748	.93769	40
21	.29821	.95450	.31482	.94915	.33134	.94351	.34775	.93759	39
22	.29849	.95441	.31510	.94906	.33161	.94342	.34803	.93748	38
23	.29876	.95433	.31537	.94897	.33189	.94332	.34830	.93738	37
24	.29904	.95424	.31565	.94888	.33216	.94322	.34857	.93728	36
25	.29932	.95415	.31593	.94878	.33244	.94313	.34884	.93718	35
26	.29960	.95407	.31620	.94869	.33271	.94303	.34912	.93708	34
27	.29987	.95398	.31648	.94860	.33298	.94293	.34939	.93698	33
28	.30015	.95389	.31675	.94851	.33326	.94284	.34966	.93688	32
29	.30043	.95380	.31703	.94842	.33353	.94274	.34993	.93677	31
30	.30071	.95372	.31730	.94832	.33381	.94264	.35021	.93667	30
31	.30098	.95363	.31758	.94823	.33408	.94254	.35048	.93657	29
32	.30126	.95354	.31786	.94814	.33436	.94245	.35075	.93647	28
33	.30154	.95345	.31813	.94805	.33463	.94235	.35102	.93637	27
34	.30182	.95337	.31841	.94795	.33490	.94225	.35130	.93626	26
35	.30209	.95328	.31868	.94786	.33518	.94215	.35157	.93616	25
36	.30237	.95319	.31896	.94777	.33545	.94206	.35184	.93606	24
37	.30265	.95310	.31923	.94768	.33573	.94196	.35211	.93596	23
38	.30292	.95301	.31951	.94758	.33600	.94186	.35239	.93585	22
39	.30320	.95293	.31979	.94749	.33627	.94176	.35266	.93575	21
40	.30348	.95284	.32006	.94740	.33655	.94167	.35293	.93565	20
41	.30376	.95275	.32034	.94730	.33682	.94157	.35320	.93555	19
42	.30403	.95266	.32061	.94721	.33710	.94147	.35347	.93544	18
43	.30431	.95257	.32089	.94712	.33737	.94137	.35375	.93534	17
44	.30459	.95248	.32116	.94702	.33764	.94127	.35402	.93524	16
45	.30486	.95240	.32144	.94693	.33792	.94118	.35429	.93514	15
46	.30514	.95231	.32171	.94684	.33819	.94108	.35456	.93503	14
47	.30542	.95222	.32199	.94674	.33846	.94098	.35484	.93493	13
48	.30570	.95213	.32227	.94665	.33874	.94088	.35511	.93483	12
49	.30597	.95204	.32254	.94656	.33901	.94078	.35538	.93472	11
50	.30625	.95195	.32282	.94646	.33929	.94068	.35565	.93462	10
51	.30653	.95186	.32309	.94637	.33956	.94058	.35592	.93452	9
52	.30680	.95177	.32337	.94627	.33983	.94049	.35619	.93441	8
53	.30708	.95168	.32364	.94618	.34011	.94039	.35647	.93431	7
54	.30736	.95159	.32392	.94609	.34038	.94029	.35674	.93420	6
55	.30763	.95150	.32419	.94599	.34065	.94019	.35701	.93410	5
56	.30791	.95142	.32447	.94590	.34093	.94009	.35728	.93400	4
57	.30819	.95133	.32474	.94580	.34120	.93999	.35755	.93389	3
58	.30846	.95124	.32502	.94571	.34147	.93989	.35782	.93379	2
59	.30874	.95115	.32529	.94561	.34175	.93979	.35810	.93368	1
60	.30902	.95106	.32557	.94552	.34202	.93969	.35837	.93358	0
′	72° Cosine	72° Sine	71° Cosine	71° Sine	70° Cosine	70° Sine	69° Cosine	69° Sine	′

TABLE XX.—*Continued*

′	21° Sine	21° Cosine	22° Sine	22° Cosine	23° Sine	23° Cosine	24° Sine	24° Cosine	′
0	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	60
1	.35864	.93348	.37488	.92707	.39100	.92039	.40700	.91343	59
2	.35891	.93337	.37515	.92697	.39127	.92028	.40727	.91331	58
3	.35918	.93327	.37542	.92686	.39153	.92016	.40753	.91319	57
4	.35945	.93316	.37569	.92675	.39180	.92005	.40780	.91307	56
5	.35973	.93306	.37595	.92664	.39207	.91994	.40806	.91295	55
6	.36000	.93295	.37622	.92653	.39234	.91982	.40833	.91283	54
7	.36027	.93285	.37649	.92642	.39260	.91971	.40860	.91272	53
8	.36054	.93274	.37676	.92631	.39287	.91959	.40886	.91260	52
9	.36081	.93264	.37703	.92620	.39314	.91948	.40913	.91248	51
10	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	.36135	.93243	.37757	.92598	.39367	.91925	.40966	.91224	49
12	.36162	.93232	.37784	.92587	.39394	.91914	.40992	.91212	48
13	.36190	.93222	.37811	.92576	.39421	.91902	.41019	.91200	47
14	.36217	.93211	.37838	.92565	.39448	.91891	.41045	.91188	46
15	.36244	.93201	.37865	.92554	.39474	.91879	.41072	.91176	45
16	.36271	.93190	.37892	.92543	.39501	.91868	.41098	.91164	44
17	.36298	.93180	.37919	.92532	.39528	.91856	.41125	.91152	43
18	.36325	.93169	.37946	.92521	.39555	.91845	.41151	.91140	42
19	.36352	.93159	.37973	.92510	.39581	.91833	.41178	.91128	41
20	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	40
21	.36406	.93137	.38026	.92488	.39635	.91810	.41231	.91104	39
22	.36434	.93127	.38053	.92477	.39661	.91799	.41257	.91092	38
23	.36461	.93116	.38080	.92466	.39688	.91787	.41284	.91080	37
24	.36488	.93106	.38107	.92455	.39715	.91775	.41310	.91068	36
25	.36515	.93095	.38134	.92444	.39741	.91764	.41337	.91056	35
26	.36542	.93084	.38161	.92432	.39768	.91752	.41363	.91044	34
27	.36569	.93074	.38188	.92421	.39795	.91741	.41390	.91032	33
28	.36596	.93063	.38215	.92410	.39822	.91729	.41416	.91020	32
29	.36623	.93052	.38241	.92399	.39848	.91718	.41443	.91008	31
30	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	.36677	.93031	.38295	.92377	.39902	.91694	.41496	.90984	29
32	.36704	.93020	.38322	.92366	.39928	.91683	.41522	.90972	28
33	.36731	.93010	.38349	.92355	.39955	.91671	.41549	.90960	27
34	.36758	.92999	.38376	.92343	.39982	.91660	.41575	.90948	26
35	.36785	.92988	.38403	.92332	.40008	.91648	.41602	.90936	25
36	.36812	.92978	.38430	.92321	.40035	.91636	.41628	.90924	24
37	.36839	.92967	.38456	.92310	.40062	.91625	.41655	.90911	23
38	.36867	.92956	.38483	.92299	.40088	.91613	.41681	.90899	22
39	.36894	.92945	.38510	.92287	.40115	.91601	.41707	.90887	21
40	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	.36948	.92924	.38564	.92265	.40168	.91578	.41760	.90863	19
42	.36975	.92913	.38591	.92254	.40195	.91566	.41787	.90851	18
43	.37002	.92902	.38617	.92243	.40221	.91555	.41813	.90839	17
44	.37029	.92892	.38644	.92231	.40248	.91543	.41840	.90826	16
45	.37056	.92881	.38671	.92220	.40275	.91531	.41866	.90814	15
46	.37083	.92870	.38698	.92209	.40301	.91519	.41892	.90802	14
47	.37110	.92859	.38725	.92198	.40328	.91508	.41919	.90790	13
48	.37137	.92849	.38752	.92186	.40355	.91496	.41945	.90778	12
49	.37164	.92838	.38778	.92175	.40381	.91484	.41972	.90766	11
50	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	.37218	.92816	.38832	.92152	.40434	.91461	.42024	.90741	9
52	.37245	.92805	.38859	.92141	.40461	.91449	.42051	.90729	8
53	.37272	.92794	.38886	.92130	.40488	.91437	.42077	.90717	7
54	.37299	.92784	.38912	.92119	.40514	.91425	.42104	.90704	6
55	.37326	.92773	.38939	.92107	.40541	.91414	.42130	.90692	5
56	.37353	.92762	.38966	.92096	.40567	.91402	.42156	.90680	4
57	.37380	.92751	.38993	.92085	.40594	.91390	.42183	.90668	3
58	.37407	.92740	.39020	.92073	.40621	.91378	.42209	.90655	2
59	.37434	.92729	.39046	.92062	.40647	.91366	.42235	.90643	1
60	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	68°		67°		66°		65°		

TABLE XX.—*Continued*

	25°		26°		27°		28°		
′	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	′
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	60
1	.42288	.90618	.43863	.89867	.45425	.89087	.46973	.88281	59
2	.42315	.90606	.43889	.89854	.45451	.89074	.46999	.88267	58
3	.42341	.90594	.43916	.89841	.45477	.89061	.47024	.88254	57
4	.42367	.90582	.43942	.89828	.45503	.89048	.47050	.88240	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	55
6	.42420	.90557	.43994	.89803	.45554	.89021	.47101	.88213	54
7	.42446	.90545	.44020	.89790	.45580	.89008	.47127	.88199	53
8	.42473	.90532	.44046	.89777	.45606	.88995	.47153	.88185	52
9	.42499	.90520	.44072	.89764	.45632	.88981	.47178	.88172	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	50
11	.42552	.90495	.44124	.89739	.45684	.88955	.47229	.88144	49
12	.42578	.90483	.44151	.89726	.45710	.88942	.47255	.88130	48
13	.42604	.90470	.44177	.89713	.45736	.88928	.47281	.88117	47
14	.42631	.90458	.44203	.89700	.45762	.88915	.47306	.88103	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	45
16	.42683	.90433	.44255	.89674	.45813	.88888	.47358	.88075	44
17	.42709	.90421	.44281	.89662	.45839	.88875	.47383	.88062	43
18	.42736	.90408	.44307	.89649	.45865	.88862	.47409	.88048	42
19	.42762	.90396	.44333	.89636	.45891	.88848	.47434	.88034	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	40
21	.42815	.90371	.44385	.89610	.45942	.88822	.47486	.88006	39
22	.42841	.90358	.44411	.89597	.45968	.88808	.47511	.87993	38
23	.42867	.90346	.44437	.89584	.45994	.88795	.47537	.87979	37
24	.42894	.90334	.44464	.89571	.46020	.88782	.47562	.87965	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	35
26	.42946	.90309	.44516	.89545	.46072	.88755	.47614	.87937	34
27	.42972	.90296	.44542	.89532	.46097	.88741	.47639	.87923	33
28	.42999	.90284	.44568	.89519	.46123	.88728	.47665	.87909	32
29	.43025	.90271	.44594	.89506	.46149	.88715	.47690	.87896	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	30
31	.43077	.90246	.44646	.89480	.46201	.88688	.47741	.87868	29
32	.43104	.90233	.44672	.89467	.46226	.88674	.47767	.87854	28
33	.43130	.90221	.44698	.89454	.46252	.88661	.47793	.87840	27
34	.43156	.90208	.44724	.89441	.46278	.88647	.47818	.87826	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	25
36	.43209	.90183	.44776	.89415	.46330	.88620	.47869	.87798	24
37	.43235	.90171	.44802	.89402	.46355	.88607	.47895	.87784	23
38	.43261	.90158	.44828	.89389	.46381	.88593	.47920	.87770	22
39	.43287	.90146	.44854	.89376	.46407	.88580	.47946	.87756	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	20
41	.43340	.90120	.44906	.89350	.46458	.88553	.47997	.87729	19
42	.43366	.90108	.44932	.89337	.46484	.88539	.48022	.87715	18
43	.43392	.90095	.44958	.89324	.46510	.88526	.48048	.87701	17
44	.43418	.90082	.44984	.89311	.46536	.88512	.48073	.87687	16
45	.43445	.90070	.45010	.89298	.46561	.88499	.48099	.87673	15
46	.43471	.90057	.45036	.89285	.46587	.88485	.48124	.87659	14
47	.43497	.90045	.45062	.89272	.46613	.88472	.48150	.87645	13
48	.43523	.90032	.45088	.89259	.46639	.88458	.48175	.87631	12
49	.43549	.90019	.45114	.89245	.46664	.88445	.48201	.87617	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	10
51	.43602	.89994	.45166	.89219	.46716	.88417	.48252	.87589	9
52	.43628	.89981	.45192	.89206	.46742	.88404	.48277	.87575	8
53	.43654	.89968	.45218	.89193	.46767	.88390	.48303	.87561	7
54	.43680	.89956	.45243	.89180	.46793	.88377	.48328	.87546	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	5
56	.43733	.89930	.45295	.89153	.46844	.88349	.48379	.87518	4
57	.43759	.89918	.45321	.89140	.46870	.88336	.48405	.87504	3
58	.43785	.89905	.45347	.89127	.46896	.88322	.48430	.87490	2
59	.43811	.89892	.45373	.89114	.46921	.88308	.48456	.87476	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	0
′	COSINE	SINE	COSINE	SINE	COSINE	SINE	COSINE	SINE	′
	64°		63°		62°		61°		

TABLE XX.—*Continued*

′	29° Sine	29° Cosine	30° Sine	30° Cosine	31° Sine	31° Cosine	32° Sine	32° Cosine	′
0	.48481	.87462	.50000	.86603	.51504	.85717	.52992	.84805	60
1	.48506	.87448	.50025	.86588	.51529	.85702	.53017	.84789	59
2	.48532	.87434	.50050	.86573	.51554	.85687	.53041	.84774	58
3	.48557	.87420	.50076	.86559	.51579	.85672	.53066	.84759	57
4	.48583	.87406	.50101	.86544	.51604	.85657	.53091	.84743	56
5	.48608	.87391	.50126	.86530	.51628	.85642	.53115	.84728	55
6	.48634	.87377	.50151	.86515	.51653	.85627	.53140	.84712	54
7	.48659	.87363	.50176	.86501	.51678	.85612	.53164	.84697	53
8	.48684	.87349	.50201	.86486	.51703	.85597	.53189	.84681	52
9	.48710	.87335	.50227	.86471	.51728	.85582	.53214	.84666	51
10	.48735	.87321	.50252	.86457	.51753	.85567	.53238	.84650	50
11	.48761	.87306	.50277	.86442	.51778	.85551	.53263	.84635	49
12	.48786	.87292	.50302	.86427	.51803	.85536	.53288	.84619	48
13	.48811	.87278	.50327	.86413	.51828	.85521	.53312	.84604	47
14	.48837	.87264	.50352	.86398	.51852	.85506	.53337	.84588	46
15	.48862	.87250	.50377	.86384	.51877	.85491	.53361	.84573	45
16	.48888	.87235	.50403	.86369	.51902	.85476	.53386	.84557	44
17	.48913	.87221	.50428	.86354	.51927	.85461	.53411	.84542	43
18	.48938	.87207	.50453	.86340	.51952	.85446	.53435	.84526	42
19	.48964	.87193	.50478	.86325	.51977	.85431	.53460	.84511	41
20	.48989	.87178	.50503	.86310	.52002	.85416	.53484	.84495	40
21	.49014	.87164	.50528	.86295	.52026	.85401	.53509	.84480	39
22	.49040	.87150	.50553	.86281	.52051	.85385	.53534	.84464	38
23	.49065	.87136	.50578	.86266	.52076	.85370	.53558	.84448	37
24	.49090	.87121	.50603	.86251	.52101	.85355	.53583	.84433	36
25	.49116	.87107	.50628	.86237	.52126	.85340	.53607	.84417	35
26	.49141	.87093	.50654	.86222	.52151	.85325	.53632	.84402	34
27	.49166	.87079	.50679	.86207	.52175	.85310	.53656	.84386	33
28	.49192	.87064	.50704	.86192	.52200	.85294	.53681	.84370	32
29	.49217	.87050	.50729	.86178	.52225	.85279	.53705	.84355	31
30	.49242	.87036	.50754	.86163	.52250	.85264	.53730	.84339	30
31	.49268	.87021	.50779	.86148	.52275	.85249	.53754	.84324	29
32	.49293	.87007	.50804	.86133	.52299	.85234	.53779	.84308	28
33	.49318	.86993	.50829	.86119	.52324	.85218	.53804	.84292	27
34	.49344	.86978	.50854	.86104	.52349	.85203	.53828	.84277	26
35	.49369	.86964	.50879	.86089	.52374	.85188	.53853	.84261	25
36	.49394	.86949	.50904	.86074	.52399	.85173	.53877	.84245	24
37	.49419	.86935	.50929	.86059	.52423	.85157	.53902	.84230	23
38	.49445	.86921	.50954	.86045	.52448	.85142	.53926	.84214	22
39	.49470	.86906	.50979	.86030	.52473	.85127	.53951	.84198	21
40	.49495	.86892	.51004	.86015	.52498	.85112	.53975	.84182	20
41	.49521	.86878	.51029	.86000	.52522	.85096	.54000	.84167	19
42	.49546	.86863	.51054	.85985	.52547	.85081	.54024	.84151	18
43	.49571	.86849	.51079	.85970	.52572	.85066	.54049	.84135	17
44	.49596	.86834	.51104	.85956	.52597	.85051	.54073	.84120	16
45	.49622	.86820	.51129	.85941	.52621	.85035	.54097	.84104	15
46	.49647	.86805	.51154	.85926	.52646	.85020	.54122	.84088	14
47	.49672	.86791	.51179	.85911	.52671	.85005	.54146	.84072	13
48	.49697	.86777	.51204	.85896	.52696	.84989	.54171	.84057	12
49	.49723	.86762	.51229	.85881	.52720	.84974	.54195	.84041	11
50	.49748	.86748	.51254	.85866	.52745	.84959	.54220	.84025	10
51	.49773	.86733	.51279	.85851	.52770	.84943	.54244	.84009	9
52	.49798	.86719	.51304	.85836	.52794	.84928	.54269	.83994	8
53	.49824	.86704	.51329	.85821	.52819	.84913	.54293	.83978	7
54	.49849	.86690	.51354	.85806	.52844	.84897	.54317	.83962	6
55	.49874	.86675	.51379	.85792	.52869	.84882	.54342	.83946	5
56	.49899	.86661	.51404	.85777	.52893	.84866	.54366	.83930	4
57	.49924	.86646	.51429	.85762	.52918	.84851	.54391	.83915	3
58	.49950	.86632	.51454	.85747	.52943	.84836	.54415	.83899	2
59	.49975	.86617	.51479	.85732	.52967	.84820	.54440	.83883	1
60	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	60°		59°		58°		57°		

TABLE XX.—*Continued*

′	33° Sine	33° Cosine	34° Sine	34° Cosine	35° Sine	35° Cosine	36° Sine	36° Cosine	′
0	.54464	.83867	.55919	.82904	.57358	.81915	.58779	.80902	60
1	.54488	.83851	.55943	.82887	.57381	.81899	.58802	.80885	59
2	.54513	.83835	.55968	.82871	.57405	.81882	.58826	.80867	58
3	.54537	.83819	.55992	.82855	.57429	.81865	.58849	.80850	57
4	.54561	.83804	.56016	.82839	.57453	.81848	.58873	.80833	56
5	.54586	.83788	.56040	.82822	.57477	.81832	.58896	.80816	55
6	.54610	.83772	.56064	.82806	.57501	.81815	.58920	.80799	54
7	.54635	.83756	.56088	.82790	.57524	.81798	.58943	.80782	53
8	.54659	.83740	.56112	.82773	.57548	.81782	.58967	.80765	52
9	.54683	.83724	.56136	.82757	.57572	.81765	.58990	.80748	51
10	.54708	.83708	.56160	.82741	.57596	.81748	.59014	.80730	50
11	.54732	83692	.56184	.82724	.57619	.81731	.59037	.80713	49
12	.54756	.83676	.56208	.82708	.57643	.81714	.59061	.80696	48
13	.54781	.83660	.56232	.82692	.57667	.81698	.59084	.80679	47
14	.54805	.83645	.56256	.82675	.57691	.81681	.59108	.80662	46
15	.54829	.83629	.56280	.82659	.57715	.81664	.59131	.80644	45
16	.54854	.83613	.56305	.82643	.57738	.81647	.59154	.80627	44
17	.54878	.83597	.56329	.82626	.57762	.81631	.59178	.80610	43
18	.54902	.83581	.56353	.82610	.57786	.81614	.59201	.80593	42
19	.54927	.83565	.56377	.82593	.57810	.81597	.59225	.80576	41
20	.54951	.83549	.56401	.82577	.57833	.81580	.59248	.80558	40
21	.54975	.83533	.56425	.82561	.57857	.81563	.59272	.80541	39
22	.54999	.83517	.56449	.82544	.57881	.81546	.59295	.80524	38
23	.55024	.83501	.56473	.82528	.57904	.81530	.59318	.80507	37
24	.55048	.83485	.56497	.82511	.57928	.81513	.59342	.80489	36
25	.55072	.83469	.56521	.82495	.57952	.81496	.59365	.80472	35
26	.55097	.83453	.56545	.82478	.57976	.81479	.59389	.80455	34
27	.55121	.83437	.56569	.82462	.57999	.81462	.59412	.80438	33
28	.55145	.83421	.56593	.82446	.58023	.81445	.59436	.80420	32
29	.55169	.83405	.56617	.82429	.58047	.81428	.59459	.80403	31
30	.55194	.83389	.56641	.82413	.58070	.81412	.59482	.80386	30
31	.55218	.83373	.56665	.82396	.58094	.81395	.59506	.80368	29
32	.55242	.83356	.56689	.82340	.58118	.81378	.59529	.80351	28
33	.55266	.83340	.56713	.82363	.58141	.81361	.59552	.80334	27
34	.55291	.83324	.56736	.82347	.58165	.81344	.59576	.80316	26
35	.55315	.83308	.56760	.82330	.58189	.81327	.59599	.80299	25
36	.55339	.83292	.56784	.82314	.58212	.81310	.59622	.80282	24
37	.55363	.83276	.56808	.82297	.58236	.81293	.59646	.80264	23
38	.55388	.83260	.56832	.82281	.58260	.81276	.59669	.80247	22
39	.55412	.83244	.56856	.82264	.58283	.81259	.59693	.80230	21
40	.55436	.83228	.56880	.82248	.58307	.81242	.59716	.80212	20
41	.55460	.83212	.56904	.82231	.58330	.81225	.59739	.80195	19
42	.55484	.83195	.56928	.82214	.58354	.81208	.59763	.80178	18
43	.55509	.83179	.56952	.82198	.58378	.81191	.59786	.80160	17
44	.55533	.83163	.56976	.82181	.58401	.81174	.59809	.80143	16
45	.55557	.83147	.57000	.82165	.58425	.81157	.59832	.80125	15
46	.55581	.83131	.57024	.82148	.58449	.81140	.59856	.80108	14
47	.55605	.83115	.57047	.82132	.58472	.81123	.59879	.80091	13
48	.55630	.83098	.57071	.82115	.58496	.81106	.59902	.80073	12
49	.55654	.83082	.57095	.82098	.58519	.81089	.59926	.80056	11
50	.55678	.83066	.57119	.82082	.58543	.81072	.59949	.80038	10
51	.55702	.83050	.57143	.82065	.58567	.81055	.59972	.80021	9
52	.55726	.83034	.57167	.82048	.58590	.81038	.59995	.80003	8
53	.55750	.83017	.57191	.82032	.58614	.81021	.60019	.79986	7
54	.55775	.83001	.57215	.82015	.58637	.81004	.60042	.79968	6
55	.55799	.82985	.57238	.81999	.58661	.80987	.60065	.79951	5
56	.55823	.82969	.57262	.81982	.58684	.80970	.60089	.79934	4
57	.55847	.82953	.57286	.81965	.58708	.80953	.60112	.79916	3
58	.55871	.82936	.57310	.81949	.58731	.80936	.60135	.79899	2
59	.55895	.82920	.57334	.81932	.58755	.80919	.60158	.79881	1
60	.55919	.82904	.57358	.81915	.58779	.80902	.60182	.79864	0
′	Cosine 56°	Sine 56°	Cosine 55°	Sine 55°	Cosine 54°	Sine 54°	Cosine 53°	Sine 53°	′

Table XX.—*Continued*

′	37° Sine	37° Cosine	38° Sine	38° Cosine	39° Sine	39° Cosine	40° Sine	40° Cosine	′
0	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	60
1	.60205	.79846	.61589	.78783	.62955	.77696	.64301	.76586	59
2	.60228	.79829	.61612	.78765	.62977	.77678	.64323	.76567	58
3	.60251	.79811	.61635	.78747	.63000	.77660	.64346	.76548	57
4	.60274	.79793	.61658	.78729	.63022	.77641	.64368	.76530	56
5	.60298	.79776	.61681	.78711	.63045	.77623	.64390	.76511	55
6	.60321	.79758	.61704	.78694	.63068	.77605	.64412	.76492	54
7	.60344	.79741	.61726	.78676	.63090	.77586	.64435	.76473	53
8	.60367	.79723	.61749	.78658	.63113	.77568	.64457	.76455	52
9	.60390	.79706	.61772	.78640	.63135	.77550	.64479	.76436	51
10	.60414	.79688	.61795	.78622	.63158	.77531	.64501	.76417	50
11	.60437	.79671	.61818	.78604	.63180	.77513	.64524	.76398	49
12	.60460	.79653	.61841	.78586	.63203	.77494	.64546	.76380	48
13	.60483	.79635	.61864	.78568	.63225	.77476	.64568	.76361	47
14	.60506	.79618	.61887	.78550	.63248	.77458	.64590	.76342	46
15	.60529	.79600	.61909	.78532	.63271	.77439	.64612	.76323	45
16	.60553	.79583	.61932	.78514	.63293	.77421	.64635	.76304	44
17	.60576	.79565	.61955	.78496	.63316	.77402	.64657	.76286	43
18	.60599	.79547	.61978	.78478	.63338	.77384	.64679	.76267	42
19	.60622	.79530	.62001	.78460	.63361	.77366	.64701	.76248	41
20	.60645	.79512	.62024	.78442	.63383	.77347	.64723	.76229	40
21	.60668	.79494	.62046	.78424	.63406	.77329	.64746	.76210	39
22	.60691	.79477	.62069	.78405	.63428	.77310	.64768	.76192	38
23	.60714	.79459	.62092	.78387	.63451	.77292	.64790	.76173	37
24	.60738	.79441	.62115	.78369	.63473	.77273	.64812	.76154	36
25	.60761	.79424	.62138	.78351	.63496	.77255	.64834	.76135	35
26	.60784	.79406	.62160	.78333	.63518	.77236	.64856	.76116	34
27	.60807	.79388	.62183	.78315	.63540	.77218	.64878	.76097	33
28	.60830	.79371	.62206	.78297	.63563	.77199	.64901	.76078	32
29	.60853	.79353	.62229	.78279	.63585	.77181	.64923	.76059	31
30	.60876	.79335	.62251	.78261	.63608	.77162	.64945	.76041	30
31	.60899	.79318	.62274	.78243	.63630	.77144	.64967	.76022	29
32	.60922	.79300	.62297	.78225	.63653	.77125	.64989	.76003	28
33	.60945	.79282	.62320	.78206	.63675	.77107	.65011	.75984	27
34	.60968	.79264	.62342	.78188	.63698	.77088	.65033	.75965	26
35	.60991	.79247	.62365	.78170	.63720	.77070	.65055	.75946	25
36	.61015	.79229	.62388	.78152	.63742	.77051	.65077	.75927	24
37	.61038	.79211	.62411	.78134	.63765	.77033	.65100	.75908	23
38	.61061	.79193	.62433	.78116	.63787	.77014	.65122	.75889	22
39	.61084	.79176	.62456	.78098	.63810	.76996	.65144	.75870	21
40	.61107	.79158	.62479	.78079	.63832	.76977	.65166	.75851	20
41	.61130	.79140	.62502	.78061	.63854	.76959	.65188	.75832	19
42	.61153	.79122	.62524	.78043	.63877	.76940	.65210	.75813	18
43	.61176	.79105	.62547	.78025	.63899	.76921	.65232	.75794	17
44	.61199	.79087	.62570	.78007	.63922	.76903	.65254	.75775	16
45	.61222	.79069	.62592	.77988	.63944	.76884	.65276	.75756	15
46	.61245	.79051	.62615	.77970	.63966	.76866	.65298	.75738	14
47	.61268	.79033	.62638	.77952	.63989	.76847	.65320	.75719	13
48	.61291	.79016	.62660	.77934	.64011	.76828	.65342	.75700	12
49	.61314	.78998	.62683	.77916	.64033	.76810	.65364	.75680	11
50	.61337	.78980	.62706	.77897	.64056	.76791	.65386	.75661	10
51	.61360	.78962	.62728	.77879	.64078	.76772	.65408	.75642	9
52	.61383	.78944	.62751	.77861	.64100	.76754	.65430	.75623	8
53	.61406	.78926	.62774	.77843	.64123	.76735	.65452	.75604	7
54	.61429	.78908	.62796	.77824	.64145	.76717	.65474	.75585	6
55	.61451	.78891	.62819	.77806	.64167	.76698	.65496	.75566	5
56	.61474	.78873	.62842	.77788	.64190	.76679	.65518	.75547	4
57	.61497	.78855	.62864	.77769	.64212	.76661	.65540	.75528	3
58	.61520	.78837	.62887	.77751	.64234	.76642	.65562	.75509	2
59	.61543	.78819	.62909	.77733	.64256	.76623	.65584	.75490	1
60	.61566	.78801	.62932	.77715	.64279	.76604	.65606	.75471	0
′	52° Cosine	52° Sine	51° Cosine	51° Sine	50° Cosine	50° Sine	49° Cosine	49° Sine	′

TABLE XX.—*Concluded*

′	41° Sine	41° Cosine	42° Sine	42° Cosine	43° Sine	43° Cosine	44° Sine	44° Cosine	′
0	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	60
1	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.65694	.75395	.66999	.74227	.68285	.73056	.69549	.71853	56
5	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.65891	.75222	.67194	.74061	.68476	.72377	.69737	.71671	47
14	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.66088	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.66197	.74953	.67495	.73787	.68772	.72597	.70029	.71386	33
28	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.66306	.74857	.67602	.73688	.68878	.72497	.70132	.71284	28
33	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	25
36	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.66436	.74741	.67730	.73570	.69004	.72377	.70257	.71162	22
39	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.66501	.74683	.67795	.73511	.69067	.72317	.70319	.71100	19
42	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.66870	.74353	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
′	Cosine 48°	Sine 48°	Cosine 47°	Sine 47°	Cosine 46°	Sine 46°	Cosine 45°	Sine 45°	′

Table XXI.—Natural Tangents and Cotangents

	0°		1°		2°		3°		
′	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	′
0	.00000	Infinite.	.01746	57.2900	.03492	28.6363	.05241	19.0811	60
1	.00029	3437.750	.01775	56.3506	.03521	28.3994	.05270	18.9755	59
2	.00058	1718.870	.01804	55.4415	.03550	28.1664	.05299	18.8711	58
3	.00087	1145.920	.01833	54.5613	.03579	27.9372	.05328	18.7678	57
4	.00116	859.436	.01862	53.7086	.03609	27.7117	.05357	18.6656	56
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	55
6	.00175	572.957	.01920	52.0807	.03667	27.2715	.05416	18.4645	54
7	.00204	491.106	.01949	51.3032	.03696	27.0566	.05445	18.3655	53
8	.00233	429.718	.01978	50.5485	.03725	26.8450	.05474	18.2677	52
9	.00262	381.971	.02007	49.8157	.03754	26.6367	.05503	18.1708	51
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	50
11	.00320	312.521	.02066	48.4121	.03812	26.2296	.05562	17.9802	49
12	.00349	286.478	.02095	47.7395	.03842	26.0307	.05591	17.8863	48
13	.00378	264.441	.02124	47.0853	.03871	25.8348	.05620	17.7934	47
14	.00407	245.552	.02153	46.4489	.03900	25.6418	.05649	17.7015	46
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	45
16	.00465	214.858	.02211	45.2261	.03958	25.2644	.05708	17.5205	44
17	.00495	202.219	.02240	44.6386	.03987	25.0798	.05737	17.4314	43
18	.00524	190.984	.02269	44.0661	.04016	24.8978	.05766	17.3432	42
19	.00553	180.932	.02298	43.5081	.04046	24.7185	.05795	17.2558	41
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	40
21	.00611	163.700	.02357	42.4335	.04104	24.3675	.05854	17.0837	39
22	.00640	156.259	.02386	41.9158	.04133	24.1957	.05883	16.9990	38
23	.00669	149.465	.02415	41.4106	.04162	24.0263	.05912	16.9150	37
24	.00698	143.237	.02444	40.9174	.04191	23.8593	.05941	16.8319	36
25	.00727	137.507	.02473	40.4358	.04220	23.6945	.05970	16.7496	35
26	.00756	132.219	.02502	39.9655	.04250	23.5321	.05999	16.6681	34
27	.00785	127.321	.02531	39.5059	.04279	23.3718	.06029	16.5874	33
28	.00814	122.774	.02560	39.0568	.04308	23.2137	.06058	16.5075	32
29	.00844	118.540	.02589	38.6177	.04337	23.0577	.06087	16.4283	31
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	30
31	.00902	110.892	.02648	37.7686	.04395	22.7519	.06145	16.2722	29
32	.00931	107.426	.02677	37.3579	.04424	22.6020	.06175	16.1952	28
33	.00960	104.171	.02706	36.9560	.04454	22.4541	.06204	16.1190	27
34	.00989	101.107	.02735	36.5627	.04483	22.3081	.06233	16.0435	26
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	25
36	.01047	95.4895	.02793	35.8006	.04541	22.0217	.06291	15.8945	24
37	.01076	92.9085	.02822	35.4313	.04570	21.8813	.06321	15.8211	23
38	.01105	90.4633	.02851	35.0695	.04599	21.7426	.06350	15.7483	22
39	.01135	88.1436	.02881	34.7151	.04628	21.6056	.06379	15.6762	21
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	20
41	.01193	83.8435	.02939	34.0273	.04687	21.3369	.06437	15.5340	19
42	.01222	81.8470	.02968	33.6935	.04716	21.2049	.06467	15.4638	18
43	.01251	79.9434	.02997	33.3662	.04745	21.0747	.06496	15.3943	17
44	.01280	78.1263	.03026	33.0452	.04774	20.9460	.06525	15.3254	16
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	15
46	.01338	74.7292	.03084	32.4213	.04832	20.6932	.06584	15.1893	14
47	.01367	73.1390	.03114	32.1181	.04862	20.5691	.06613	15.1222	13
48	.01396	71.6151	.03143	31.8205	.04891	20.4465	.06642	15.0557	12
49	.01425	70.1533	.03172	31.5284	.04920	20.3253	.06671	14.9898	11
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	10
51	.01484	67.4019	.03230	30.9599	.04978	20.0872	.06730	14.8596	9
52	.01513	66.1055	.03259	30.6833	.05007	19.9702	.06759	14.7954	8
53	.01542	64.8580	.03288	30.4116	.05037	19.8546	.06788	14.7317	7
54	.01571	63.6567	.03317	30.1446	.05066	19.7403	.06817	14.6685	6
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	5
56	.01629	61.3829	.03376	29.6245	.05124	19.5156	.06876	14.5438	4
57	.01658	60.3058	.03405	29.3711	.05153	19.4051	.06905	14.4823	3
58	.01687	59.2659	.03434	29.1220	.05182	19.2959	.06934	14.4212	2
59	.01716	58.2612	.03463	28.8771	.05212	19.1879	.06963	14.3607	1
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	0
′	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	′
	89°		88°		87°		86°		

TABLE XXI.—*Continued*

′	4° Tan.	4° Co-tan.	5° Tan.	5° Co-tan.	6° Tan.	6° Co-tan.	7° Tan.	7° Co-tan.	′
0	.06993	14.3007	.08749	11.4301	.10510	9.51436	.12278	8.14435	60
1	.07022	14.2411	.08778	11.3919	.10540	9.48781	.12308	8.12481	59
2	.07051	14.1821	.08807	11.3540	.10569	9.46141	.12338	8.10536	58
3	.07080	14.1235	.08837	11.3163	.10599	9.43515	.12367	8.08600	57
4	.07110	14.0655	.08866	11.2789	.10628	9.40904	.12397	8.06674	56
5	.07139	14.0079	.08895	11.2417	.10657	9.38307	.12426	8.04756	55
6	.07168	13.9507	.08925	11.2048	.10687	9.35724	.12456	8.02848	54
7	.07197	13.8940	.08954	11.1681	.10716	9.33154	.12485	8.00948	53
8	.07227	13.8378	.08983	11.1316	.10746	9.30599	.12515	7.99058	52
9	.07256	13.7821	.09013	11.0954	.10775	9.28058	.12544	7.97176	51
10	.07285	13.7267	.09042	11.0594	.10805	9.25530	.12574	7.95302	50
11	.07314	13.6719	.09071	11.0237	.10834	9.23016	.12603	7.93438	49
12	.07344	13.6174	.09101	10.9882	.10863	9.20516	.12633	7.91582	48
13	.07373	13.5634	.09130	10.9529	.10893	9.18028	.12662	7.89734	47
14	.07402	13.5098	.09159	10.9178	.10922	9.15554	.12692	7.87895	46
15	.07431	13.4566	.09189	10.8829	.10952	9.13093	.12722	7.86064	45
16	.07461	13.4039	.09218	10.8483	.10981	9.10646	.12751	7.84242	44
17	.07490	13.3515	.09247	10.8139	.11011	9.08211	.12781	7.82428	43
18	.07519	13.2996	.09277	10.7797	.11040	9.05789	.12810	7.80622	42
19	.07548	13.2480	.09306	10.7457	.11070	9.03379	.12840	7.78825	41
20	.07578	13.1969	.09335	10.7119	.11099	9.00983	.12869	7.77035	40
21	.07607	13.1461	.09365	10.6783	.11128	8.98598	.12899	7.75254	39
22	.07636	13.0958	.09394	10.6450	.11158	8.96227	.12929	7.73480	38
23	.07665	13.0458	.09423	10.6118	.11187	8.93867	.12958	7.71715	37
24	.07695	12.9962	.09453	10.5789	.11217	8.91520	.12988	7.69957	36
25	.07724	12.9469	.09482	10.5462	.11246	8.89185	.13017	7.68208	35
26	.07753	12.8981	.09511	10.5136	.11276	8.86862	.13047	7.66466	34
27	.07782	12.8496	.09541	10.4813	.11305	8.84551	.13076	7.64732	33
28	.07812	12.8014	.09570	10.4491	.11335	8.82252	.13106	7.63005	32
29	.07841	12.7536	.09600	10.4172	.11364	8.79964	.13136	7.61287	31
30	.07870	12.7062	.09629	10.3854	.11394	8.77689	.13165	7.59575	30
31	.07899	12.6591	.09658	10.3538	.11423	8.75425	.13195	7.57872	29
32	.07929	12.6124	.09688	10.3224	.11452	8.73172	.13224	7.56176	28
33	.07958	12.5660	.09717	10.2913	.11482	8.70931	.13254	7.54487	27
34	.07987	12.5199	.09746	10.2602	.11511	8.68701	.13284	7.52806	26
35	.08017	12.4742	.09776	10.2294	.11541	8.66482	.13313	7.51132	25
36	.08046	12.4288	.09805	10.1988	.11570	8.64275	.13343	7.49465	24
37	.08075	12.3838	.09834	10.1683	.11600	8.62078	.13372	7.47806	23
38	.08104	12.3390	.09864	10.1381	.11629	8.59893	.13402	7.46154	22
39	.08134	12.2946	.09893	10.1080	.11659	8.57718	.13432	7.44509	21
40	.08163	12.2505	.09923	10.0780	.11688	8.55555	.13461	7.42871	20
41	.08192	12.2067	.09952	10.0483	.11718	8.53402	.13491	7.41240	19
42	.08221	12.1632	.09981	10.0187	.11747	8.51259	.13521	7.39616	18
43	.08251	12.1201	.10011	9.98931	.11777	8.49128	.13550	7.37999	17
44	.08280	12.0772	.10040	9.96007	.11806	8.47007	.13580	7.36389	16
45	.08309	12.0346	.10069	9.93101	.11836	8.44896	.13609	7.34786	15
46	.08339	11.9923	.10099	9.90211	.11865	8.42795	.13639	7.33190	14
47	.08368	11.9504	.10128	9.87338	.11895	8.40705	.13669	7.31600	13
48	.08397	11.9087	.10158	9.84482	.11924	8.38625	.13698	7.30018	12
49	.08427	11.8673	.10187	9.81641	.11954	8.36555	.13728	7.28442	11
50	.08456	11.8262	.10216	9.78817	.11983	8.34496	.13758	7.26873	10
51	.08485	11.7853	.10246	9.76009	.12013	8.32446	.13787	7.25310	9
52	.08514	11.7448	.10275	9.73217	.12042	8.30406	.13817	7.23754	8
53	.08544	11.7045	.10305	9.70441	.12072	8.28376	.13846	7.22204	7
54	.08573	11.6645	.10334	9.67680	.12101	8.26355	.13876	7.20661	6
55	.08602	11.6248	.10363	9.64935	.12131	8.24345	.13906	7.19125	5
56	.08632	11.5853	.10393	9.62205	.12160	8.22344	.13935	7.17594	4
57	.08661	11.5461	.10422	9.59490	.12190	8.20352	.13965	7.16071	3
58	.08690	11.5072	.10452	9.56791	.12219	8.18370	.13995	7.14553	2
59	.08720	11.4685	.10481	9.54106	.12249	8.16398	.14024	7.13042	1
60	.08749	11.4301	.10510	9.51436	.12278	8.14435	.14054	7.11537	0
′	Co-tan. 85°	Tan. 85°	Co-tan. 84°	Tan. 84°	Co-tan. 83°	Tan. 83°	Co-tan. 82°	Tan. 82°	′

TABLE XXI.—*Continued*

′	8° Tan.	8° Co-tan.	9° Tan.	9° Co-tan.	10° Tan.	10° Co-tan.	11° Tan.	11° Co-tan.	′
0	.14054	7.11537	.15838	6.31375	.17633	5.67128	.19438	5.14455	60
1	.14084	7.10038	.15868	6.30189	.17663	5.66165	.19468	5.13658	59
2	.14113	7.08546	.15898	6.29007	.17693	5.65205	.19498	5.12862	58
3	.14143	7.07059	.15928	6.27829	.17723	5.64248	.19529	5.12069	57
4	.14173	7.05579	.15958	6.26655	.17753	5.63295	.19559	5.11279	56
5	.14202	7.04105	.15988	6.25486	.17783	5.62344	.19589	5.10490	55
6	.14232	7.02637	.16017	6.24321	.17813	5.61397	.19619	5.09704	54
7	.14262	7.01174	.16047	6.23160	.17843	5.60452	.19649	5.08921	53
8	.14291	6.99718	.16077	6.22003	.17873	5.59511	.19680	5.08139	52
9	.14321	6.98268	.16107	6.20851	.17903	5.58573	.19710	5.07360	51
10	.14351	6.96823	.16137	6.19703	.17933	5.57638	.19740	5.06584	50
11	.14381	6.95385	.16167	6.18559	.17963	5.56706	.19770	5.05809	49
12	.14410	6.93952	.16196	6.17419	.17993	5.55777	.19801	5.05037	48
13	.14440	6.92525	.16226	6.16283	.18023	5.54851	.19831	5.04267	47
14	.14470	6.91104	.16256	6.15151	.18053	5.53927	.19861	5.03499	46
15	.14499	6.89688	.16286	6.14023	.18083	5.53007	.19891	5.02734	45
16	.14529	6.88278	.16316	6.12899	.18113	5.52090	.19921	5.01971	44
17	.14559	6.86874	.16346	6.11779	.18143	5.51176	.19952	5.01210	43
18	.14588	6.85475	.16376	6.10664	.18173	5.50264	.19982	5.00451	42
19	.14618	6.84082	.16405	6.09552	.18203	5.49356	.20012	4.99695	41
20	.14648	6.82694	.16435	6.08444	.18233	5.48451	.20042	4.98940	40
21	.14678	6.81312	.16465	6.07340	.18263	5.47548	.20073	4.98188	39
22	.14707	6.79936	.16495	6.06240	.18293	5.46648	.20103	4.97438	38
23	.14737	6.78564	.16525	6.05143	.18323	5.45751	.20133	4.96690	37
24	.14767	6.77199	.16555	6.04051	.18353	5.44857	.20164	4.95945	36
25	.14796	6.75838	.16585	6.02962	.18383	5.43966	.20194	4.95201	35
26	.14826	6.74483	.16615	6.01878	.18414	5.43077	.20224	4.94460	34
27	.14856	6.73133	.16645	6.00797	.18444	5.42192	.20254	4.93721	33
28	.14886	6.71789	.16674	5.99720	.18474	5.41309	.20285	4.92984	32
29	.14915	6.70450	.16704	5.98646	.18504	5.40429	.20315	4.92249	31
30	.14945	6.69116	.16734	5.97576	.18534	5.39552	.20345	4.91516	30
31	.14975	6.67787	.16764	5.96510	.18564	5.38677	.20376	4.90785	29
32	.15005	6.66463	.16794	5.95448	.18594	5.37805	.20406	4.90056	28
33	.15034	6.65144	.16824	5.94390	.18624	5.36936	.20436	4.89330	27
34	.15064	6.63831	.16854	5.93335	.18654	5.36070	.20466	4.88605	26
35	.15094	6.62523	.16884	5.92283	.18684	5.35206	.20497	4.87882	25
36	.15124	6.61219	.16914	5.91235	.18714	5.34345	.20527	4.87162	24
37	.15153	6.59921	.16944	5.90191	.18745	5.33487	.20557	4.86444	23
38	.15183	6.58627	.16974	5.89151	.18775	5.32631	.20588	4.85727	22
39	.15213	6.57339	.17004	5.88114	.18805	5.31778	.20618	4.85013	21
40	.15243	6.56055	.17033	5.87080	.18835	5.30928	.20648	4.84300	20
41	.15272	6.54777	.17063	5.86051	.18865	5.30080	.20679	4.83590	19
42	.15302	6.53503	.17093	5.85024	.18895	5.29235	.20709	4.82882	18
43	.15332	6.52234	.17123	5.84001	.18925	5.28393	.20739	4.82175	17
44	.15362	6.50970	.17153	5.82982	.18955	5.27553	.20770	4.81471	16
45	.15391	6.49710	.17183	5.81966	.18986	5.26715	.20800	4.80769	15
46	.15421	6.48456	.17213	5.80953	.19016	5.25880	.20830	4.80068	14
47	.15451	6.47206	.17243	5.79944	.19046	5.25048	.20861	4.79370	13
48	.15481	6.45961	.17273	5.78938	.19076	5.24218	.20891	4.78673	12
49	.15511	6.44720	.17303	5.77936	.19106	5.23391	.20921	4.77978	11
50	.15540	6.43484	.17333	5.76937	.19136	5.22566	.20952	4.77286	10
51	.15570	6.42253	.17363	5.75941	.19166	5.21744	.20982	4.76595	9
52	.15600	6.41026	.17393	5.74949	.19197	5.20925	.21013	4.75906	8
53	.15630	6.39804	.17423	5.73960	.19227	5.20107	.21043	4.75219	7
54	.15660	6.38587	.17453	5.72974	.19257	5.19293	.21073	4.74534	6
55	.15689	6.37374	.17483	5.71992	.19287	5.18480	.21104	4.73851	5
56	.15719	6.36165	.17513	5.71013	.19317	5.17671	.21134	4.73170	4
57	.15749	6.34961	.17543	5.70037	.19347	5.16863	.21164	4.72490	3
58	.15779	6.33761	.17573	5.69064	.19378	5.16058	.21195	4.71813	2
59	.15809	6.32566	.17603	5.68094	.19408	5.15256	.21225	4.71137	1
60	.15838	6.31375	.17633	5.67128	.19438	5.14455	.21256	4.70463	0
′	81° Co-tan.	81° Tan.	80° Co-tan.	80° Tan.	79° Co-tan.	79° Tan.	78° Co-tan.	78° Tan.	′

TABLE XXI.—*Continued*

′	12° Tan.	12° Co-tan.	13° Tan.	13° Co-tan.	14° Tan.	14° Co-tan.	15° Tan.	15° Co-tan.	′
0	.21256	4.70463	.23087	4.33148	.24933	4.01078	.26795	3.73205	60
1	.21286	4.69791	.23117	4.32573	.24964	4.00582	.26826	3.72771	59
2	.21316	4.69121	.23148	4.32001	.24995	4.00086	.26857	3.72338	58
3	.21347	4.68452	.23179	4.31430	.25026	3.99592	.26888	3.71907	57
4	.21377	4.67786	.23209	4.30860	.25056	3.99099	.26920	3.71476	56
5	.21408	4.67121	.23240	4.30291	.25087	3.98607	.26951	3.71046	55
6	.21438	4.66458	.23271	4.29724	.25118	3.98117	.26982	3.70616	54
7	.21469	4.65797	.23301	4.29159	.25149	3.97627	.27013	3.70188	53
8	.21499	4.65138	.23332	4.28595	.25180	3.97139	.27044	3.69761	52
9	.21529	4.64480	.23363	4.28032	.25211	3.96651	.27076	3.69335	51
10	.21560	4.63825	.23393	4.27471	.25242	3.96165	.27107	3.68909	50
11	.21590	4.63171	.23424	4.26911	.25273	3.95680	.27138	3.68485	49
12	.21621	4.62518	.23455	4.26352	.25304	3.95196	.27169	3.68061	48
13	.21651	4.61868	.23485	4.25795	.25335	3.94713	.27201	3.67638	47
14	.21682	4.61219	.23516	4.25239	.25366	3.94232	.27232	3.67217	46
15	.21712	4.60572	.23547	4.24685	.25397	3.93751	.27263	3.66796	45
16	.21743	4.59927	.23578	4.24132	.25428	3.93271	.27294	3.66376	44
17	.21773	4.59283	.23608	4.23580	.25459	3.92793	.27326	3.65957	43
18	.21804	4.58641	.23639	4.23030	.25490	3.92316	.27357	3.65538	42
19	.21834	4.58001	.23670	4.22481	.25521	3.91839	.27388	3.65121	41
20	.21864	4.57363	.23700	4.21933	.25552	3.91364	.27419	3.64705	40
21	.21895	4.56726	.23731	4.21387	.25583	3.90890	.27451	3.64289	39
22	.21925	4.56091	.23762	4.20842	.25614	3.90417	.27482	3.63874	38
23	.21956	4.55458	.23793	4.20298	.25645	3.89945	.27513	3.63461	37
24	.21986	4.54826	.23823	4.19756	.25676	3.89474	.27545	3.63048	36
25	.22017	4.54196	.23854	4.19215	.25707	3.89004	.27576	3.62636	35
26	.22047	4.53568	.23885	4.18675	.25738	3.88536	.27607	3.62224	34
27	.22078	4.52941	.23916	4.18137	.25769	3.88068	.27638	3.61814	33
28	.22108	4.52316	.23946	4.17600	.25800	3.87601	.27670	3.61405	32
29	.22139	4.51693	.23977	4.17064	.25831	3.87136	.27701	3.60996	31
30	.22169	4.51071	.24008	4.16530	.25862	3.86671	.27732	3.60588	30
31	.22200	4.50451	.24039	4.15997	.25893	3.86208	.27764	3.60181	29
32	.22231	4.49832	.24069	4.15465	.25924	3.85745	.27795	3.59775	28
33	.22261	4.49215	.24100	4.14934	.25955	3.85284	.27826	3.59370	27
34	.22292	4.48600	.24131	4.14405	.25986	3.84824	.27858	3.58966	26
35	.22322	4.47986	.24162	4.13877	.26017	3.84364	.27889	3.58562	25
36	.22353	4.47374	.24193	4.13350	.26048	3.83906	.27920	3.58160	24
37	.22383	4.46764	.24223	4.12825	.26079	3.83449	.27952	3.57758	23
38	.22414	4.46155	.24254	4.12301	.26110	3.82992	.27983	3.57357	22
39	.22444	4.45548	.24285	4.11778	.26141	3.82537	.28015	3.56957	21
40	.22475	4.44942	.24316	4.11256	.26172	3.82083	.28046	3.56557	20
41	.22505	4.44338	.24347	4.10736	.26203	3.81630	.28077	3.56159	19
42	.22536	4.43735	.24377	4.10216	.26235	3.81177	.28109	3.55761	18
43	.22567	4.43134	.24408	4.09699	.26266	3.80726	.28140	3.55364	17
44	.22597	4.42534	.24439	4.09182	.26297	3.80276	.28172	3.54968	16
45	.22628	4.41936	.24470	4.08666	.26328	3.79827	.28203	3.54573	15
46	.22658	4.41340	.24501	4.08152	.26359	3.79378	.28234	3.54179	14
47	.22689	4.40745	.24532	4.07639	.26390	3.78931	.28266	3.53785	13
48	.22719	4.40152	.24562	4.07127	.26421	3.78485	.28297	3.53393	12
49	.22750	4.39560	.24593	4.06616	.26452	3.78040	.28329	3.53001	11
50	.22781	4.38969	.24624	4.06107	.26483	3.77595	.28360	3.52609	10
51	.22811	4.38381	.24655	4.05599	.26515	3.77152	.28391	3.52219	9
52	.22842	4.37793	.24686	4.05092	.26546	3.76709	.28423	3.51829	8
53	.22872	4.37207	.24717	4.04586	.26577	3.76268	.28454	3.51441	7
54	.22903	4.36623	.24747	4.04081	.26608	3.75828	.28486	3.51053	6
55	.22934	4.36040	.24778	4.03578	.26639	3.75388	.28517	3.50666	5
56	.22964	4.35459	.24809	4.03075	.26670	3.74950	.28549	3.50279	4
57	.22995	4.34879	.24840	4.02574	.26701	3.74512	.28580	3.49894	3
58	.23026	4.34300	.24871	4.02074	.26733	3.74075	.28612	3.49509	2
59	.23056	4.33723	.24902	4.01576	.26764	3.73640	.28643	3.49125	1
60	.23087	4.33148	.24933	4.01078	.26795	3.73205	.28675	3.48741	0
′	77° Co-tan.	77° Tan.	76° Co-tan.	76° Tan.	75° Co-tan.	75° Tan.	74° Co-tan.	74° Tan.	′

TABLE XXI.—*Continued*

′	16° Tan.	16° Co-tan.	17° Tan.	17° Co-tan.	18° Tan.	18° Co-tan.	19° Tan.	19° Co-tan.	′
0	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	60
1	.28706	3.48359	.30605	3.26745	.32524	3.07464	.34465	2.90147	59
2	.28738	3.47977	.30637	3.26406	.32556	3.07160	.34498	2.89873	58
3	.28769	3.47596	.30669	3.26067	.32588	3.06857	.34530	2.89600	57
4	.28800	3.47216	.30700	3.25729	.32621	3.06554	.34563	2.89327	56
5	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	.28864	3.46458	.30764	3.25055	.32685	3.05950	.34628	2.88783	54
7	.28895	3.46080	.30796	3.24719	.32717	3.05649	.34661	2.88511	53
8	.28927	3.45703	.30828	3.24383	.32749	3.05349	.34693	2.88240	52
9	.28958	3.45327	.30860	3.24049	.32782	3.05049	.34726	2.87970	51
10	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2.87700	50
11	.29021	3.44576	.30923	3.23381	.32846	3.04450	.34791	2.87430	49
12	.29053	3.44202	.30955	3.23048	.32878	3.04152	.34824	2.87161	48
13	.29084	3.43829	.30987	3.22715	.32911	3.03854	.34856	2.86892	47
14	.29116	3.43456	.31019	3.22384	.32943	3.03556	.34889	2.86624	46
15	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	.29179	3.42713	.31083	3.21722	.33007	3.02963	.34954	2.86089	44
17	.29210	3.42343	.31115	3.21392	.33040	3.02667	.34987	2.85822	43
18	.29242	3.41973	.31147	3.21063	.33072	3.02372	.35019	2.85555	42
19	.29274	3.41604	.31178	3.20734	.33104	3.02077	.35052	2.85289	41
20	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	.29337	3.40869	.31242	3.20079	.33169	3.01489	.35117	2.84758	39
22	.29368	3.40502	.31274	3.19752	.33201	3.01196	.35150	2.84494	38
23	.29400	3.40136	.31306	3.19426	.33233	3.00903	.35183	2.84229	37
24	.29432	3.39771	.31338	3.19100	.33266	3.00611	.35216	2.83965	36
25	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	.29495	3.39042	.31402	3.18451	.33330	3.00028	.35281	2.83439	34
27	.29526	3.38679	.31434	3.18127	.33363	2.99738	.35314	2.83176	33
28	.29558	3.38317	.31466	3.17804	.33395	2.99447	.35346	2.82914	32
29	.29590	3.37955	.31498	3.17481	.33427	2.99158	.35379	2.82653	31
30	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	.29653	3.37234	.31562	3.16838	.33492	2.98580	.35445	2.82130	29
32	.29685	3.36875	.31594	3.16517	.33524	2.98292	.35477	2.81870	28
33	.29716	3.36516	.31626	3.16197	.33557	2.98004	.35510	2.81610	27
34	.29748	3.36158	.31658	3.15877	.33589	2.97717	.35543	2.81350	26
35	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	.29811	3.35443	.31722	3.15240	.33654	2.97144	.35608	2.80833	24
37	.29843	3.35087	.31754	3.14922	.33686	2.96858	.35641	2.80574	23
38	.29875	3.34732	.31786	3.14605	.33718	2.96573	.35674	2.80316	22
39	.29906	3.34377	.31818	3.14288	.33751	2.96288	.35707	2.80059	21
40	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	.29970	3.33670	.31882	3.13656	.33816	2.95721	.35772	2.79545	19
42	.30001	3.33317	.31914	3.13341	.33848	2.95437	.35805	2.79289	18
43	.30033	3.32965	.31946	3.13027	.33881	2.95155	.35838	2.79033	17
44	.30065	3.32614	.31978	3.12713	.33913	2.94872	.35871	2.78778	16
45	.30097	3.32264	.32010	3.12400	.33945	2.94590	.35904	2.78523	15
46	.30128	3.31914	.32042	3.12087	.33978	2.94309	.35937	2.78269	14
47	.30160	3.31565	.32074	3.11775	.34010	2.94028	.35969	2.78014	13
48	.30192	3.31216	.32106	3.11464	.34043	2.93748	.36002	2.77761	12
49	.30224	3.30868	.32139	3.11153	.34075	2.93468	.36035	2.77507	11
50	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	.30287	3.30174	.32203	3.10532	.34140	2.92910	.36101	2.77002	9
52	.30319	3.29829	.32235	3.10223	.34173	2.92632	.36134	2.76750	8
53	.30351	3.29483	.32267	3.09914	.34205	2.92354	.36167	2.76498	7
54	.30382	3.29139	.32299	3.09606	.34238	2.92076	.36199	2.76247	6
55	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	.30446	3.28452	.32363	3.08991	.34303	2.91523	.36265	2.75746	4
57	.30478	3.28109	.32396	3.08685	.34335	2.91246	.36298	2.75496	3
58	.30509	3.27767	.32428	3.08379	.34368	2.90971	.36331	2.75246	2
59	.30541	3.27426	.32460	3.08073	.34400	2.90696	.36364	2.74997	1
60	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
′	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	Co-tan.	Tan.	′
	73°		72°		71°		70°		

TABLE XXI.—*Continued*

′	20° Tan.	20° Co-tan.	21° Tan.	21° Co-tan.	22° Tan.	22° Co-tan.	23° Tan.	23° Co-tan.	′
0	.36397	2.74748	.38386	2.60509	.40403	2.47509	.42447	2.35585	60
1	.36430	2.74499	.38420	2.60283	.40436	2.47302	.42482	2.35395	59
2	.36463	2.74251	.38453	2.60057	.40470	2.47095	.42516	2.35205	58
3	.36496	2.74004	.38487	2.59831	.40504	2.46888	.42551	2.35015	57
4	.36529	2.73756	.38520	2.59606	.40538	2.46682	.42585	2.34825	56
5	.36562	2.73509	.38553	2.59381	.40572	2.46476	.42619	2.34636	55
6	.36595	2.73263	.38587	2.59156	.40606	2.46270	.42654	2.34447	54
7	.36628	2.73017	.38620	2.58932	.40640	2.46065	.42688	2.34258	53
8	.36661	2.72771	.38654	2.58708	.40674	2.45860	.42722	2.34069	52
9	.36694	2.72526	.38687	2.58484	.40707	2.45655	.42757	2.33881	51
10	.36727	2.72281	.38721	2.58261	.40741	2.45451	.42791	2.33693	50
11	.36760	2.72036	.38754	2.58038	.40775	2.45246	.42826	2.33505	49
12	.36793	2.71792	.38787	2.57815	.40809	2.45043	.42860	2.33317	48
13	.36826	2.71548	.38821	2.57593	.40843	2.44839	.42894	2.33130	47
14	.36859	2.71305	.38854	2.57371	.40877	2.44636	.42929	2.32943	46
15	.36892	2.71062	.38888	2.57150	.40911	2.44433	.42963	2.32756	45
16	.36925	2.70819	.38921	2.56928	.40945	2.44230	.42998	2.32570	44
17	.36958	2.70577	.38955	2.56707	.40979	2.44027	.43032	2.32383	43
18	.36991	2.70335	.38988	2.56487	.41013	2.43825	.43067	2.32197	42
19	.37024	2.70094	.39022	2.56266	.41047	2.43623	.43101	2.32012	41
20	.37057	2.69853	.39055	2.56046	.41081	2.43422	.43136	2.31826	40
21	.37090	2.69612	.39089	2.55827	.41115	2.43220	.43170	2.31641	39
22	.37124	2.69371	.39122	2.55608	.41149	2.43019	.43205	2.31456	38
23	.37157	2.69131	.39156	2.55389	.41183	2.42819	.43239	2.31271	37
24	.37190	2.68892	.39190	2.55170	.41217	2.42618	.43274	2.31086	36
25	.37223	2.68653	.39223	2.54952	.41251	2.42418	.43308	2.30902	35
26	.37256	2.68414	.39257	2.54734	.41285	2.42218	.43343	2.30718	34
27	.37289	2.68175	.39290	2.54516	.41319	2.42019	.43378	2.30534	33
28	.37322	2.67937	.39324	2.54299	.41353	2.41819	.43412	2.30351	32
29	.37355	2.67700	.39357	2.54082	.41387	2.41620	.43447	2.30167	31
30	.37388	2.67462	.39391	2.53865	.41421	2.41421	.43481	2.29984	30
31	.37422	2.67225	.39425	2.53648	.41455	2.41223	.43516	2.29801	29
32	.37455	2.66989	.39458	2.53432	.41490	2.41025	.43550	2.29619	28
33	.37488	2.66752	.39492	2.53217	.41524	2.40827	.43585	2.29437	27
34	.37521	2.66516	.39526	2.53001	.41558	2.40629	.43620	2.29254	26
35	.37554	2.66281	.39559	2.52786	.41592	2.40432	.43654	2.29073	25
36	.37588	2.66046	.39593	2.52571	.41626	2.40235	.43689	2.28891	24
37	.37621	2.65811	.39626	2.52357	.41660	2.40038	.43724	2.28710	23
38	.37654	2.65576	.39660	2.52142	.41694	2.39841	.43758	2.28528	22
39	.37687	2.65342	.39694	2.51929	.41728	2.39645	.43793	2.28348	21
40	.37720	2.65109	.39727	2.51715	.41763	2.39449	.43828	2.28167	20
41	.37754	2.64875	.39761	2.51502	.41797	2.39253	.43862	2.27987	19
42	.37787	2.64642	.39795	2.51289	.41831	2.39058	.43897	2.27806	18
43	.37820	2.64410	.39829	2.51076	.41865	2.38862	.43932	2.27626	17
44	.37853	2.64177	.39862	2.50864	.41899	2.38668	.43966	2.27447	16
45	.37887	2.63945	.39896	2.50652	.41933	2.38473	.44001	2.27267	15
46	.37920	2.63714	.39930	2.50440	.41968	2.38279	.44036	2.27088	14
47	.37953	2.63483	.39963	2.50229	.42002	2.38084	.44071	2.26909	13
48	.37986	2.63252	.39997	2.50018	.42036	2.37891	.44105	2.26730	12
49	.38020	2.63021	.40031	2.49807	.42070	2.37697	.44140	2.26552	11
50	.38053	2.62791	.40065	2.49597	.42105	2.37504	.44175	2.26374	10
51	.38086	2.62561	.40098	2.49386	.42139	2.37311	.44210	2.26196	9
52	.38120	2.62332	.40132	2.49177	.42173	2.37118	.44244	2.26018	8
53	.38153	2.62103	.40166	2.48967	.42207	2.36925	.44279	2.25840	7
54	.38186	2.61874	.40200	2.48758	.42242	2.36733	.44314	2.25663	6
55	.38220	2.61646	.40234	2.48549	.42276	2.36541	.44349	2.25486	5
56	.38253	2.61418	.40267	2.48340	.42310	2.36349	.44384	2.25309	4
57	.38286	2.61190	.40301	2.48132	.42345	2.36158	.44418	2.25132	3
58	.38320	2.60963	.40335	2.47924	.42379	2.35967	.44453	2.24956	2
59	.38353	2.60736	.40369	2.47716	.42413	2.35776	.44488	2.24780	1
60	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	0
′	69° Co-tan.	69° Tan.	68° Co-tan.	68° Tan.	67° Co-tan.	67° Tan.	66° Co-tan.	66° Tan.	′

TABLE XXI.—*Continued*

′	24° Tan.	24° Co-tan.	25° Tan.	25° Co-tan.	26° Tan.	26° Co-tan.	27° Tan.	27° Co-tan.	
0	.44523	2.24604	.46631	2.14451	.48773	2.05030	.50953	1.96261	60
1	.44558	2.24428	.46666	2.14288	.48809	2.04879	.50989	1.96120	59
2	.44593	2.24252	.46702	2.14125	.48845	2.04728	.51026	1.95979	58
3	.44627	2.24077	.46737	2.13963	.48881	2.04577	.51063	1.95838	57
4	.44662	2.23902	.46772	2.13801	.48917	2.04426	.51099	1.95698	56
5	.44697	2.23727	.46808	2.13639	.48953	2.04276	.51136	1.95557	55
6	.44732	2.23553	.46843	2.13477	.48989	2.04125	.51173	1.95417	54
7	.44767	2.23378	.46879	2.13316	.49026	2.03975	.51209	1.95277	53
8	.44802	2.23204	.46914	2.13154	.49062	2.03825	.51246	1.95137	52
9	.44837	2.23030	.46950	2.12993	.49098	2.03675	.51283	1.94997	51
10	.44872	2.22857	.46985	2.12832	.49134	2.03526	.51319	1.94858	50
11	.44907	2.22683	.47021	2.12671	.49170	2.03376	.51356	1.94718	49
12	.44942	2.22510	.47056	2.12511	.49206	2.03227	.51393	1.94579	48
13	.44977	2.22337	.47092	2.12350	.49242	2.03078	.51430	1.94440	47
14	.45012	2.22164	.47128	2.12190	.49278	2.02929	.51467	1.94301	46
15	.45047	2.21992	.47163	2.12030	.49315	2.02780	.51503	1.94162	45
16	.45082	2.21819	.47199	2.11871	.49351	2.02631	.51540	1.94023	44
17	.45117	2.21647	.47234	2.11711	.49387	2.02483	.51577	1.93885	43
18	.45152	2.21475	.47270	2.11552	.49423	2.02335	.51614	1.93746	42
19	.45187	2.21304	.47305	2.11392	.49459	2.02187	.51651	1.93608	41
20	.45222	2.21132	.47341	2.11233	.49495	2.02039	.51688	1.93470	40
21	.45257	2.20961	.47377	2.11075	.49532	2.01891	.51724	1.93332	39
22	.45292	2.20790	.47412	2.10916	.49568	2.01743	.51761	1.93195	38
23	.45327	2.20619	.47448	2.10758	.49604	2.01596	.51798	1.93057	37
24	.45362	2.20449	.47483	2.10600	.49640	2.01449	.51835	1.92920	36
25	.45397	2.20278	.47519	2.10442	.49677	2.01302	.51872	1.92782	35
26	.45432	2.20108	.47555	2.10284	.49713	2.01155	.51909	1.92645	34
27	.45467	2.19938	.47590	2.10126	.49749	2.01008	.51946	1.92508	33
28	.45502	2.19769	.47626	2.09969	.49786	2.00862	.51983	1.92371	32
29	.45537	2.19599	.47662	2.09811	.49822	2.00715	.52020	1.92235	31
30	.45573	2.19430	.47698	2.09654	.49858	2.00569	.52057	1.92098	30
31	.45608	2.19261	.47733	2.09498	.49894	2.00423	.52094	1.91962	29
32	.45643	2.19092	.47769	2.09341	.49931	2.00277	.52131	1.91826	28
33	.45678	2.18923	.47805	2.09184	.49967	2.00131	.52168	1.91690	27
34	.45713	2.18755	.47840	2.09028	.50004	1.99986	.52205	1.91554	26
35	.45748	2.18587	.47876	2.08872	.50040	1.99841	.52242	1.91418	25
36	.45784	2.18419	.47912	2.08716	.50076	1.99695	.52279	1.91282	24
37	.45819	2.18251	.47948	2.08560	.50113	1.99550	.52316	1.91147	23
38	.45854	2.18084	.47984	2.08405	.50149	1.99406	.52353	1.91012	22
39	.45889	2.17916	.48019	2.08250	.50185	1.99261	.52390	1.90876	21
40	.45924	2.17749	.48055	2.08094	.50222	1.99116	.52427	1.90741	20
41	.45960	2.17582	.48091	2.07939	.50258	1.98972	.52464	1.90607	19
42	.45995	2.17416	.48127	2.07785	.50295	1.98828	.52501	1.90472	18
43	.46030	2.17249	.48163	2.07630	.50331	1.98684	.52538	1.90337	17
44	.46065	2.17083	.48198	2.07476	.50368	1.98540	.52575	1.90203	16
45	.46101	2.16917	.48234	2.07321	.50404	1.98396	.52613	1.90069	15
46	.46136	2.16751	.48270	2.07167	.50441	1.98253	.52650	1.89935	14
47	.46171	2.16585	.48306	2.07014	.50477	1.98110	.52687	1.89801	13
48	.46206	2.16420	.48342	2.06860	.50514	1.97966	.52724	1.89667	12
49	.46242	2.16255	.48378	2.06706	.50550	1.97823	.52761	1.89533	11
50	.46277	2.16090	.48414	2.06553	.50587	1.97680	.52798	1.89400	10
51	.46312	2.15925	.48450	2.06400	.50623	1.97538	.52836	1.89266	9
52	.46348	2.15760	.48486	2.06247	.50660	1.97395	.52873	1.89133	8
53	.46383	2.15596	.48521	2.06094	.50696	1.97253	.52910	1.89000	7
54	.46418	2.15432	.48557	2.05942	.50733	1.97111	.52947	1.88867	6
55	.46454	2.15268	.48593	2.05790	.50769	1.96969	.52984	1.88734	5
56	.46489	2.15104	.48629	2.05637	.50806	1.96827	.53022	1.88602	4
57	.46525	2.14940	.48665	2.05485	.50843	1.96685	.53059	1.88469	3
58	.46560	2.14777	.48701	2.05333	.50879	1.96544	.53096	1.88337	2
59	.46595	2.14614	.48737	2.05182	.50916	1.96402	.53134	1.88205	1
60	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	0
′	65° Co-tan.	65° Tan.	64° Co-tan.	64° Tan.	63° Co-tan.	63° Tan.	62° Co-tan.	62° Tan.	′

TABLE XXI.—*Continued*

	28°		29°		30°		31°		
	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	
0	.53171	1.88073	.55431	1.80405	.57735	1.73205	.60086	1.66428	60
1	.53208	1.87941	.55469	1.80281	.57774	1.73089	.60126	1.66318	59
2	.53246	1.87809	.55507	1.80158	.57813	1.72973	.60165	1..6209	58
3	.53283	1.87677	.55545	1.80034	.57851	1.72857	.60205	1.66099	57
4	.53320	1.87546	.55583	1.79911	.57890	1.72741	.60245	1.65990	56
5	.53358	1.87415	.55621	1.79788	.57929	1.72625	.60284	1.65881	55
6	.53395	1.87283	.55659	1.79665	.57968	1.72509	.60324	1.65772	54
7	.53432	1.87152	.55697	1.79542	.58007	1.72393	.60364	1.65663	53
8	.53470	1.87021	.55736	1.79419	.58046	1.72278	.60403	1.65534	52
9	.53507	1.86891	.55774	1.79296	.58085	1.72163	.60443	1.65445	51
10	.53545	1.86760	.55812	1.79174	.58124	1.72047	.60483	1.65337	50
11	.53582	1.86630	.55850	1.79051	.58162	1.71932	.60522	1.65228	49
12	.53620	1.86499	.55888	1.78929	.58201	1.71817	.60562	1.65120	48
13	.53657	1.86369	.55926	1.78807	.58240	1.71702	.60602	1.65011	47
14	.53694	1.86239	.55964	1.78685	.58279	1.71588	.60642	1.64903	46
15	.53732	1.86109	.56003	1.78563	.58318	1.71473	.60681	1.64795	45
16	.53769	1.85979	.56041	1.78441	.58357	1.71358	.60721	1.64687	44
17	.53807	1.85850	.56079	1.78319	.58396	1.71244	.60761	1.64579	43
18	.53844	1.85720	.56117	1.78198	.58435	1.71129	.60801	1.64471	42
19	.53882	1.85591	.56156	1.78077	.58474	1.71015	.60841	1.64363	41
20	.53920	1.85462	.56194	1.77955	.58513	1.70901	.60881	1.64256	40
21	.53957	1.85333	.56232	1.77834	.58552	1.70787	.60921	1.64148	39
22	.53995	1.85204	.56270	1.77713	.58591	1.70673	.60960	1.64041	38
23	.54032	1.85075	.56309	1.77592	.58631	1.70560	.61000	1.63934	37
24	.54070	1.84946	.56347	1.77471	.58670	1.70446	.61040	1.63826	36
25	.54107	1.84818	.56385	1.77351	.58709	1.70332	.61080	1.63719	35
26	.54145	1.84689	.56424	1.77230	.58748	1.70219	.61120	1.63612	34
27	.54183	1.84561	.56462	1.77110	.58787	1.70106	.61160	1.63505	33
28	.54220	1.84433	.56500	1.76990	.58826	1.69992	.61200	1.63398	32
29	.54258	1.84305	.56539	1.76869	.58865	1.69879	.61240	1.63292	31
30	.54296	1.84177	.56577	1.76749	.58904	1.69766	.61280	1.63185	30
31	.54333	1.84049	.56616	1.76630	.58944	1.69653	.61320	1.63079	29
32	.54371	1.83922	.56654	1.76510	.58983	1.69541	.61360	1.62972	28
33	.54409	1.83794	.56693	1.76390	.59022	1.69428	.61400	1.62866	27
34	.54446	1.83667	.56731	1.76271	.59061	1.69316	.61440	1.62760	26
35	.54484	1.83540	.56769	1.76151	.59101	1.69203	.61480	1.62654	25
36	.54522	1.83413	.56808	1.76032	.59140	1.69091	.61520	1.62548	24
37	.54560	1.83286	.56846	1.75913	.59179	1.68979	.61561	1.62442	23
38	.54597	1.83159	.56885	1.75794	.59218	1.68866	.61601	1.62336	22
39	.54635	1.83033	.56923	1.75675	.59258	1.68754	.61641	1.62230	21
40	.54673	1.82906	.56962	1.75556	.59297	1.68643	.61681	1.62125	20
41	.54711	1.82780	.57000	1.75437	.59336	1.68531	.61721	1.62019	19
42	.54748	1.82654	.57039	1.75319	.59376	1.68419	.61761	1.61914	18
43	.54786	1.82528	.57078	1.75200	.59415	1.68308	.61801	1.61808	17
44	.54824	1.82402	.57116	1.75082	.59454	1.68196	.61842	1.61703	16
45	.54862	1.82276	.57155	1.74964	.59494	1.68085	.61882	1.61598	15
46	.54900	1.82150	.57193	1.74846	.59533	1.67974	.61922	1.61493	14
47	.54938	1.82025	.57232	1.74728	.59573	1.67863	.61962	1.61388	13
48	.54975	1.81899	.57271	1.74610	.59612	1.67752	.62003	1.61283	12
49	.55013	1.81774	.57309	1.74492	.59651	1.67641	.62043	1.61179	11
50	.55051	1.81649	.57348	1.74375	.59691	1.67530	.62083	1.61074	10
51	.55089	1.81524	.57386	1.74257	.59730	1.67419	.62124	1.60970	9
52	.55127	1.81399	.57425	1.74140	.59770	1.67309	.62164	1.60865	8
53	.55165	1.81274	.57464	1.74022	.59809	1.67198	.62204	1.60761	7
54	.55203	1.81150	.57503	1.73905	.59849	1.67088	.62245	1.60657	6
55	.55241	1.81025	.57541	1.73788	.59888	1.66978	.62285	1.60553	5
56	.55279	1.80901	.57580	1.73671	.59928	1.66867	.62325	1.60449	4
57	.55317	1.80777	.57619	1.73555	.59967	1.66757	.62366	1.60345	3
58	.55355	1.80653	.57657	1.73438	.60007	1.66647	.62406	1.60241	2
59	.55393	1.80529	.57696	1.73321	.60046	1.66538	.62446	1.60137	1
60	.55431	1.80405	.57735	1.73205	.60086	1.66428	.62487	1.60033	0
	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	CO-TAN.	TAN.	
	61°		60°		59°		58°		

TABLE XXI.—*Continued*

′	32° Tan.	32° Co-tan.	33° Tan.	33° Co-tan.	34° Tan.	34° Co-tan.	35° Tan.	35° Co-tan.	′
0	.62487	1.60033	.64941	1.53986	.67451	1.48256	.70021	1.42815	60
1	.62527	1.59930	.64982	1.53888	.67493	1.48163	.70064	1.42726	59
2	.62568	1.59826	.65023	1.53791	.67536	1.48070	.70107	1.42638	58
3	.62608	1.59723	.65065	1.53693	.67578	1.47977	.70151	1.42550	57
4	.62649	1.59620	.65106	1.53595	.67620	1.47885	.70194	1.42462	56
5	.62689	1.59517	.65148	1.53497	.67663	1.47792	.70238	1.42374	55
6	.62730	1.59414	.65189	1.53400	.67705	1.47699	.70281	1.42286	54
7	.62770	1.59311	.65231	1.53302	.67748	1.47607	.70325	1.42198	53
8	.62811	1.59208	.65272	1.53205	.67790	1.47514	.70368	1.42110	52
9	.62852	1.59105	.65314	1.53107	.67832	1.47422	.70412	1.42022	51
10	.62892	1.59002	.65355	1.53010	.67875	1.47330	.70455	1.41934	50
11	.62933	1.58900	.65397	1.52913	.67917	1.47238	.70499	1.41847	49
12	.62973	1.58797	.65438	1.52816	.67960	1.47146	.70542	1.41759	48
13	.63014	1.58695	.65480	1.52719	.68002	1.47053	.70586	1.41672	47
14	.63055	1.58593	.65521	1.52622	.68045	1.46962	.70629	1.41584	46
15	.63095	1.58490	.65563	1.52525	.68088	1.46870	.70673	1.41497	45
16	.63136	1.58388	.65604	1.52429	.68130	1.46778	.70717	1.41409	44
17	.63177	1.58286	.65646	1.52332	.68173	1.46686	.70760	1.41322	43
18	.63217	1.58184	.65688	1.52235	.68215	1.46595	.70804	1.41235	42
19	.63258	1.58083	.65729	1.52139	.68258	1.46503	.70848	1.41148	41
20	.63299	1.57981	.65771	1.52043	.68301	1.46411	.70891	1.41061	40
21	.63340	1.57879	.65813	1.51946	.68343	1.46320	.70935	1.40974	39
22	.63380	1.57778	.65854	1.51850	.68386	1.46229	.70979	1.40887	38
23	.63421	1.57676	.65896	1.51754	.68429	1.46137	.71023	1.40800	37
24	.63462	1.57575	.65938	1.51658	.68471	1.46046	.71066	1.40714	36
25	.63503	1.57474	.65980	1.51562	.68514	1.45955	.71110	1.40627	35
26	.63544	1.57372	.66021	1.51466	.68557	1.45864	.71154	1.40540	34
27	.63584	1.57271	.66063	1.51370	.68600	1.45773	.71198	1.40454	33
28	.63625	1.57170	.66105	1.51275	.68642	1.45682	.71242	1.40367	32
29	.63666	1.57069	.66147	1.51179	.68685	1.45592	.71285	1.40281	31
30	.63707	1.56969	.66189	1.51084	.68728	1.45501	.71329	1.40195	30
31	.63748	1.56868	.66230	1.50988	.68771	1.45410	.71373	1.40109	29
32	.63789	1.56767	.66272	1.50893	.68814	1.45320	.71417	1.40022	28
33	.63830	1.56667	.66314	1.50797	.68857	1.45229	.71461	1.39936	27
34	.63871	1.56566	.66356	1.50702	.68900	1.45139	.71505	1.39850	26
35	.63912	1.56466	.66398	1.50607	.68942	1.45049	.71549	1.39764	25
36	.63953	1.56366	.66440	1.50512	.68985	1.44958	.71593	1.39679	24
37	.63994	1.56265	.66482	1.50417	.69028	1.44868	.71637	1.39593	23
38	.64035	1.56165	.66524	1.50322	.69071	1.44778	.71681	1.39507	22
39	.64076	1.56065	.66566	1.50228	.69114	1.44688	.71725	1.39421	21
40	.64117	1.55966	.66608	1.50133	.69157	1.44598	.71769	1.39336	20
41	.64158	1.55866	.66650	1.50038	.69200	1.44508	.71813	1.39250	19
42	.64199	1.55766	.66692	1.49944	.69243	1.44418	.71857	1.39165	18
43	.64240	1.55666	.66734	1.49849	.69286	1.44329	.71901	1.39079	17
44	.64281	1.55567	.66776	1.49755	.69329	1.44239	.71946	1.38994	16
45	.64322	1.55467	.66818	1.49661	.69372	1.44149	.71990	1.38909	15
46	.64363	1.55368	.66860	1.49566	.69416	1.44060	.72034	1.38824	14
47	.64404	1.55269	.66902	1.49472	.69459	1.43970	.72078	1.38738	13
48	.64446	1.55170	.66944	1.49378	.69502	1.43881	.72122	1.38653	12
49	.64487	1.55071	.66986	1.49284	.69545	1.43792	.72166	1.38568	11
50	.64528	1.54972	.67028	1.49190	.69588	1.43703	.72211	1.38484	10
51	.64569	1.54873	.67071	1.49097	.69631	1.43614	.72255	1.38399	9
52	.64610	1.54774	.67113	1.49003	.69675	1.43525	.72299	1.38314	8
53	.64652	1.54675	.67155	1.48909	.69718	1.43436	.72344	1.38229	7
54	.64693	1.54576	.67197	1.48816	.69761	1.43347	.72388	1.38145	6
55	.64734	1.54478	.67239	1.48722	.69804	1.43258	.72432	1.38060	5
56	.64775	1.54379	.67282	1.48629	.69847	1.43169	.72477	1.37976	4
57	.64817	1.54281	.67324	1.48536	.69891	1.43080	.72521	1.37891	3
58	.64858	1.54183	.67366	1.48442	.69934	1.42992	.72565	1.37807	2
59	.64899	1.54085	.67409	1.48349	.69977	1.42903	.72610	1.37722	1
60	.64941	1.53986	.67451	1.48256	.70021	1.42815	.72654	1.37638	0
′	57° Co-tan.	57° Tan.	56° Co-tan	56° Tan.	55° Co-tan.	55° Tan.	54° Co-tan.	54° Tan.	′

TABLE XXI.—*Continued*

′	36° Tan.	36° Co-tan.	37° Tan.	37° Co-tan.	38° Tan.	38° Co-tan.	39° Tan.	39° Co-tan.	′
0	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	60
1	.72699	1.37554	.75401	1.32624	.78175	1.27917	.81027	1.23416	59
2	.72743	1.37470	.75447	1.32544	.78222	1.27841	.81075	1.23343	58
3	.72788	1.37386	.75492	1.32464	.78269	1.27764	.81123	1.23270	57
4	.72832	1.37302	.75538	1.32384	.78316	1.27688	.81171	1.23196	56
5	.72877	1.37218	.75584	1.32304	.78363	1.27611	.81220	1.23123	55
6	.72921	1.37134	.75629	1.32224	.78410	1.27535	.81268	1.23050	54
7	.72966	1.37050	.75675	1.32144	.78457	1.27458	.81316	1.22977	53
8	.73010	1.36967	.75721	1.32064	.78504	1.27382	.81364	1.22904	52
9	.73055	1.36883	.75767	1.31984	.78551	1.27306	.81413	1.22831	51
10	.73100	1.36800	.75812	1.31904	.78598	1.27230	.81461	1.22758	50
11	.73144	1.36716	.75858	1.31825	.78645	1.27153	.81510	1.22685	49
12	.73189	1.36633	.75904	1.31745	.78692	1.27077	.81558	1.22612	48
13	.73234	1.36549	.75950	1.31666	.78739	1.27001	.81606	1.22539	47
14	.73278	1.36466	.75996	1.31586	.78786	1.26925	.81655	1.22467	46
15	.73323	1.36383	.76042	1.31507	.78834	1.26849	.81703	1.22394	45
16	.73368	1.36300	.76088	1.31427	.78881	1.26774	.81752	1.22321	44
17	.73413	1.36217	.76134	1.31348	.78928	1.26698	.81800	1.22249	43
18	.73457	1.36133	.76180	1.31269	.78975	1.26622	.81849	1.22176	42
19	.73502	1.36051	.76226	1.31190	.79022	1.26546	.81898	1.22104	41
20	.73547	1.35968	.76272	1.31110	.79070	1.26471	.81946	1.22031	40
21	.73592	1.35885	.76318	1.31031	.79117	1.26395	.81995	1.21959	39
22	.73637	1.35802	.76364	1.30952	.79164	1.26319	.82044	1.21886	38
23	.73681	1.35719	.76410	1.30873	.79212	1.26244	.82092	1.21814	37
24	.73726	1.35637	.76456	1.30795	.79259	1.26169	.82141	1.21742	36
25	.73771	1.35554	.76502	1.30716	.79306	1.26093	.82190	1.21670	35
26	.73816	1.35472	.76548	1.30637	.79354	1.26018	.82238	1.21598	34
27	.73861	1.35389	.76594	1.30558	.79401	1.25943	.82287	1.21526	33
28	.73906	1.35307	.76640	1.30480	.79449	1.25867	.82336	1.21454	32
29	.73951	1.35224	.76686	1.30401	.79496	1.25792	.82385	1.21382	31
30	.73996	1.35142	.76733	1.30323	.79544	1.25717	.82434	1.21310	30
31	.74041	1.35060	.76779	1.30244	.79591	1.25642	.82483	1.21238	29
32	.74086	1.34978	.76825	1.30166	.79639	1.25567	.82531	1.21166	28
33	.74131	1.34896	.76871	1.30087	.79686	1.25492	.82580	1.21094	27
34	.74176	1.34814	.76918	1.30009	.79734	1.25417	.82629	1.21023	26
35	.74221	1.34732	.76964	1.29931	.79781	1.25343	.82678	1.20951	25
36	.74267	1.34650	.77010	1.29853	.79829	1.25268	.82727	1.20879	24
37	.74312	1.34568	.77057	1.29775	.79877	1.25193	.82776	1.20808	23
38	.74357	1.34487	.77103	1.29696	.79924	1.25118	.82825	1.20736	22
39	.74402	1.34405	.77149	1.29618	.79972	1.25044	.82874	1.20665	21
40	.74447	1.34323	.77196	1.29541	.80020	1.24969	.82923	1.20593	20
41	.74492	1.34242	.77242	1.29463	.80067	1.24895	.82972	1.20522	19
42	.74538	1.34160	.77289	1.29385	.80115	1.24820	.83022	1.20451	18
43	.74583	1.34079	.77335	1.29307	.80163	1.24746	.83071	1.20379	17
44	.74628	1.33998	.77382	1.29229	.80211	1.24672	.83120	1.20308	16
45	.74674	1.33916	.77428	1.29152	.80258	1.24597	.83169	1.20237	15
46	.74719	1.33835	.77475	1.29074	.80306	1.24523	.83218	1.20166	14
47	.74764	1.33754	.77521	1.28997	.80354	1.24449	.83268	1.20095	13
48	.74810	1.33673	.77568	1.28919	.80402	1.24375	.83317	1.20024	12
49	.74855	1.33592	.77615	1.28842	.80450	1.24301	.83366	1.19953	11
50	.74900	1.33511	.77661	1.28764	.80498	1.24227	.83415	1.19882	10
51	.74946	1.33430	.77708	1.28687	.80546	1.24153	.83465	1.19811	9
52	.74991	1.33349	.77754	1.28610	.80594	1.24079	.83514	1.19740	8
53	.75037	1.33268	.77801	1.28533	.80642	1.24005	.83564	1.19669	7
54	.75082	1.33187	.77848	1.28456	.80690	1.23931	.83613	1.19599	6
55	.75128	1.33107	.77895	1.28379	.80738	1.23858	.83662	1.19528	5
56	.75173	1.33026	.77941	1.28302	.80786	1.23784	.83712	1.19457	4
57	.75219	1.32946	.77988	1.28225	.80834	1.23710	.83761	1.19387	3
58	.75264	1.32865	.78035	1.28148	.80882	1.23637	.83811	1.19316	2
59	.75310	1.32785	.78082	1.28071	.80930	1.23563	.83860	1.19246	1
60	.75355	1.32704	.78129	1.27994	.80978	1.23490	.83910	1.19175	0
′	Co-tan. 53°	Tan. 53°	Co-tan. 52°	Tan. 52°	Co-tan. 51°	Tan. 51°	Co-tan. 50°	Tan 50°	′

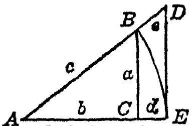
TABLE XXI.—*Continued.*

′	40° Tan.	40° Co-tan.	41° Tan.	41° Co-tan.	42° Tan.	42° Co-tan.	43° Tan.	43° Co-tan.	′
0	.83910	1.19175	.86929	1.15037	.90040	1.11061	.93252	1.07237	60
1	.83960	1.19105	.86980	1.14969	.90093	1.10996	.93306	1.07174	59
2	.84009	1.19035	.87031	1.14902	.90146	1.10931	.93360	1.07112	58
3	.84059	1.18964	.87082	1.14834	.90199	1.10867	.93415	1.07049	57
4	.84108	1.18894	.87133	1.14767	.90251	1.10802	.93469	1.06987	56
5	.84158	1.18824	.87184	1.14699	.90304	1.10737	.93524	1.06925	55
6	.84208	1.18754	.87236	1.14632	.90357	1.10672	.93578	1.06862	54
7	.84258	1.18684	.87287	1.14565	.90410	1.10607	.93633	1.06800	53
8	.84307	1.18614	.87338	1.14498	.90463	1.10543	.93688	1.06738	52
9	.84357	1.18544	.87389	1.14430	.90516	1.10478	.93742	1.06676	51
10	.84407	1.18474	.87441	1.14363	.90569	1.10414	.93797	1.06613	50
11	.84457	1.18404	.87492	1.14296	.90621	1.10349	.93852	1.06551	49
12	.84507	1.18334	.87543	1.14229	.90674	1.10285	.93906	1.06489	48
13	.84556	1.18264	.87595	1.14162	.90727	1.10220	.93961	1.06427	47
14	.84606	1.18194	.87646	1.14095	.90781	1.10156	.94016	1.06365	46
15	.84656	1.18125	.87698	1.14028	.90834	1.10091	.94071	1.06303	45
16	.84706	1.18055	.87749	1.13961	.90887	1.10027	.94125	1.06241	44
17	.84756	1.17986	.87801	1.13894	.90940	1.09963	.94180	1.06179	43
18	.84806	1.17916	.87852	1.13828	.90993	1.09899	.94235	1.06117	42
19	.84856	1.17846	.87904	1.13761	.91046	1.09834	.94290	1.06056	41
20	.84906	1.17777	.87955	1.13694	.91099	1.09770	.94345	1.05994	40
21	.84956	1.17708	.88007	1.13627	.91153	1.09706	.94400	1.05932	39
22	.85006	1.17638	.88059	1.13561	.91206	1.09642	.94455	1.05870	38
23	.85057	1.17569	.88110	1.13494	.91259	1.09578	.94510	1.05809	37
24	.85107	1.17500	.88162	1.13428	.91313	1.09514	.94565	1.05747	36
25	.85157	1.17430	.88214	1.13361	.91366	1.09450	.94620	1.05685	35
26	.85207	1.17361	.88265	1.13295	.91419	1.09386	.94676	1.05624	34
27	.85257	1.17292	.88317	1.13228	.91473	1.09322	.94731	1.05562	33
28	.85307	1.17223	.88369	1.13162	.91526	1.09258	.94786	1.05501	32
29	.85358	1.17154	.88421	1.13096	.91580	1.09195	.94841	1.05439	31
30	.85408	1.17085	.88473	1.13029	.91633	1.09131	.94896	1.05378	30
31	.85458	1.17016	.88524	1.12963	.91687	1.09067	.94952	1.05317	29
32	.85509	1.16947	.88576	1.12897	.91740	1.09003	.95007	1.05255	28
33	.85559	1.16878	.88628	1.12831	.91794	1.08940	.95062	1.05194	27
34	.85609	1.16809	.88680	1.12765	.91847	1.08876	.95118	1.05133	26
35	.85660	1.16741	.88732	1.12699	.91901	1.08813	.95173	1.05072	25
36	.85710	1.16672	.88784	1.12633	.91955	1.08749	.95229	1.05010	24
37	.85761	1.16603	.88836	1.12567	.92008	1.08686	.95284	1.04949	23
38	.85811	1.16535	.88888	1.12501	.92062	1.08622	.95340	1.04888	22
39	.85862	1.16466	.88940	1.12435	.92116	1.08559	.95395	1.04827	21
40	.85912	1.16398	.88992	1.12369	.92170	1.08496	.95451	1.04766	20
41	.85963	1.16329	.89045	1.12303	.92224	1.08432	.95506	1.04705	19
42	.86014	1.16261	.89097	1.12238	.92277	1.08369	.95562	1.04644	18
43	.86064	1.16192	.89149	1.12172	.92331	1.08306	.95618	1.04583	17
44	.86115	1.16124	.89201	1.12106	.92385	1.08243	.95673	1.04522	16
45	.86166	1.16056	.89253	1.12041	.92439	1.08179	.95729	1.04461	15
46	.86216	1.15987	.89306	1.11975	.92493	1.08116	.95785	1.04401	14
47	.86267	1.15919	.89358	1.11909	.92547	1.08053	.95841	1.04340	13
48	.86318	1.15851	.89410	1.11844	.92601	1.07990	.95897	1.04279	12
49	.86368	1.15783	.89463	1.11778	.92655	1.07927	.95952	1.04218	11
50	.86419	1.15715	.89515	1.11713	.92709	1.07864	.96008	1.04158	10
51	.86470	1.15647	.89567	1.11648	.92763	1.07801	.96064	1.04097	9
52	.86521	1.15579	.89620	1.11582	.92817	1.07738	.96120	1.04036	8
53	.86572	1.15511	.89672	1.11517	.92872	1.07676	.96176	1.03976	7
54	.86623	1.15443	.89725	1.11452	.92926	1.07613	.96232	1.03915	6
55	.86674	1.15375	.89777	1.11387	.92980	1.07550	.96288	1.03855	5
56	.86725	1.15308	.89830	1.11321	.93034	1.07487	.96344	1.03794	4
57	.86776	1.15240	.89883	1.11256	.93088	1.07425	.96400	1.03734	3
58	.86827	1.15172	.89935	1.11191	.93143	1.07362	.96457	1.03674	2
59	.86878	1.15104	.89988	1.11126	.93197	1.07299	.96513	1.03613	1
60	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	0
′	49° Co-tan.	49° Tan.	48° Co-tan.	48° Tan.	47° Co-tan.	47° Tan.	46° Co-tan.	46° Tan.	′

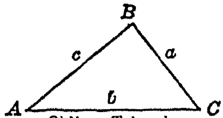
TABLE XXI.—*Concluded*

	44°				44°				44°		
′	TAN.	CO-TAN.	′	′	TAN.	CO-TAN.	′	′	TAN.	CO-TAN.	′
0	.96569	1.03553	60	21	.97756	1.02295	39	41	.98901	1.01112	19
1	.96625	1.03493	59	22	.97813	1.02236	38	42	.98958	1.01053	18
2	.96681	1.03433	58	23	.97870	1.02176	37	43	.99016	1.00994	17
3	.96738	1.03372	57	24	.97927	1.02117	36	44	.99073	1.00935	16
4	.96794	1.03312	56	25	.97984	1.02057	35	45	.99131	1.00876	15
5	.96850	1.03252	55	26	.98041	1.01998	34	46	.99189	1.00818	14
6	.96907	1.03192	54	27	.98098	1.01939	33	47	.99247	1.00759	13
7	.96963	1.03132	53	28	.98155	1.01879	32	48	.99304	1.00701	12
8	.97020	1.03072	52	29	.98213	1.01820	31	49	.99362	1.00642	11
9	.97076	1.03012	51	30	.98270	1.01761	30	50	.99420	1.00583	10
10	.97133	1.02952	50	31	.98327	1.01702	29	51	.99478	1.00525	9
11	.97189	1.02892	49	32	.98384	1.01642	28	52	.99536	1.00467	8
12	.97246	1.02832	48	33	.98441	1.01583	27	53	.99594	1.00408	7
13	.97302	1.02772	47	34	.98499	1.01524	26	54	.99652	1.00350	6
14	.97359	1.02713	46	35	.98556	1.01465	25	55	.99710	1.00291	5
15	.97416	1.02653	45	36	.98613	1.01406	24	56	.99768	1.00233	4
16	.97472	1.02593	44	37	.98671	1.01347	23	57	.99826	1.00175	3
17	.97529	1.02533	43	38	.98728	1.01288	22	58	.99884	1.00116	2
18	.97586	1.02474	42	39	.98786	1.01229	21	59	.99942	1.00058	1
19	.97643	1.02414	41	40	.98843	1.01170	20	60	1	1	0
20	.97700	1.02355	40								
′	CO-TAN.	TAN.	′	′	CO-TAN.	TAN.	′	′	CO-TAN.	TAN.	′
	45°				45°				45°		

Table XXII.—Trigonometric Formulas



Right Triangle



Oblique Triangle

Right Triangles

$$\sin A = \frac{a}{c} = \cos B \qquad \sec A = \frac{c}{b} = \operatorname{cosec} B$$

$$\cos A = \frac{b}{c} = \sin B \qquad \operatorname{cosec} A = \frac{c}{a} = \sec B$$

$$\tan A = \frac{a}{b} = \cot B \qquad \operatorname{vers} A = \frac{c-b}{c} = \frac{d}{c}$$

$$\cot A = \frac{b}{a} = \tan B \qquad \operatorname{exsec} A = \frac{e}{c}$$

$$a = c \sin A = c \cos B = b \tan A = b \cot B = \sqrt{c^2 - b^2}$$

$$b = c \cos A = c \sin B = a \cot A = a \tan B = \sqrt{c^2 - a^2}$$

$$c = \frac{a}{\sin A} = \frac{a}{\cos B} = \frac{b}{\sin B} = \frac{b}{\cos A} = \frac{d}{\operatorname{vers} A} = \frac{e}{\operatorname{exsec} A} = \sqrt{a^2 + b^2}$$

$$d = c \operatorname{vers} A \qquad e = c \operatorname{exsec} A$$

Oblique Triangles

Given	Sought	Formulas
A, B, a	b, c	$b = \frac{a}{\sin A}\cdot\sin B \qquad c = \frac{a}{\sin A}\cdot\sin (A + B)$
A, a, b	B, c	$\sin B = \frac{\sin A}{a}\cdot b \qquad c = \frac{a}{\sin A}\cdot\sin C$
C, a, b	$\frac{1}{2}(A + B)$	$\frac{1}{2}(A + B) = 90° - \frac{1}{2}C$
	$\frac{1}{2}(A - B)$	$\tan\frac{1}{2}(A - B) = \frac{a - b}{a + b}\cdot\tan\frac{1}{2}(A + B)$
a, b, c	A	If $s = \frac{1}{2}(a + b + c)$, $\sin \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$
		$\cos\frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$, $\tan\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
		$\sin A = 2\frac{\sqrt{s(s-a)(s-b)(s-c)}}{bc}$
		$\operatorname{vers} A = \frac{2(s-b)(s-c)}{bc}$
	area	*area* $= \sqrt{s(s-a)(s-b)(s-c)}$
C, a, b	*area*	*area* $= \frac{1}{2}ab \sin C$
A, B, C, a	*area*	*area* $= \frac{a^2 \sin B \cdot \sin C}{2 \sin A}$

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